QUANTITATIVE HELLY-TYPE THEOREMS VIA SPARSE APPROXIMATION

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Abstract. We prove the following sparse approximation result for polytopes. Assume that Q is a polytope in John's position. Then there exist at most $2d$ vertices of Q whose convex hull Q' satisfies $Q \subseteq -2d^2\, Q'$. As a consequence, we retrieve the best bound for the quantitative Helly-type result for the volume, achieved by Brazitikos, and improve on the strongest bound for the quantitative Helly-type theorem for the diameter, shown by Ivanov and Naszódi: We prove that given a finite family $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$ with intersection K, we may select at most $2d$ members of $\mathcal{F}$ such that their intersection has volume at most $(cd)^{3d/2} \operatorname{vol} K$, and it has diameter at most $2d^2 \operatorname{diam} K$, for some absolute constant $c > 0$.

1. History and results

Helly's theorem, dated from 1923 [H23], is a cornerstone result in convex geometry. Its finitary version states that the intersection of a finite family of convex sets is empty if and only if there exists a subfamily of $d + 1$ sets such that its intersection is empty. In 1982, Bárány, Katchalski and Pach [BKP82] introduced the following quantitative versions of Helly's theorem: there exist positive constants $v(d), \delta(d)$ such that for a finite family $\mathcal{F}$ of convex bodies, one may select $2d$ members such that their intersection has volume at most $v(d) \operatorname{vol}(\bigcap \mathcal{F})$, or has diameter at most $\delta(d) \operatorname{diam}(\bigcap \mathcal{F})$.

The problem of finding the optimal values of $\delta(d)$ and $v(d)$ has enjoyed special interest in recent years (see e.g. the excellent survey article [BK21]). In [BKP82] (see also [BKP84]) the authors proved that $v(d) \leq d^{2d^2}$ and $\delta(d) \leq d^{2d}$, and they conjectured that $v(d) \approx d^{c_1 d}$ and $\delta(d) \approx c_2 d^{1/2}$ for some positive constants $c_1, c_2 > 0$.

For the volume problem, in a breakthrough paper, Naszódi [N16] proved that $v(d) \leq e^{d+1} d^{2d+\frac{1}{2}}$, while $v(d) \geq d^{d/2}$ must hold. Improving upon his ideas, Brazitikos [B17] found the current best upper bound for volume: $v(d) \leq (cd)^{3d/2}$ for a constant $c > 0$.

For the diameter question, Brazitikos [B18] proved the first polynomial bound on $\delta(d)$ by showing that $\delta(d) \leq cd^{11/2}$ for some $c > 0$. In 2021, Ivanov and Naszódi [IN21] found the best known upper bound, $\delta(d) \leq (2d)^3$, and also proved that $\delta(d) \geq cd^{1/2}$. Thus, the value conjectured in [BKP82] for $\delta(d)$ would be asymptotically sharp.

In the present note, we show that given a finite family $\mathcal{F}$ of closed convex sets, one can select at most $2d$ members such that their intersection sits inside a scaled version of $\bigcap \mathcal{F}$ for

Date: August 17, 2021.

2020 *Mathematics Subject Classification.* 52A35, 52A27.

Key words and phrases. Helly-type theorem, volume, diameter, sparse approximation, John's ellipsoid.

Research of the second named author was supported by NKFIH grant KKP-133819 and by the EFOP-3.6.1-16-2016-00008 project, which in turn has been supported by the European Union, co-financed by the European Social Fund.

a suitable location of the origin. Clearly, it suffices to prove this statement for the special case when $\mathcal{F}$ consists of closed halfspaces intersecting in a convex body. As an application, we obtain an improvement on the diameter bound, $\delta(d) \leq 2d^2$, and retrieve the best known bound for $v(d)$. The crux of the argument is the following sparse approximation result for polytopes, which is a strengthening of Theorem 2 in [IN21].

Theorem 1. *Let $\lambda > 0$ and $Q \subset \mathbb{R}^d$ be a convex polytope such that $Q \subseteq -\lambda Q$. Then there exist at most $2d$ vertices of Q whose convex hull Q' satisfies*

$$Q \subseteq -(\lambda + 2)d\, Q'.$$

We immediately obtain the following corollary.

Corollary 2. *Assume that $Q = -Q$ is a symmetric convex polytope in $\mathbb{R}^d$. Then we may select a set of at most $2d$ vertices of Q with convex hull Q' such that*

$$Q \subseteq 3d\, Q'.$$

As usual, let B^d denote the unit ball of $\mathbb{R}^d$ and let S^{d-1} be the unit sphere of $\mathbb{R}^d$. A standard consequence of Fritz John's theorem [J48] states that if $K \subset \mathbb{R}^d$ is a convex body in John's position, that is, the largest volume ellipsoid inscribed in K is B^d, then $B^d \subseteq K \subseteq dB^d \subseteq -dK$ (see e.g. [B97]). Thus, we derive

Corollary 3. *Assume that $Q \subset \mathbb{R}^d$ is a convex polytope in John's position. Then there exists a subset of at most $2d$ vertices of Q whose convex hull Q' satisfies*

$$Q \subseteq -2d^2\, Q'.$$

For a family of sets $\{K_1, \ldots, K_n\} \subset \mathbb{R}^d$ and for a subset $\sigma \subset [n] = \{1, \ldots, n\}$, we will use the notation

$$K_\sigma = \bigcap_{i \in \sigma} K_i,$$

as in [IN21]. Also, $\binom{[n]}{\leq k}$ stands for the set of all subsets of $[n]$ with cardinality at most k. Using this terminology, we are ready to state our quantitative Helly-type result.

Theorem 4. *Let $\{K_1, \ldots, K_n\}$ be a family of closed convex sets in $\mathbb{R}^d$ with $d \geq 2$ such that their intersection $K = K_1 \cap \cdots \cap K_n$ is a convex body. Then there exists a $\sigma \in \binom{[n]}{\leq 2d}$ such that*

$$\operatorname{vol}_d K_\sigma \leq (cd)^{3d/2} \operatorname{vol}_d K \quad \text{and} \quad \operatorname{diam} K_\sigma \leq 2d^2 \operatorname{diam} K$$

for some constant $c > 0$.

To conclude the section we formulate the following conjecture, which was essentially stated already in [BKP82]. This would imply the asymptotically sharp bound for $v(d)$.

Conjecture 5. *Assume that $\{u_1, \ldots, u_n\} \subset S^{d-1}$ is a set of unit vectors satisfying the conditions of Fritz John's theorem. That is, there exist positive numbers $\alpha_1, \ldots, \alpha_n$ for which $\sum_{i=1}^n \alpha_i u_i = 0$ and $\sum_{i=1}^n \alpha_i u_i \otimes u_i = I_d$, the identity operator on $\mathbb{R}^d$. Then there exists a subset $\sigma \subset [n]$ with cardinality at most $2d$ so that*

$$B^d \subset c\, d \operatorname{conv}\{u_i : i \in \sigma\}$$

with an absolute constant $c > 0$.

That the above estimate would be asymptotically sharp is shown by taking $n = d + 1$ and letting $\{u_1, \ldots, u_n\}$ to be the set of vertices of a regular simplex inscribed in S^{d-1}.

2. Proofs

Proof of Theorem 1. The condition $Q \subseteq -\lambda Q$ ensures that $0 \in \operatorname{int} Q$. Among all simplices with d vertices from the vertices of Q and one vertex at the origin, consider a simplex $S = \operatorname{conv}\{0, v_1, \ldots, v_d\}$ with maximal volume. We write S in the form

$$S = \left\{ x \in \mathbb{R}^d : x = \alpha_1 v_1 + \ldots + \alpha_d v_d \text{ for } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{d} \alpha_i \leq 1 \right\}. \tag{1}$$

For every $i = 1, \ldots, d$, let H_i be the hyperplane spanned by $\{0, v_1, \ldots, v_d\} \setminus \{v_i\}$, and let L_i be the strip between the hyperplanes $v_i + H_i$ and $-v_i + H_i$. Define $P = \bigcap_{i \in [d]} L_i$ (see Figure 1).

Note that

$$P = \{x \in \mathbb{R}^d : \operatorname{vol}_d(\operatorname{conv}(\{0, x, v_1, \ldots, v_d\} \setminus \{v_i\}) \leq \operatorname{vol}_d(S) \text{ for all } i = 1, \ldots, d\}. \tag{2}$$

This follows from the volume formula

$$\operatorname{vol}_d(\operatorname{conv}\{0, w_1, \ldots, w_d\}) = \frac{1}{d!} \big| \det(w_1\, w_2\, \cdots\, w_d) \big|$$

for arbitrary $w_1, \ldots, w_d \in \mathbb{R}^d$, which implies that for all $x \in \mathbb{R}^d$ of the form $x = cv_i + w$ with $w \in H_i$, $i = 1, \ldots, d$,

$$\operatorname{vol}_d(\operatorname{conv}(\{0, x, v_1, \ldots, v_d\} \setminus \{v_i\}) = |c| \operatorname{vol}_d(S).$$

Next, we show that

$$P = \{x \in \mathbb{R}^d : x = \beta_1 v_1 + \ldots + \beta_d v_d \text{ for } \beta_i \in [-1, 1]\}. \tag{3}$$

Indeed, since $v_1, \ldots, v_d$ are linearly independent, we may consider the linear transformation A with $A(v_i) = e_i$ for $i = 1, \ldots, d$. Note that

$$A(P) = A\left(\bigcap_{i \in [d]} L_i \right) = \bigcap_{i \in [d]} A(L_i) = \{x \in \mathbb{R}^d : x = \beta_1 e_1 + \cdots + \beta_d e_d \text{ for } \beta_i \in [-1, 1]\}.$$

Thus, (3) holds.

Since S is chosen maximally, equation (2) shows that for any vertex w of Q, $w \in P$. By convexity,

$$Q \subseteq P. \tag{4}$$

Let $S' = -2dS + (v_1 + \ldots + v_d)$. By (1),

$$S' = \left\{ x \in \mathbb{R}^d : x = \gamma_1 v_1 + \ldots + \gamma_d v_d \text{ for } \gamma_i \leq 1 \text{ and } \sum_{i=1}^{d} \gamma_i \geq -d \right\}. \tag{5}$$

Then, from (3) and (5),

$$P \subseteq S'. \tag{6}$$

Let $u = \frac{1}{d}(v_1 + \ldots + v_d)$ be the centroid of the facet $\operatorname{conv}\{v_1, \ldots, v_d\}$ of S. Let y be the intersection of the ray from 0 through $-u$ and the boundary of Q. By Carathéodory's theorem, we can choose $k \leq d$ vertices $\{v'_1, \ldots, v'_k\}$ of Q such that $y \in \operatorname{conv}\{v'_1, \ldots, v'_k\}$. Set $Q' = \operatorname{conv}\{v_1, \ldots, v_d, v'_1, \ldots, v'_k\}$.

Note that $[y, u] \subseteq Q'$, which implies $0 \in Q'$. Thus,

$$S \subseteq Q'. \tag{7}$$

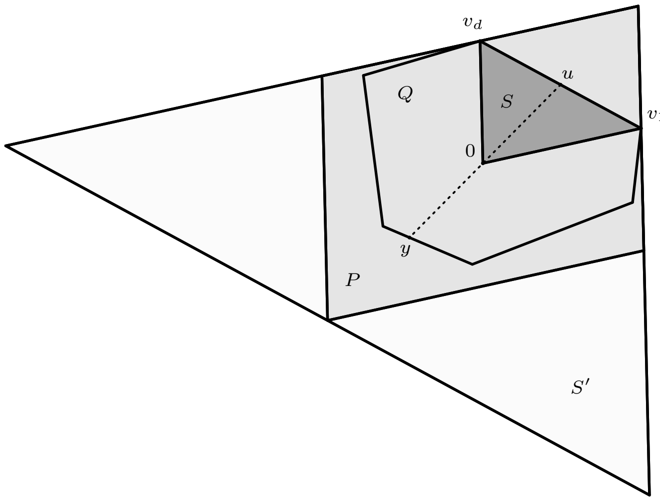

FIGURE 1.

Since $Q \subseteq -\lambda Q$, we have that $-u \in \lambda Q$. Since λy is on the boundary of λQ, we also have that $-u \in [0, \lambda y]$. We know that $0, \lambda y \in \lambda Q'$, so

$$u \in -\lambda Q'. \tag{8}$$

Combining (4), (6), (7), and (8):

$$Q \subseteq P \subseteq S' = -2dS + du \subseteq -2d\,Q' - \lambda d\,Q' = -(\lambda + 2)d\,Q'. \tag{9}$$ □

Proof of Theorem 4. As shown in [BKP82], we may assume that $\{K_1, \ldots, K_n\}$ consists of closed halfspaces such that $K = \bigcap K_i$ is a d-dimensional polytope. Let T be the affine transformation sending K to John's position. Let $\widetilde{K}_i = TK_i$ for $i \in [n]$, $\widetilde{K} = TK$, and for some $\sigma \subset [n]$, let $\widetilde{K}_\sigma = \bigcap_{i \in \sigma} \widetilde{K}_i$. We claim that there exists $\sigma \in \binom{[n]}{\leq 2d}$ such that the following two properties hold:

$$\widetilde{K}_\sigma \subseteq -2d^2 \widetilde{K}, \text{ and} \tag{10}$$

$$\mathrm{vol}_d\, \widetilde{K}_\sigma \leq (cd)^{3d/2}\, \mathrm{vol}_d\, \widetilde{K} \tag{11}$$

for some constant $c > 0$. Statements (10) and (11) are affine invariant, so this will suffice to prove Theorem 4.

Recall that since $\widetilde{K}$ is in John's position, $B^d \subseteq \widetilde{K} \subseteq dB^d$ (see [B97] or [GLMP04, Theorem 5.1]). Setting $Q = (\widetilde{K})^\circ$, this yields that $\frac{1}{d} B^d \subseteq Q \subseteq B^d$. In particular, $Q \subseteq -dQ$. Hence, we may apply Theorem 1 to Q with $\lambda = d$, we obtain a subset of at most $2d$ vertices of Q such that their convex hull Q' satisfies

$$Q \subseteq -(d+2)dQ' \subseteq -2d^2 Q'. \tag{12}$$

The family of closed halfspaces supporting the facets of $(Q')^\circ$ is a subset of $\{\widetilde{K}_1, \ldots, \widetilde{K}_n\}$ with at most $2d$ elements. Thus, we can choose $\sigma \in \binom{[n]}{\leq 2d}$ such that $\widetilde{K}_\sigma = (Q')^\circ$. Taking the polar of (12), we obtain

$$\widetilde{K}_\sigma \subseteq -(d+2)d\widetilde{K} \subseteq -2d^2 \widetilde{K},$$

which shows (10).

Let P be the parallelotope enclosing Q from the proof of Theorem 1 and set $P' = -\frac{1}{2d^2}P$. Statement (9) implies

$$Q' \supseteq P'.$$

Since S is chosen maximally, the volume of S is at least the volume of the simplex obtained from the Dvoretzky-Rogers lemma [DR50] (see also [N16, Lemma 1.4]):

$$\text{vol}_d(S) \geq \frac{1}{\sqrt{d!}d^{d/2}}. \tag{13}$$

Using (13),

$$\text{vol}_d(P') = (2d^2)^{-d}\,\text{vol}_d(P) = (2d^2)^{-d} \cdot 2^d d!\,\text{vol}_d(S) \geq d^{-5d/2}(d!)^{1/2}. \tag{14}$$

Note that P' is centrally symmetric, so we can use the Blaschke-Santaló inequality (see [AGM15, Theorem 1.5.10]) for P':

$$\text{vol}_d(P') \cdot \text{vol}_d((P')^\circ) \leq \text{vol}_d(B_2^d)^2. \tag{15}$$

Using the inclusions $\widetilde{K} \supseteq B_2^d$ and $\widetilde{K}_\sigma = (Q')^\circ \subseteq (P')^\circ$, as well as (14) and (15):

$$\frac{\text{vol}_d \widetilde{K}_\sigma}{\text{vol}_d \widetilde{K}} \leq \frac{\text{vol}_d((P')^\circ)}{\text{vol}_d(B_2^d)} \leq \frac{\text{vol}_d(B_2^d)}{\text{vol}_d(P')} \leq \frac{\pi^{d/2}d^{5d/2}(d!)^{-1/2}}{\Gamma((d/2)+1)} \leq (cd)^{3d/2}$$

for some absolute constant $c > 0$. This shows (11) and concludes the proof. □

3. Acknowledgements

This research was done under the auspices of the Budapest Semesters in Mathematics program.

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