VECTOR SUM-INTERSECTION THEOREMS

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Abstract. We introduce the following generalization of set intersection via characteristic vectors: for $n, q, s, t \geq 1$ a family $\mathcal{F} \subseteq \{0, 1, \ldots, q\}^n$ of vectors is said to be *s-sum t-intersecting* if for any distinct $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ there exist at least t coordinates, where the entries of $\mathbf{x}$ and $\mathbf{y}$ sum up to at least s, i.e. $|\{i : x_i + y_i \geq s\}| \geq t$. The original set intersection corresponds to the case $q = 1, s = 2$.

We address analogs of several variants of classical results in this setting: the Erdős–Ko–Rado theorem or the theorem of Bollobás on intersecting set pairs.

1. Introduction

Many problems in extremal finite set theory ask for the maximum size of a family (or some other combinatorial object) that satisfies some intersection property. When members of the family examined are subsets of $[n] := \{1, 2, \ldots, n\}$, then there is a one-to-one correspondence between a set F and its 0-1 characteristic vector $\mathbf{x}_F$ of length n, that has a 1-entry in its ith coordinate if and only if $i \in F$ for $i \in [n]$. So one can say that two sets F and G intersect, if the sum of their characteristic vectors (as vectors in $\mathbb{Z}^n$) contains a 2 in some coordinate. The goal of this paper is the introduction of a notion of intersection that generalizes set intersection (translated to sum of characteristic vectors) to a type of intersection among q-ary vectors.

To do so for $q, n \geq 1$ we introduce the notation $Q^n := \{0, 1, \ldots, q\}^n$, and we will consider it as a subset of $\mathbb{Z}^n$ (so addition is *not* modulo $q + 1$). We will denote the vectors by boldface letters and the ith coordinate of the vector $\mathbf{x}$ will be denoted by x_i.

There exist intersection results in the literature for vectors (under the name of integer sequences) with several types of definition for intersection, we mention two of them: the *permutation-type* definition is that $\mathbf{x}, \mathbf{y} \in \{0, 1, \ldots, q\}^n$ intersect if there exists i with $x_i = y_i$ and more generally $|\mathbf{x} \cap_{\text{perm}} \mathbf{y}| = |\{i : x_i = y_i\}|$; for results about this type of intersection see e.g. [9, 10]. The *multiset-type* definition corresponds to multisets represented by vectors and in this case for $\mathbf{x}, \mathbf{y} \in \{0, 1, \ldots, q\}^n$ we have $|\mathbf{x} \cap_{\text{multi}} \mathbf{y}| = \sum_i \min\{x_i, y_i\}$; for corresponding results see e.g. [11, 12].

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As we mentioned earlier, our next definition generalizes set intersection based on the fact that F and G intersect if and only if there exists $i \in [n]$ such that $(\mathbf{x}_F)_i + (\mathbf{x}_G)_i \geq 2$ holds.

Definition 1.1. For integers $n, q, s \geq 1$ and two vectors $\mathbf{x}, \mathbf{y} \in Q^n$, we define the *size of their s-sum intersection* as $|\mathbf{x} \cap_s \mathbf{y}| = |\{i : x_i + y_i \geq s\}|$.

For $t \geq 1$ we say that $\mathbf{x}, \mathbf{y} \in Q^n$ are *s-sum t-intersecting*, if $|\mathbf{x} \cap_s \mathbf{y}| \geq t$. More generally, $\mathcal{F} \subset Q^n$ is *s-sum t-intersecting* if any two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ are s-sum t-intersecting.

In case of $t = 1$ we just simply write *s-sum intersecting* instead of s-sum 1-intersecting.

Note that in the case of $q = 1$ and $s = 2$ we get back the same notions for sets.

We will consider analogs of the Erdős–Ko–Rado theorem and theorems about Bollobás's intersecting set-pair systems. To be able to state our results first we need to define uniformity for families of vectors. One has several options: as in the case of multisets and many other types of problems, we can work with the *weight/rank* $\sum_{i=1}^n x_i$ of $\mathbf{x} \in Q^n$ and say that for an integer $r \geq 0$ a family $\mathcal{F} \subseteq Q^n$ is *r-rank uniform* if $r(\mathbf{x}) := \sum_{i=1}^n x_i = r$ for all $\mathbf{x} \in \mathcal{F}$. Another possible notion for the size of a vector is *the size of its support*, i.e. $|\{i : x_i \neq 0\}|$. We say that $\mathcal{F} \subseteq Q^n$ is *r-support uniform* if the size of the support of every $\mathbf{x} \in \mathcal{F}$ is r.

Notation. We use the following notations.

• For any set X, we denote by $\binom{X}{r}$ the family of all r-subsets of X and 2^X denotes the power set of X.

• For a set $F \subset [n]$ we denote its complement, i.e. $[n] \setminus F$ by $\overline{F}$ and for $\mathcal{F}$ a family of subsets of $[n]$ we introduce the notation $\overline{\mathcal{F}} := \{\overline{F} : F \in \mathcal{F}\}$. For any vector $\mathbf{x} \in Q^n$ let us define its 'complement' $\overline{\mathbf{x}}$ as $\overline{x}_i := q - x_i$ for all $i \in [n]$ and for $\mathcal{F}$ a family of vectors in Q^n let us introduce the notation $\overline{\mathcal{F}} := \{\overline{\mathbf{x}} : \mathbf{x} \in \mathcal{F}\}$.

• For $\mathbf{x} \in Q^n$ we denote its support by $S_{\mathbf{x}}$.

• For two functions $f, g : \mathbb{N} \to \mathbb{N}$ we say that $f = O(g)$, if there is a constant c and an $n_0 \in \mathbb{N}$ such that $f \leq cg$ for all $n \geq n_0$.

Structure of the paper.

The structure of the paper is the following. In Subsection 1.1 we state various results about s-sum intersecting families of vectors, while in Subsection 1.2 we list our result about intersecting vector pairs. In Section 2 and Section 3 we prove our results about intersecting vector and intersecting vector pairs, respectively. In Section 4—as concluding results—we give a new intersection definition to provide analogs of some results that would not work with s-sum intersection.

1.1. **Results on intersecting families of vectors.** Let us start with stating the seminal result of Erdős, Ko and Rado [5].

Theorem 1.2 (Erdős, Ko, Rado [5]). *For $n, r \geq 1$ with $2r \leq n$ if $\mathcal{F} \subseteq \binom{[n]}{r}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. Moreover, if $2r < n$ and $|\mathcal{F}| = \binom{n-1}{r-1}$, then $\mathcal{F} = \mathcal{F}_x := \{F : x \in F \in \binom{[n]}{r}\}$ holds for some $x \in [n]$.*

Furthermore, for any $1 \leq t < r$ there exists $n_0 = n_0(r,t)$ such that if $\mathcal{F} \subseteq \binom{[n]}{r}$ is t-intersecting, then $|\mathcal{F}| \leq \binom{n-t}{r-t}$ holds with equality if and only if $\mathcal{F} = \{F : T \subset F \in \binom{[n]}{r}\}$ for some $T \in \binom{[n]}{t}$.

The exact value of the smallest possible $n_0(r,t)$ was obtained by Frankl [7] and Wilson [22]. The largest possible size of an r-uniform t-intersecting family for all values of $n, t, r \geq 1$ was determined by Ahlswede and Khatchatrian [1].

Our first result is a generalization of the Erdős–Ko–Rado (or EKR, in short) theorem for r-support uniform families.

Theorem 1.3. *For any $q, s \geq 2$ and integer $r \geq 1$, there exists $n(q,s,r) \in \mathbb{N}$ such that if $\mathcal{F} \subseteq Q^n$ is r-support uniform s-sum intersecting with $n \geq n(q,r,s)$, then*

$$|\mathcal{F}| \leq \begin{cases} (q - \frac{s}{2} + 1)q^{r-1}\binom{n-1}{r-1} & \text{if } s \text{ is even,} \\ 1 + (q - \lceil \frac{s}{2} \rceil + 1)\sum_{i=1}^{r}\binom{n-i}{r-i}q^{r-i} & \text{if } s \text{ is odd,} \end{cases} \tag{1}$$

and these bounds are best possible.

The statement and proof of Theorem 1.3 can be adjusted for the r-rank uniform case, too. We only provide the statement and the proof in the special case $s = q+1$ that works for all meaningful values of n.

Before stating our theorem, observe that if both $\mathbf{x}, \mathbf{y} \in Q^n$ have rank less than $\frac{q+1}{2}$, then they cannot $(q+1)$-sum intersect, while if both of them have rank greater than $\frac{qn}{2}$ then they always $(q+1)$-sum intersect. We denote by $Q(n,r)$ the set of all vectors in Q^n of rank r.

Theorem 1.4. *Let $n, q, r \geq 1$ and $\mathcal{F} \subseteq Q^n$ be an r-rank uniform $(q+1)$-sum intersecting family with $\frac{q+1}{2} \leq r \leq \frac{qn}{2}$. Then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=\frac{q+1}{2}}^{q} |Q(n-1, r-j)| & \text{if } q+1 \text{ is even,} \\ 1 + \sum_{j=\lceil\frac{q+1}{2}\rceil}^{q} \sum_{i=1}^{\lfloor\frac{2(r-1)}{q}\rfloor} |Q(n-i, r-j-\frac{(i-1)q}{2})| & \text{if } q+1 \text{ is odd,} \end{cases} \tag{2}$$

and these bounds are best possible.

Now we continue with s-sum t-intersecting families with $t \geq 1$. We give the constructions that will be shown to be extremal for r-support uniform s-sum t-intersecting families.

Construction 1.5. For any $n, q, r, t \geq 1$ with $n \geq r \geq t$ and s even with $2 \leq s \leq 2q$ and for any $T \in \binom{[n]}{t}$ let us define

$$\mathcal{F}_{n,q,s,r,T} := \left\{ \mathbf{x} \in Q^n : \ x_i \geq \frac{s}{2} \text{ for all } i \in T \right\}.$$

For $n, r, q, t \geq 1$ with $n \geq r \geq t$ and s odd with $2 < s < 2q$ let us define the following r-support uniform families:

$$\mathcal{F}_{n,q,s,r,t} :=$$
$$\bigcup_{T'\in\binom{[r]}{t-1}} \left\{\mathbf{x}_{T'} : (\forall i \in T')((x_{T'})_i > \frac{s}{2}) \wedge (\forall i \in [r]\setminus T')((x_{T'})_i = \lfloor\frac{s}{2}\rfloor) \wedge S_{x_{T'}} = [r]\right\} \cup$$
$$\bigcup_{T\in\binom{[r]}{t}} \left\{\mathbf{y}_T \in Q^n : (\forall i \in T)((y_T)_i > \lfloor\frac{s}{2}\rfloor) \wedge (\forall i \in [\max T]\setminus T)((y_T)_i = \lfloor\frac{s}{2}\rfloor) \wedge S_{y_T} = [r]\right\}.$$

The family $\mathcal{F}_{n,q,s,r,t}$ is s-sum t-intersecting as for any pair $\mathbf{x},\mathbf{y} \in \mathcal{F}_{n,q,s,r,t}$ there exist at least t coordinates $i \in [r]$, where one of x_i, y_i is at least $\lfloor\frac{s}{2}\rfloor$ while the other is at least $\lceil\frac{s}{2}\rceil$.

Let $f(n,q,s,r,t)$ denote the size of $\mathcal{F}_{n,q,s,r,t}$.

Theorem 1.6. *For any $q, s \geq 2$ and $r \geq t \geq 1$, there exists $n(q,s,r,t)$ such that if $\mathcal{F} \subseteq Q^n$ is r-support uniform s-sum t-intersecting with $n \geq n(q,s,r,t)$, then*

$$|\mathcal{F}| \leq \begin{cases} (q-\frac{s}{2}+1)^t q^{r-t}\binom{n-t}{r-t} & \text{if } s \text{ is even,} \\ f(n,q,s,r,t) & \text{if } s \text{ is odd,} \end{cases} \tag{3}$$

and these bounds are best possible as shown by the families of Construction 1.5.

1.2. **Results on intersecting pairs of vectors.** Let us continue with stating another classical result, the theorem of Bollobás on intersecting set pairs for which we prove sum-intersecting analogs.

Theorem 1.7 (Bollobás [2]). *Let $\{(A_j, B_j) : j = 1, 2, \ldots, m\}$ be pairs of sets with $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then the inequality*

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_j|+|B_j|}{|A_j|}} \leq 1$$

holds. In particular, if $|A_j| \leq a$ and $|B_j| \leq b$ for all $j = 1, 2, \ldots, m$, then $m \leq \binom{a+b}{a}$.

Now we recall some notions from the literature. Suppose $\mathcal{S} = \{(A_i, B_i) : i = 1, 2, \ldots, m\}$. Then

- $\mathcal{S}$ is called a *strong ISP-system* (shorthand for intersecting set-pair system) if
  - $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq n$, and
  - $A_i \cap B_j \neq \emptyset$ for all $1 \leq i \neq j \leq n$;
- $\mathcal{S}$ is called a *weak ISP-system* if
  - $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq n$, and
  - at least one of $A_i \cap B_j \neq \emptyset$ and $B_i \cap A_j \neq \emptyset$ holds for all $1 \leq i \neq j \leq n$.

If also $a = \max_{1\leq i\leq n} |A_i|$ and $b = \max_{1\leq i\leq n} |B_i|$, then $\mathcal{S}$ is a strong or weak (a,b)-system. Note that Theorem 1.7 is about strong ISP-systems. In its flavor the following general inequality is valid for weak ISP-systems.

Theorem 1.8 (Tuza [18])**.** *Let $0 < p < 1$ be any real number and $q = 1 - p$. If $\{(A_j, B_j) : j = 1, 2, \dots, m\}$ is a weak ISP-system, then the inequality*

$$\sum_{j=1}^{m} p^{|A_j|} q^{|B_j|} \leq 1$$

holds. Moreover, for every $a, b \in \mathbb{N}$ there exists a weak (a, b)-system for which equality holds for all $0 < p < 1$ and $q = 1 - p$.

For a general overview on ISP-systems and their applications in extremal combinatorics we refer to the two-part survey [19, 20]. Theorem 1.8 implies the upper bound $m \leq \frac{(a+b)^{a+b}}{a^a b^b}$ for weak (a, b)-systems. The best lower bounds on the maximum size of weak (a, b)-systems are due to Király, Nagy, Pálvölgyi and Visontai [16], and Wagner [21].

Now we would like to generalize these notions to vectors in the s-sum intersecting setting. Note that there is no assumption on the size of the ground set of ISP-systems: neither in Theorem 1.7 nor in the results on weak ISP-systems. Let us denote by $Q^{<\mathbb{N}} (\subset \mathbb{Z}^{<\mathbb{N}})$ the set of all $\mathbf{x} \in \{0, 1, \dots, q\}^{\mathbb{N}}$ that are 0's everywhere except for a finite number of coordinates and for $i \in \mathbb{N}$ we denote the ith coordinate by x_i (just like in the finite dimensional case). The *support* of $\mathbf{x} \in Q^{<\mathbb{N}}$ is the (finite) set of all coordinates, where $\mathbf{x}$ is not zero and we denote it by $S_{\mathbf{x}}$.

Now for $m, s \geq 1$ we say that $\{(\mathbf{x}^j, \mathbf{y}^j) \in Q^{<\mathbb{N}} \times Q^{<\mathbb{N}} : j = 1, 2, \dots, m\}$ is a *strong s-sum IVP-system* in $Q^{<\mathbb{N}}$, if $|\mathbf{x}^j \cap_s \mathbf{y}^j| = 0$ for all $j = 1, 2, \dots, m$ and $|\mathbf{x}^i \cap_s \mathbf{y}^j| \neq 0$ for all $1 \leq i \neq j \leq m$. And we say that $\{(\mathbf{x}^j, \mathbf{y}^j) \in Q^{<\mathbb{N}} \times Q^{<\mathbb{N}} : j = 1, 2, \dots, m\}$ is a *weak s-sum IVP-system* in $Q^{<\mathbb{N}}$, if for all $1 \leq i \neq j \leq m$ *at least one* pair of $\mathbf{x}^i, \mathbf{y}^j$ or $\mathbf{x}^j, \mathbf{y}^i$ is s-sum intersecting. If the support of all $\mathbf{x}^j$ have size at most a, and the support of all $\mathbf{y}^j$ have size at most b, then we will talk about *strong* and *weak s-sum (a, b)-systems*.

We start with the following observation.

Observation 1.9.

(i) If $\mathcal{F}$ is a strong/weak s-sum (a, b)-system, then for all $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ and all $i \leq m$ we have $x_i, y_i < s$.

(ii) If $\mathcal{F} \subset (\{0, 1, \dots, q\}^{<\mathbb{N}})^2$ is a strong/weak $(q + t)$-sum (a, b)-system with $t > 1$, then there exists a $(q - t + 2)$-sum strong/weak (a, b)-system $\mathcal{F}' \subset (\{0, 1, \dots, q - t + 1\}^{<\mathbb{N}})^2$ with $|\mathcal{F}| = |\mathcal{F}'|$.

Proof. If $x_i \geq s$ or $y_j \geq s$ for some $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$, then $|\mathbf{x} \cap_s \mathbf{y}| > 0$. This implies (i).

To see (ii), for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ introduce $(\mathbf{x}', \mathbf{y}')$ with $x'_i = \max\{x_i - t + 1, 0\}, y'_i = \max\{y_i - t + 1, 0\}$ for all indices i. Clearly, for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{F}$ and index j, we have $x'_j + y'_j < q + t - 2(t - 1) = q - t + 2$. Furthermore, if $|\mathbf{x}^{h_1} \cap_{q+t} \mathbf{y}^{h_2}| > 0$, then there exists an index j with $q + t \leq x^{h_1}_j + y^{h_2}_j$. So $x^{h_1,}_j + y^{h_2,}_j \geq q + t - 2(t - 1) = q - t + 2$, and thus the system $\mathcal{F}' = \{(\mathbf{x}', \mathbf{y}') : (\mathbf{x}, \mathbf{y}) \in \mathcal{F}\} \subset (\{0, 1, \dots, q - t + 1\}^{<\mathbb{N}})^2$ is a $(q - t + 2)$-sum strong/weak (a, b)-system. $\square$

Observation 1.9 means that it is enough to deal with $(q+1)$-sum IVP-systems in $\{0,1,\ldots,q\}^{<\mathbb{N}}$. To obtain bounds on their size let us introduce the notation $m(q,k)$ and $m'(q,k)$ for the maximum number of vector pairs in a strong / weak $(q+1)$-sum (k,k)-system. In particular, for $q=2$ and $s=3$ let $m(k) := m(2,k)$ denote the maximum size of a strong 3-sum (k,k)-system in $\{0,1,2\}^{<\mathbb{N}}$.

To estimate $m(k)$, we let

$$f(k) := \max \frac{(x+y+z)!}{x!\,y!\,z!},$$

where the maximum is taken over all nonnegative integers x,y,z such that $x+z \le k$ and $y+z \le k$. The following inequalities provide an almost tight bound on $m(k)$, with only a linear multiplicative error in k, while the function is exponential.

Theorem 1.10. *For every $k \ge 1$ we have*

$$f(k) \le m(k) \le k \cdot f(k)\,.$$

Finally, we determine the order of magnitude of the maximum size of strong and weak $(q+1)$-sum IVP systems in $\{0,1,\ldots,q\}^{<\mathbb{N}}$ up to a polynomial factor.

Theorem 1.11. *For any $q \ge 1$, $\lim_{k\to\infty} \sqrt[k]{m(q,k)} = \lim_{k\to\infty} \sqrt[k]{m'(q,k)} = (\sqrt{q}+1)^2$.*

A standard calculation shows that the maximum in the definition of $f(k)$ is attained when $z = (1-\frac{1}{\sqrt{2}})k + O(1)$ and $x = y = k - z$. Plugging in these values, we obtain that $f(k) = (c+o(1))\frac{1}{k}(3+\sqrt{2})^k$ for some real $c<1$. The upper bound of Theorem 1.10 on strong 3-sum (k,k)-systems is a constant factor smaller than the upper bound obtained during the proof of Theorem 1.11 on weak 3-sum (k,k)-systems.

2. Sum-intersecting families of vectors

This subsection contains the proofs of Theorem 1.3, Theorem 1.4 and Theorem 1.6.

Proposition 2.1. *For $n,q \ge 1$ if $\mathcal{F} \subseteq Q^n$ is $(q+1)$-sum intersecting, then $|\mathcal{F}| \le \lceil \frac{(q+1)^n}{2} \rceil$ and this bound is best possible.*

Proof. Note that we cannot have $\mathbf{x}$ and $\overline{\mathbf{x}}$ both belong to $\mathcal{F}$. Moreover, there exists one vector $\mathbf{x}$ with $\overline{\mathbf{x}} = \mathbf{x}$ if and only if q is even. This proves the upper bound. For the lower bound consider the family of all vectors with rank larger than $\frac{qn}{2}$ together with one vector from each pair of (the not necessarily different vectors) $\mathbf{x}, \overline{\mathbf{x}}$ of rank $\frac{qn}{2}$ (if such pairs exist). $\square$

Corollary 2.2. *For $n,q,s \ge 1$ with $q \ge s$ if $\mathcal{F} \subseteq Q^n$ is s-sum intersecting, then $|\mathcal{F}| \le (q+1)^n - s^n + \lceil \frac{s^n}{2} \rceil$ and this bound is best possible.*

Proof. If a vector contains an entry at least s, then it s-sum intersects every other vector. The number of such vectors is $(q+1)^n - s^n$, and then we apply Proposition 2.1 to the set of all other vectors. $\square$

Now we turn our attention to (rank- or support-) uniform families of vectors. We shall start with the proof of Theorem 1.4, but we need several definitions and some results from the literature.

Definition 2.3. The *shadow* $\Delta(F)$ *of a set* F is $\{G \subset F : |G| = |F| - 1\}$ and the *shadow* $\Delta(\mathcal{F})$ *of a family* $\mathcal{F}$ *of sets* is $\cup_{F \in \mathcal{F}} \Delta(F)$. If $\mathcal{F}$ is r-uniform and $0 \leq \ell < r$, then $\Delta_\ell(\mathcal{F}) := \{G : |G| = \ell \text{ and } \exists F \in \mathcal{F} \text{ s.t. } G \subset F\}$

We introduce the notation $<_{\text{colex}}$ for the *colex ordering* of all finite subsets of the positive integers. In this ordering for two finite sets A and B we have $A <_{\text{colex}} B$ if and only if the largest element of the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of A and B belongs to B.

Kruskal and Katona independently proved the following fundamental theorem.

Theorem 2.4 (Kruskal [17], Katona [15])**.** *Let $n, r, m \geq 1$ and $L_{r,m}$ be the initial segment of $\binom{[n]}{r}$ of size m with respect to the colex ordering. For any $\mathcal{F} \subseteq \binom{[n]}{r}$ of size m, we have $|\Delta(\mathcal{F})| \geq |\Delta(L_{r,m})|$.*

We can introduce the notion of shadow for vectors also.

Definition 2.5. The *shadow* $\Delta(\mathbf{x})$ *of a vector* $\mathbf{x} \in Q^n$ is $\{\mathbf{y} < \mathbf{x} : r(\mathbf{y}) = r(\mathbf{x}) - 1\}$, where $<$ denotes the coordinate-wise ordering, i.e., for two vectors $\mathbf{x}$ and $\mathbf{y}$ we have $\mathbf{y} < \mathbf{x}$ if and only if $y_i \leq x_i$ for all $1 \leq i \leq n$ and $y_i < x_i$ for at least one i. Then for $\mathcal{F} \subseteq Q^n$ we define the *shadow* $\Delta(\mathcal{F})$ *of* $\mathcal{F}$ as $\cup_{\mathbf{x} \in \mathcal{F}} \Delta(\mathbf{x})$ and for r-rank uniform $\mathcal{F}$ and $\ell < r$ we let $\Delta_\ell(\mathcal{F}) = \{\mathbf{y} : r(\mathbf{y}) = \ell \text{ and } \exists \mathbf{x} \in \mathcal{F} \text{ s.t. } \mathbf{y} < \mathbf{x}\}$.

Analogously to the set case we can introduce the *colex ordering of* Q^n, i.e., for $\mathbf{x}, \mathbf{y} \in Q^n$ we have $\mathbf{x} <_{\text{colex}} \mathbf{y}$ if and only if $x_i < y_i$ where i is the largest coordinate in which $\mathbf{x}$ and $\mathbf{y}$ differ.

Clements and Lindström provided a generalization of the Kruskal-Katona theorem for the shadows of vectors we introduced in Definition 2.5.

Theorem 2.6 (Clements, Lindström [3])**.** *Let $q, r, m, n \geq 1$, and let $L_{q,r,m}$ be the initial segment of $Q(n, r)$ of size m with respect to the colex ordering. For any $\mathcal{F} \subseteq Q(n, r)$ of size m, we have $|\Delta(\mathcal{F})| \geq |\Delta(L_{r,m})|$.*

One can easily check the following properties of the colex ordering of sets and vectors, so we omit their proof.

Proposition 2.7. *Suppose $n \geq r \geq 1$.*

(i) Both in $\binom{[n]}{r}$ and in $Q(n, r)$, the shadow of an initial segment is an initial segment, so one can iterate Theorems 2.4 and 2.6 to obtain that initial segments minimize the size of (lower) shadows.

(ii) If $\mathcal{F}$ is the family of the largest m sets of $\binom{[n]}{r}$ with respect to the colex ordering, then $\overline{\mathcal{F}} = L_{n-r,m}$.

(iii) If $\mathcal{F}$ is the family of the largest m vectors of $Q(n,r)$ with respect to the colex ordering, then $\overline{\mathcal{F}} = L_{q,qn-r,m}$.

Before the proof of Theorem 1.4, first let us briefly recall the proof of the upper bound in Theorem 1.2 that uses the Kruskal–Katona shadow theorem (Theorem 2.4) and was obtained by Daykin [4] as we would like to mimic it.

Suppose contrary to the statement of Theorem 1.2 that there exists an intersecting family $\mathcal{F} \subseteq \binom{[n]}{r}$ of size larger than $\binom{n-1}{r-1}$. Consider the family $\overline{\mathcal{F}} = \{[n] \setminus F : F \in \mathcal{F}\}$ and observe that as $\mathcal{F}$ is intersecting, we must have $\mathcal{F} \cap \Delta_r(\overline{\mathcal{F}}) = \emptyset$. Clearly, $|\overline{\mathcal{F}}| = |\mathcal{F}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$, as $n \geq 2r$. Applying Theorem 2.4, any $(n-r)$-uniform family of size larger than $\binom{y}{n-r}$ has r-shadow larger than $\binom{y}{r}$. So $\binom{n}{r} = |\binom{[n]}{r}| \geq |\mathcal{F}| + |\Delta_r(\overline{\mathcal{F}})| > \binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$. This contradiction proves the upper bound in Theorem 1.2.

This proof seems to be very lucky that it includes miraculous equalities $\binom{n-1}{r-1} = \binom{n-1}{n-r}$ and $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$, so let us recite it without any calculation. Consider greedily the largest sets of $\binom{[n]}{r}$ with respect to the colex order as long as they form an intersecting family. Let $\mathcal{F}_0$ be the family when we need to stop. If $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}_0}) = \binom{[n]}{r}$, then $\mathcal{F}_0$ is a largest possible intersecting family. Indeed, if $|\mathcal{F}| > |\mathcal{F}_0|$, then as $\overline{\mathcal{F}_0}$ is an initial segment, by Proposition 2.7 (i) and (ii), we have $|\mathcal{F}| + |\Delta_r(\overline{\mathcal{F}})| > |\mathcal{F}_0| + |\Delta_r(\overline{\mathcal{F}_0})| = \binom{n}{r}$, contradiction, so $\mathcal{F}$ cannot be intersecting. To obtain the results of Theorem 1.2 about intersecting families, all we need is to observe that $\mathcal{F}_0 = \{F \in \binom{[n]}{r} : n \in F\}$ and $\Delta_r(\overline{\mathcal{F}_0}) = \{F \in \binom{[n]}{r} : n \notin F\}$.

Before the proof of Theorem 1.4 let us restate it.

Theorem 1.4. *Let $n, q, r \geq 1$ and $\mathcal{F} \subseteq Q^n$ be an r-rank uniform $(q+1)$-sum intersecting family with $\frac{q+1}{2} \leq r \leq \frac{qn}{2}$. Then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=\frac{q+1}{2}}^{q} |Q(n-1, r-j)| & \text{if } q+1 \text{ is even,} \\ 1 + \sum_{j=\lceil \frac{q+1}{2} \rceil}^{q} \sum_{i=1}^{\lfloor \frac{2(r-1)}{q} \rfloor} |Q(n-i, r-j-\frac{(i-1)q}{2})| & \text{if } q+1 \text{ is odd,} \end{cases} \tag{4}$$

and these bounds are best possible.

Proof. Clearly $\mathbf{x} \in Q^n$ does not $(q+1)$-sum intersect a vector $\mathbf{y} \in Q^n$ if and only if $\mathbf{y}$ is less than or equal to $\overline{\mathbf{x}}$ in the coordinate-wise ordering. Also, $\mathcal{F} \subseteq Q(n,r)$ is a $(q+1)$-sum intersecting family if and only if $\mathcal{F} \cap \Delta_r(\overline{\mathcal{F}})$ contains at most one vector as $\mathcal{F}$ may contain one vector $\mathbf{x}$ that does not $(q+1)$-sum intersect itself. Indeed, if $\mathbf{x} \neq \mathbf{y}$ and $|\mathbf{x} \cap_{q+1} \mathbf{y}| = 0$, then $\mathbf{x}, \mathbf{y} \in \mathcal{F} \cap \Delta_r(\overline{\mathcal{F}})$. On the other hand if $\mathbf{x}, \mathbf{y} \in \mathcal{F} \cap \Delta_r(\mathcal{F})$ and $\mathcal{F}$ is intersecting, then by the above, we must have $\mathbf{x} < \overline{\mathbf{x}}$ and $\mathbf{y} < \overline{\mathbf{y}}$. But as $|\mathbf{x} \cap_{q+1} \mathbf{y}| > 0$, there must exist an index i with $x_i + y_i \geq q+1$, so either x_i or y_i, say x_i, is at least $\frac{q+1}{2}$. But then $x_i > q - x_i = \overline{x}_i$ - a contradiction.

The reasoning of Daykin stays valid with a little modification, if for the maximal $(q+1)$-sum intersecting family $\mathcal{F}_0 \subseteq Q(n,r)$ consisting of largest vectors with respect to the colex ordering we have both $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}_0}) = Q(n,r)$ *and* $|\Delta_r(\overline{\mathcal{F}}_0)| < |\Delta_r(\overline{\mathcal{F}}_0^+)|$, where $\mathcal{F}_0^+$ is the initial segment of the colex ordering of $Q(n, qn-r)$ one larger than $\overline{\mathcal{F}}_0$. Indeed, if $\mathcal{F}$ was an r-rank uniform $(q+1)$-sum intersecting family larger than $\mathcal{F}_0$, then by the following series of inequalities:

$$|\mathcal{F} \cup \Delta_r(\overline{\mathcal{F}})| \geq |\mathcal{F}| + |\Delta_r(\overline{\mathcal{F}})| - 1 \geq |\mathcal{F}_0| + 1 + |\Delta_r(\overline{\mathcal{F}}_0)| + 1 - 1 = |Q(n,r)| + 1$$

we would get a contradiction.

And this is exactly the case: for the maximal $(q+1)$-sum intersecting family $\mathcal{F}_0 \subseteq Q(n,r)$ consisting of largest vectors with respect to the colex ordering, we prove that we have both $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}_0}) = Q(n,r)$ *and* $|\Delta(\overline{\mathcal{F}}_0)| < |\Delta(\overline{\mathcal{F}}_0^+)|$, where $\mathcal{F}_0^+$ is the one larger initial segment of the colex ordering of $Q(n, qn-r)$ than $\overline{\mathcal{F}}_0$.

Suppose first that $q+1 = 2k$. Then $\mathcal{F}_0 = \{\mathbf{x} \in Q(n,r) : x_n \geq k\}$, $\overline{\mathcal{F}}_0 = \{\mathbf{x} \in Q(n, qn-r) : x_n < k\}$ and clearly $\Delta_r(\overline{\mathcal{F}}_0) = \{\mathbf{x} \in Q(n,r) : x_n < k\} = Q(n,r) \setminus \mathcal{F}_0$ and since $\overline{\mathcal{F}}_0^+$ contains a vector $\mathbf{x}$ with $x_n = k$, its r-shadow is strictly larger than that of $\overline{\mathcal{F}}_0$.

Suppose next $q+1 = 2k+1$. Then

$$\mathcal{F}_0 = \bigcup_{j=0}^{\lfloor \frac{r-1}{k} \rfloor} \{\mathbf{x} \in Q(n,r) : x_n = x_{n-1} = \cdots = x_{n-j+1} = k, x_{n-j} > k\} \cup \{\mathbf{x}^*\},$$

where $x^*_n = x^*_{n-1} = \cdots = x^*_{n-\lfloor \frac{r-1}{k} \rfloor} = k$, $x^*_{n-\lfloor \frac{r-1}{k} \rfloor - 1} \equiv r \pmod{k}$ and all other entries are 0. Observe that $\mathbf{x}^*$ does not $(q+1)$-sum intersect itself. To see that $\mathcal{F}_0 \cup \Delta_r(\overline{\mathcal{F}}_0) = Q(n,r)$ holds, one only has to observe that any vector $\mathbf{y} \in Q(n,r) \setminus \mathcal{F}_0$ with $y_n = y_{n-1} = \cdots = y_{n-\lfloor \frac{r-1}{k} \rfloor} = k$ belongs to $\Delta_r(\overline{\mathbf{x}^*})$. Also, any vector $\mathbf{y} \in Q(n,r)$ with $\overline{\mathbf{x}^*} <_{\text{colex}} \mathbf{y}$ has an entry larger than k in the last $\lfloor \frac{r-1}{k} \rfloor$ coordinates, so $|\Delta_r(\overline{\mathcal{F}}_0^+)| > |\Delta_r(\overline{\mathcal{F}}_0)|$.

So we are done with the proof of Theorem 1.4. □

Let us continue with the proof of Theorem 1.3. Before that we cite two well-known stability-type results that we use during the proof.

Theorem 2.8 (Hilton, Milner [13])**.** *If $\mathcal{F} \subseteq \binom{[n]}{r}$ is an intersecting family with $n \geq 2r+1$ and $\cap_{F \in \mathcal{F}} F = \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.*

Theorem 2.9 (Frankl [6])**.** *Let $\mathcal{F} \subseteq \binom{[n]}{r}$ be a t-intersecting family with $|\cap_{F \in \mathcal{F}} F| < t$. If n is large enough, then $|\mathcal{F}| \leq \max\{|\mathcal{F}_1|, |\mathcal{F}_2|\}$, where*

$$\mathcal{F}_1 = \left\{F \in \binom{[n]}{r} : [t] \subset F, F \cap [t+1, r+1] \neq \emptyset\right\} \cup \binom{[r+1]}{r}$$

and

$$\mathcal{F}_2 = \left\{F \in \binom{[n]}{r} : |F \cap [t+2]| \geq t+1\right\}.$$

Now let us restate Theorem 1.3.

Theorem 1.3. *For any $q, s \geq 2$ and integer $r \geq 1$, there exists $n(q,s,r) \in \mathbb{N}$ such that if $\mathcal{F} \subseteq Q^n$ is r-support uniform s-sum intersecting with $n \geq n(q,r,s)$, then*

$$|\mathcal{F}| \leq \begin{cases} (q - \frac{s}{2} + 1)q^{r-1}\binom{n-1}{r-1} & \text{if } s \text{ is even,} \\ 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^{r} \binom{n-i}{r-i} q^{r-i} & \text{if } s \text{ is odd,} \end{cases} \tag{5}$$

and these bounds are best possible.

Proof of Theorem 1.3. Suppose first that s is even. The constructions showing that the bound is best possible are $\mathcal{F}_{n,q,s,r,i} = \{\mathbf{x} \in Q^n : \frac{s}{2} \leq x_i\}$. To see the upper bound, let $\mathcal{F}$ be an r-support uniform s-sum intersecting family and let $\mathcal{S}_{\mathcal{F}}$ denote the family of supports in $\mathcal{F}$. For a fixed support S, the number of vectors having S as support is bounded by a constant (depending on $|S|$, r and q), therefore, by Theorem 2.8, unless all supports in $\mathcal{S}_{\mathcal{F}}$ share a common element i, we have $|\mathcal{F}| = O(n^{r-2}) < \binom{n-1}{r-1}$ if n is large enough. So we can suppose that there exists an index i that belongs to all supports. Assume next that there exists $\mathbf{x} \in \mathcal{F}$ with $x_i < \frac{s}{2}$. Then consider the subfamily $\mathcal{F}' = \{\mathbf{y} \in \mathcal{F} : y_i \leq \frac{s}{2}\}$. As vectors in $\mathcal{F}'$ must all s-sum intersect $\mathbf{x}$, but they do not s-sum intersect it at coordinate i, therefore their supports must intersect the support of $\mathbf{x}$ in some coordinate other than i. Therefore, we obtain $|\mathcal{F}'| = O(n^{r-2})$. But then

$$|\mathcal{F}| \leq |\mathcal{F}'| + (q - \frac{s}{2})q^{r-1}\binom{n-1}{r-1} < (q - \frac{s}{2} + 1)q^{r-1}\binom{n-1}{r-1}$$

if n is large enough. We obtained that either $\mathcal{F}$ is smaller than the claimed bound or $\mathcal{F} \subseteq \mathcal{F}_{n,q,s,r,i}$ for some index i.

Suppose next that s is odd. The extremal families are defined via ordered r-tuples $(s_1, s_2, \ldots, s_r)$ the following way:

$$\mathcal{F}_{n,q,s,(s_1,s_2,\ldots,s_r)} = \{\mathbf{x}\} \cup \bigcup_{i=1}^{r} \{\mathbf{y} \in Q^n : y_1 = y_2 \cdots = y_{i-1} = \lfloor \frac{s}{2} \rfloor, y_i \geq \frac{s}{2}\},$$

where $\mathbf{x}$ is the vector with $x_{s_i} = \lfloor \frac{s}{2} \rfloor$ for all $1 \leq i \leq r$ and $x_j = 0$ otherwise. To prove the upper bound, we proceed by induction on r. If $r = 1$, then all supports of an r-support uniform s-sum intersecting family $\mathcal{F}$ must be the same singleton $\{i\}$. If m is the minimum entry over all vectors in $\mathcal{F}$ at coordinate i, then all other entries must be at least $s - m$, so the number of vectors is at most $\min\{q - m + 1, q - (s - m)\}$. This is maximized if $m = \lfloor \frac{s}{2} \rfloor$ and the claimed bound follows. Let $r > 1$, and $\mathcal{F} \subseteq Q^n$ be an r-support uniform, s-sum intersecting family. Then just as in the even s case, using Theorem 2.8, we obtain that $|\mathcal{F}| = O(n^{r-2})$ unless all sets in $\mathcal{S}_{\mathcal{F}}$ share a common element s_1. If there exists a vector $\mathbf{z} \in \mathcal{F}$ with $z_{s_1} < \lfloor \frac{s}{2} \rfloor$, then also just as in the even s case, we obtain that $\mathcal{F}' = \{\mathbf{y} \in \mathcal{F} : y_{s_1} \leq \lceil \frac{s}{2} \rceil\}$ is of size $O(n^{r-2})$ and thus $\mathcal{F}$ is smaller than the claimed bound if n is large enough. So we can assume that for all vectors $\mathbf{z} \in \mathcal{F}$, we have $z_{s_1} \geq \lfloor \frac{s}{2} \rfloor$. The number of those vectors $\mathbf{z}$ with $z_{s_1} \geq \lceil \frac{s}{2} \rceil$ is $(q - \lceil \frac{s}{2} \rceil + 1)q^{r-1}\binom{n-1}{r-1}$, while the family $\mathcal{F}^* = \{\mathbf{z}' : z_{s_1} = \lfloor \frac{s}{2} \rfloor\}$ is $(r-1)$-support uniform, s-sum intersecting, where $\mathbf{z}'$ is the vector obtained from $\mathbf{z}$ by removing its s_1st entry. By induction, we obtain

$$|\mathcal{F}^*| \leq 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^{r-1} q^{r-1-i} \binom{n-1-i}{r-1-i}$$

and so

$$\begin{aligned} |\mathcal{F}| &\leq |\mathcal{F}^*| + (q - \lceil \frac{s}{2} \rceil + 1) q^{r-1} \binom{n-1}{r-1} \\ &\leq 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^{r-1} q^{r-1-i} \binom{n-1-i}{r-1-i} + (q - \lceil \frac{s}{2} \rceil + 1) q^{r-1} \binom{n-1}{r-1} \\ &= 1 + (q - \lceil \frac{s}{2} \rceil + 1) \sum_{i=1}^{r} q^{r-i} \binom{n-i}{r-i}, \end{aligned}$$

as claimed. □

Before the proof of Theorem 1.6 we prove the following for the size of the Construction 1.5.

Proposition 2.10. *Suppose that n, q, s, r, t are integers with the assumptions on them as in Construction 1.5.*

(i) If $r \geq 2t$, then

$$f(n,q,s,r,t) =$$
$$\binom{n-t}{r-t} q^{r-t} (q - \lfloor \frac{s}{2} \rfloor)^t + (q - \lfloor \frac{s}{2} \rfloor)^{|S|} \sum_{S \subsetneq [t]} f(n-t, q, s, r-t, t-|S|).$$

(ii) If $t < r < 2t$, then

$$f(n,q,s,r,t) =$$
$$\binom{n-t}{r-t} q^{r-t} (q - \lfloor \frac{s}{2} \rfloor)^t + \binom{t}{2t-r-1} + (q - \lfloor \frac{s}{2} \rfloor)^{|S|} \sum_{S \subsetneq [t], |S| \geq 2t-r} f(n-t, q, s, r-t, t-|S|).$$

Proof. In both cases, the first term of the right-hand side stands for those vectors for which $x_i > \frac{s}{2}$ for all $1 \leq i \leq t$. In (i), the big sum partitions the other vectors according to which of the first t entries have value more than $s/2$. They all should contain at least $t - |S|$ other entries larger than $s/2$, out of the remaining $r - t$ support entries.

In (ii), the big summation can neglect small subsets of $[t]$ because if $|S| < 2t - r$ then $|S| + r - t < t$. The middle term stands for those $\mathbf{x}_{T'}$s where T' contains exactly $2t - r - 1$ elements from $[t]$ (the others must contain more). □

Now we continue with the proof of Theorem 1.6 that we restate.

Theorem 1.6. *For any $q, s \geq 2$ and $r \geq t \geq 1$, there exists $n(q,s,r,t)$ such that if $\mathcal{F} \subseteq Q^n$ is r-support uniform s-sum t-intersecting with $n \geq n(q,s,r,t)$, then*

$$|\mathcal{F}| \leq \begin{cases} (q - \frac{s}{2} + 1)^t q^{r-t}\binom{n-t}{r-t} & \text{if } s \text{ is even,} \\ f(n,q,s,r,t) & \text{if } s \text{ is odd,} \end{cases} \tag{6}$$

and these bounds are best possible as shown by the families of Construction 1.5.

Proof of Theorem 1.6. Suppose first s is even. To see the upper bound, let $\mathcal{F}$ be an r-support uniform s-sum t-intersecting family and let $\mathcal{S}_\mathcal{F}$ denote the family of supports in $\mathcal{F}$. For a fixed support S, the number of vectors having S as support is bounded by a constant (depending on $|S|$, r and q), therefore, by Theorem 2.9, unless all supports in $\mathcal{S}_\mathcal{F}$ share all elements of a t-subset T of $[n]$, we have $|\mathcal{F}| = O(n^{r-t-1}) < \binom{n-t}{r-t}$ if n is large enough. So we can suppose that there exists a t-subset T that is contained in all supports. Assume next that there exists $\mathbf{x} \in \mathcal{F}$ with $x_i < \frac{s}{2}$ for some $i \in T$. Then consider the subfamily $\mathcal{F}' = \{\mathbf{y} \in \mathcal{F} : y_i \leq \frac{s}{2}\}$. As vectors in $\mathcal{F}'$ must all s-sum t-intersect $\mathbf{x}$, but they do not s-sum intersect it at coordinate i, therefore their supports must intersect the support of $\mathbf{x}$ in some coordinate outside T. Therefore, we obtain $|\mathcal{F}'| = O(n^{r-t-1})$. But then

$$|\mathcal{F}| \leq |\mathcal{F}'| + (q - \frac{s}{2})(q - \frac{s}{2})^{t-1} q^{r-t}\binom{n-t}{r-t} < (q - \frac{s}{2} + 1)^t q^{r-t}\binom{n-t}{r-t}$$

if n is large enough. We obtained that either $\mathcal{F}$ is smaller than the claimed bound or $\mathcal{F} \subseteq \mathcal{F}_{n,q,s,r,T}$ for some t-subset T.

Suppose next s is odd. We proceed by induction on $r + t$ and observe that in all cases, the family of supports must be t-intersecting. The case $t = 1$ is covered by Theorem 1.3. Let $\mathcal{F} \subseteq Q^n$ be an s-sum t-intersecting r-support uniform family. We consider three cases according to the relationship of r and t.

Case I: $r = t$.

The assumption $r = t$ implies that all supports in $\mathcal{F}$ are identical, say the support is S. Therefore, for any $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ and $i \in S$ we must have $x_i + y_i \geq s$. In particular, for any $i \in S$ there is at most one $\mathbf{x} \in \mathcal{F}$ with $x_i < s/2$. So

$$|\mathcal{F}| \leq \binom{t}{t-1}(q - \lfloor s/2 \rfloor)^{t-1} + (q - \lfloor s/2 \rfloor)^t,$$

as claimed.

Case II: $t < r < 2t$.

The family $\mathcal{S}_\mathcal{F}$ of supports is t-intersecting, so unless all supports of $\mathcal{F}$ share t elements, $|\mathcal{F}| = O(n^{r-t-1})$ holds by Theorem 2.9. Let T be the set of these t elements, and for any $S \subset T$ let $\mathcal{F}_S$ denote the family of those vectors in $\mathcal{F}$ for which $x_i \geq s/2$ for all $i \in S$, and $1 < x_i \leq s/2$ for all $i \in T \setminus S$. As all supports contain T, we have $\mathcal{F} = \cup_{S \subset T} \mathcal{F}_S$. Clearly, $|\mathcal{F}_T| \leq q^{r-t}\binom{n-t}{r-t}(q - \lfloor s/2 \rfloor)^t$.

Consider next all subsets S with $2t-r \le |S| < t$. For any such $S \subsetneq T$, let $\mathcal{F}'_S = \{\mathbf{x}' : \mathbf{x} \in \mathcal{F}_S\}$, where $\mathbf{x}'$ is the vector obtained from $\mathbf{x}$ by deleting the coordinates belonging to T. So $\mathcal{F}'_S$ is $(r-t)$-support uniform s-sum $(t-|S|)$-intersecting, and thus by induction, we have

$$|\mathcal{F}_S| \le (q - \lfloor s/2 \rfloor)^{|S|} |\mathcal{F}'_S| \le (q - \lfloor s/2 \rfloor)^{|S|} f(n-t, q, s, r-t, t-|S|).$$

Finally, consider all subsets $S \subset T$ with $|S| < 2t-r$. As $|S| + r - t < 2t - r + r - t = t$, we must have $|\mathcal{F}_S| \le 1$ for all such S. Observe that for any $(r+1-t)$-subset $Z \subset T$ there exists at most one subset $S \subset T$ with $Z \cap S = \emptyset$ and $\mathcal{F}_S \neq \emptyset$. Indeed, if $\mathbf{x} \in \mathcal{F}_S$, $\mathbf{y} \in \mathcal{F}_{S'}$, then $\mathbf{x}$ and $\mathbf{y}$ can only s-sum intersect in at most $r-t$ coordinates outside T and in at most $t-(r+1-t) = 2t-r-1$ coordinates within T, so $|\mathbf{x} \cap_s \mathbf{y}| \le t-1$, a contradiction. Therefore

$$\sum_{S \subset T, |S| < 2t-r} |\mathcal{F}_S| \le \binom{t}{r+1-t} = \binom{t}{2t-r-1}.$$

Adding up these bounds for all $|\mathcal{F}_S|$ we obtain the desired bound on $|\mathcal{F}|$ by Proposition 2.10 (ii).

Case III: $2t \le r$.

The family $\mathcal{S}_\mathcal{F}$ of supports is t-intersecting, so unless all supports of $\mathcal{F}$ share t elements, we have $|\mathcal{F}| = O(n^{r-t-1})$ by Theorem 2.9. Let T be the set of these t elements, and for any $S \subset T$ let $\mathcal{F}_S$ denote the family of those vectors $\mathbf{x} \in \mathcal{F}$ for which $x_i \ge s/2$ for all $i \in S$, and $1 < x_i \le s/2$ for all $i \in T \setminus S$. As all supports contain T, we have $\mathcal{F} = \cup_{S \subset T} \mathcal{F}_S$. Clearly, $|\mathcal{F}_T| \le q^{r-t}\binom{n-t}{r-t}(q - \lfloor s/2 \rfloor)^t$. For any $S \subsetneq T$, let $\mathcal{F}'_S = \{\mathbf{x}' : \mathbf{x} \in \mathcal{F}_S\}$, where $\mathbf{x}'$ is the vector obtained from $\mathbf{x}$ by deleting the coordinates belonging to T. So $\mathcal{F}'_S$ is $(r-t)$-support uniform s-sum $(t-|S|)$-intersecting, and thus by induction, we have

$$|\mathcal{F}_S| \le (q - \lfloor s/2 \rfloor)^{|S|} |\mathcal{F}'_S| \le (q - \lfloor s/2 \rfloor)^{|S|} f(n-t, q, s, r-t, t-|S|).$$

Adding up these bounds for all $|\mathcal{F}_S|$ we obtain the desired bound on $|\mathcal{F}|$ by Proposition 2.10 (i). □

3. Intersecting vector pairs

In this section we provide proofs for Theorem 1.10 and Theorem 1.11.

Let us start with a general construction.

Construction 3.1. Let $c \le a \le b$ and $3 \le s < 2q$ be integers and fix a set X of size $a+b-c$. For any 3-partition $A \cup B \cup C = X$ with $|A| = a-c$, $|B| = b-c$, $|C| = c$, we define the pairs $\mathbf{x}^{A,B,C}$ and $\mathbf{y}^{A,B,C}$ with

$$x_i^{A,B,C} = y_i^{A,B,C} = \lceil s/2 \rceil - 1 \text{ if } i \in C,$$
$$x_i^{A,B,C} = \lfloor s/2 \rfloor + 1,\ y_i^{A,B,C} = 0 \text{ if } i \in A$$

and

$$x_i^{A,B,C} = 0,\ y_i^{A,B,C} = \lfloor s/2 \rfloor + 1 \text{ if } i \in B.$$

Note that $\{(\mathbf{x}^{A,B,C}, \mathbf{y}^{A,B,C}) : A \cup B \cup C = X, |A| = a-c, |B| = b-c, |C| = c\}$ is a strong s-sum IVP-system of cardinality $\binom{a+b-c}{b-c}\binom{a}{c}$.

More generally, let $\alpha_0,\alpha_1,\dots,\alpha_q$ be positive integers with $\sum_{i=0}^{q-1}\alpha_i\le a$ and $\sum_{i=1}^{q}\alpha_i\le b$. Set $N=\sum_{i=0}^{q}\alpha_i\le a$, and define

$$\{(\mathbf{x}^{A_0,A_1,\dots,A_q},\mathbf{y}^{A_0,A_1,\dots,A_q}):[N]=\bigsqcup_{i=0}^{q}A_i,\ |A_i|=\alpha_i\}\ ,$$

where $x_j^{A_0,A_1,\dots,A_q}=q-y_j^{A_0,A_1,\dots,A_q}=i$ if and only if $j\in A_i$.

Observe that the above is a strong (a,b)-system. Indeed, by definition we have that $x_j^{A_0,A_1,\dots,A_q}+y_j^{A_0,A_1,\dots,A_q}=q$ for any $j\in N$ and partition $A_0,A_1,\dots,A_q$ with $|A_i|=\alpha_i$ and so $|\mathbf{x}^{A_0,A_1,\dots,A_q}\cap_{q+1}\mathbf{y}^{A_0,A_1,\dots,A_q}|=0$. Furthermore, if $(A_0,A_1,\dots,A_q)\ne(B_0,B_1,\dots,B_q)$, then there exists j such that $A_j\ne B_j$. We consider such j that minimizes $\min\{j,q-j\}$. By the assumption on j, we have $N_j:=\sqcup_{i=j}^{q-j}A_j=\sqcup_{i=j}^{q-j}B_j$ and there exist $i\in A_j\setminus B_j$ and $i'\in B_j\setminus A_j$. As $i,i'\in N_j$ we obtain $x_i^{A_0,A_1,\dots,A_q}+y_i^{B_0,B_1,\dots,B_q}>j+q-j$ and $x_{i'}^{B_0,B_1,\dots,B_q}+y_{i'}^{A_0,A_1,\dots,A_q}>q-j+j$. This proves that we indeed defined a strong (a,b)-system.

3.1. **Upper bound for strong 3-sum IVP-systems in $\{0,1,2\}^{<\mathbb{N}}$.** In this subsection we will prove Theorem 1.10. Let $\{(\mathbf{x}^j,\mathbf{y}^j)\mid 1\le j\le m\}$ be a strong 3-sum (k,k)-system in $\{0,1,2\}^{<\mathbb{N}}$. Let us also introduce the following further notation for $j=1,\dots,m$:

- $a_j=|S_{\mathbf{x}^j}\setminus S_{\mathbf{y}^j}|$,
- $b_j=|S_{\mathbf{y}^j}\setminus S_{\mathbf{x}^j}|$,
- $c_j=|S_{\mathbf{x}^j}\cap S_{\mathbf{y}^j}|$.

First we prove the following LYM-type theorem for 3-sum (a,b)-systems.

Theorem 3.2. *Suppose that for $a,b,m\ge 1$ $\{(\mathbf{x}^j,\mathbf{y}^j)\mid 1\le j\le m\}$ is a strong 3-sum (a,b)-system in $\{0,1,2\}^{<\mathbb{N}}$. Then*

$$\sum_{j=1}^{m}\frac{a_j!\,b_j!\,c_j!}{(a_j+b_j+c_j)!}=\sum_{j=1}^{m}\frac{1}{\binom{a_j+b_j+c_j}{a_j+b_j}\binom{a_j+b_j}{a_j}}\le\min(a,b). \tag{7}$$

Proof. Essentially we apply induction on n.

- Note first that $a_i=0$ and $b_j=0$ cannot hold simultaneously for any $1\le i\ne j\le m$. Indeed, if $S_{\mathbf{x}^i}\subset S_{\mathbf{y}^i}$ and $S_{\mathbf{y}^j}\subset S_{\mathbf{x}^j}$ then all nonzero entries in $\mathbf{x}^i$ are equal to 1, and the same holds for all nonzero entries in $\mathbf{y}^j$ as well, hence $\mathbf{x}^i\cap_3\mathbf{y}^j=\emptyset$, a contradiction. As a consequence, either $a_j>0$ for all j or $b_j>0$ for all j (or both), or there is exactly one j with $a_j=b_j=0$.
- As long as $S_{\mathbf{y}^j}\not\subseteq S_{\mathbf{x}^j}$ holds for all j:
  For every $t\in[n]$, consider the systems
  $$\{(\mathbf{x}^j,(\mathbf{y}^j)')\mid 1\le i\le m,\ t\notin S_{\mathbf{x}^j}\},$$
  where $((y^j)')_i=(y^j)_i$ for all $i\in[n]\setminus\{t\}$ and $((y^j)')_t=0$.
  These systems keep the required intersections. Denoting $b'_j=|S_{(\mathbf{y}^j)'}\setminus S_{\mathbf{x}^j}|$ we have $b'_j=b_j-1$ exactly $b_j>0$ times, and $b'_j=b_j$ exactly $n-(a_j+b_j+c_j)$ times. Taking

the sum of (7) over all t, for the term belonging to j we have

$$b_j \cdot \frac{a_j!\,(b_j-1)!\,c_j!}{(a_j+(b_j-1)+c_j)!} + (n-a_j-b_j-c_j)\cdot\frac{a_j!\,b_j!\,c_j!}{(a_j+b_j+c_j)!} = n\cdot\frac{a_j!\,b_j!\,c_j!}{(a_j+b_j+c_j)!}\,,$$

hence the overall sum for all j is n times the left-hand side of (7). Certainly the right-hand side is also multiplied by n, and the inequality follows by induction.

This step is applicable unless $b_j = 0$ holds for some j. Hence from now on assume $S_{\mathbf{y}^j} \subseteq S_{\mathbf{x}^j}$.

- As long as $S_{\mathbf{x}^j} \not\subseteq S_{\mathbf{y}^j}$ holds for all j, also including $j = i$:

  For every t consider the systems

  $$\{((\mathbf{x}^j)', \mathbf{y}^j) \mid 1 \le j \le m,\ t \notin S_{\mathbf{y}^j}\},$$

  where $((x^j)')_i = (x^j)_i$ for all $i \in [n] \setminus \{t\}$ and $((x^j)')_t = 0$.

  The argument analogous to the previous case yields the required inequality unless $a_j = 0$ holds for some j. However, then we have $a_j = b_i = 0$ which implies $j = i$. Hence for the rest of the proof assume $S_{\mathbf{x}^1} = S_{\mathbf{y}^1}$, as we can choose $i = 1$, without loss of generality. Recall that in this situation $(x^1)_i = (y^1)_i$ for all $i \in S_{\mathbf{x}^1} = S_{\mathbf{y}^1}$.
- If we omit $(\mathbf{x}^1, \mathbf{y}^1)$ from the system, the left-hand side of (7) decreases by exactly 1, as currently $c_1 = |S_{\mathbf{x}^1}|$ and $a_1 = b_1 = 0$. For every $j \neq 1$ in the remaining subsystem we have $a_j, b_j > 0$ because each $\mathbf{x}^j$ needs an entry of 2 to intersect $\mathbf{y}^1$, and each $\mathbf{y}^j$ needs an entry of 2 to intersect $\mathbf{x}^1$, while those two elements cannot be the same as $\mathbf{x}^j$ must not sum-intersect $\mathbf{y}^j$. Consequently when we repeat the above steps, once the procedure halts, the elements of $S_{\mathbf{x}^j} \setminus S_{\mathbf{y}^j}$ and of $S_{\mathbf{y}^j} \setminus S_{\mathbf{x}^j}$ will not remain there, i.e. the value of the corresponding c_j will be at most $\min(a,b) - 1$ when $a_j = b_j = 0$.
- The last halt occurs when the system contains a single vector-pair $(\mathbf{x}^j, \mathbf{y}^j)$ with $a_j = b_j = 0$ and $c_j \ge 1$. This situation is reached after performing the above procedure at most $\min(a,b) - c_j + 1 \le \min(a,b)$ times. Note that if $c_j = 1$ then the intersection conditions exclude the presence of any other vector-pair.

□

Let us repeat that $m(k)$ denotes the maximum number of vector pairs in such a strong 3-sum (k,k)-system and let

$$f(k) := \max \frac{(x+y+z)!}{x!\,y!\,z!},$$

where the maximum is taken over all nonnegative integers x, y, z such that $x + z \le k$ and $y + z \le k$. Now we are ready to prove

Theorem 1.10. *For every $k \ge 1$ we have*

$$f(k) \le m(k) \le k \cdot f(k)\,.$$

Proof of Theorem 1.10. The upper bound is a consequence of Theorem 3.2 as all terms on the left-hand side of (7) are at least $(f(k))^{-1}$. To obtain the lower bound we choose x, y, z for which $f(k)$ is attained, and choose $a = x$, $b = y$, $c = z$ in Construction 3.1. □

3.2. **Upper bound for weak $(q+1)$-sum IVP-systems in $\{0,1,\dots,q\}^{<\mathbb{N}}$.** Let $\{(\mathbf{x}^j,\mathbf{y}^j) \mid 1 \le j \le m\}$ be a weak $(q+1)$-sum IVP-system in $\{0,1,\dots,q\}^{<\mathbb{N}}$.

Observation 3.3.
(i) For any weak $(q+1)$-sum (a,b)-system $\mathcal{F}$ there exists another one $\mathcal{F}'$ with $|\mathcal{F}| = |\mathcal{F}'|$ such that for any $(\mathbf{x}^j,\mathbf{y}^j) \in \mathcal{F}'$ and i with $x_i^j + y_i^j > 0$ we have $x_i^j + y_i^j = q$.
(ii) For any strong $(q+1)$-sum (a,b)-system $\mathcal{F}$ there exists another one $\mathcal{F}'$ with $|\mathcal{F}| = |\mathcal{F}'|$ such that for any $(\mathbf{x}^j,\mathbf{y}^j) \in \mathcal{F}'$ and i with $x_i^j + y_i^j > 0$ we have $x_i^j + y_i^j = q$.

Proof. As $|\mathbf{x}^j \cap_{q+1} \mathbf{y}^j| = 0$ implies $x_i^j + y_i^j \le q$, and increasing a coordinate helps to intersect other vectors, we can replace $\mathbf{y}^j$ by $\mathbf{y}^{j,}$ with $y_i^{j,} = q - x_i^j$. □

We will say that a weak/strong $(q+1)$-sum (k,k)-system is *saturated* if it satisfies the property of Observation 3.3. For such $\mathcal{F} = \{(\mathbf{x}^j,\mathbf{y}^j) : 1 \le j \le m\}$, let us write A_i^j to denote $\{t : x_t^j = i\}$ and α_i^j to denote $|A_i^j|$.

Theorem 3.4. *Let p_i $i = 0,1,\dots,q$ be non-negative reals with $\sum_{i=0}^q p_i = 1$. If $\mathcal{F} = \{(\mathbf{x}^j,\mathbf{y}^j) : 1 \le j \le m\}$ is a saturated weak $(q+1)$-sum IVP-system, then $\sum_{j=1}^m \prod_{i=0}^q p_i^{\alpha_i^j} \le 1$ holds.*

Proof. Let $(X_0, X_1, \dots, X_q)$ be a partition of $[n]$ taken at random by the rule

$$\mathbb{P}(t \in X_0) = p_0\,, \quad \mathbb{P}(t \in X_1) = p_1\,, \quad \dots, \quad \mathbb{P}(t \in X_q) = p_q$$

applied independently for each $t \in [n] =: \bigcup_{j=1}^m (S(\mathbf{x}^j) \cup S(\mathbf{y}^j))$. For $j = 1,\dots,m$ consider the events

$$E_j = \bigwedge_{i=0}^q (A_i^j \subseteq X_i)\ .$$

We then have

$$\mathbb{P}(E_j) = \prod_{i=0}^q p_i^{\alpha_i^j}\ .$$

Observe that $\mathbb{P}(E_j \wedge E_{j'}) = 0$ holds for all $1 \le j \ne j' \le m$. Indeed, otherwise $A_i^j, A_i^{j'} \subset X_i$ holds for all $i = 0,1,\dots,q$. But then $\mathbf{x}_z^j = i$ implies $\mathbf{y}_z^j = q - i$ or $\mathbf{y}_z^j = 0$, so $|\mathbf{x}^j \cap_{q+1} \mathbf{y}^j| = 0$ and similarly $\mathbf{x}_z^{j'} = i$ implies $\mathbf{y}_z^{j'} = q - i$ or $\mathbf{y}_z^{j'} = 0$, so $|\mathbf{x}^{j'} \cap_{q+1} \mathbf{y}^{j'}| = 0$ — a contradiction to the weak ISVP-property.

Consequently the events $E_1,\dots,E_m$ mutually exclude each other, which implies that the sum of their probabilities is at most 1. □

Now we prove

Theorem 1.11. *For any $q \ge 1$ let $m(q,k)$ and $m'(q,k)$ denote the maximum size of a strong / weak $(q+1)$-sum (k,k)-system. Then $\lim_{k\to\infty} \sqrt[k]{m(q,k)} = \lim_{k\to\infty} \sqrt[k]{m'(q,k)} = (\sqrt{q}+1)^2$.*

Proof. Let us prove the upper bound first. By Observation 3.3, we can assume that $\mathcal{F}$ is saturated. Then we apply Theorem 3.4 with $p_0 = p_q = \frac{\sqrt{q}-1}{q-1}$ and $p_1 = p_2 = \cdots = p_{q-1} = p_0^2 = \frac{q+1-2\sqrt{q}}{(q-1)^2}$. (Observe that $2p_0 + (q-1)p_0^2 = 1$ as required.) As $\alpha_0^j = k - \sum_{i=1}^{q-1} \alpha_i^j = \alpha_q^j$, we obtain

$$\prod_{i=0}^{q} p_i^{\alpha_i^j} = p_0^{\alpha_0^j+\alpha_q^j} p_0^{2(k-\sum_{i=1}^{q-1}\alpha_i^j)} = p_0^{2k} .$$

Therefore, Theorem 3.4 implies $|\mathcal{F}| \le (p_0^{-2})^k = ((\frac{q-1}{\sqrt{q}-1})^2)^k = (\sqrt{q}+1)^{2k}$.

The lower bound is obtained using Construction 3.1. For fixed q and growing N, we let $\alpha_i = p_i N$ for $i = 0, 1, \dots, q$ with p_i as above in the proof of the lower bound, and so $k = \frac{p_0+(q-1)p_0^2}{2p_0+(q-1)p_0^2} N = (p_0 + (q-1)p_0^2)N$. Then the number of pairs in the construction is $\prod_{i=0}^{q} \binom{(1-\sum_{j=0}^{i-1} p_j)N}{p_i N}$. Using Stirling's formula and omitting polynomial terms, this is

$$\left[\frac{1}{p_0^{2p_0}(p_0^2)^{(q-1)p_0^2}}\right]^N = (p_0^{-2})^{(p_0+(q-1)p_0^2)N} = (p_0^{-2})^k = \left(\frac{q-1}{\sqrt{q}-1}\right)^{2k} = (\sqrt{q}+1)^{2k}.$$

Taking kth root yields the claimed lower bound. □

4. Concluding remarks

There exist lots of intersection theorems all waiting to be addressed in the sum-intersection setting. We just would like to point out one. Katona's intersection theorem [14] gives the maximum size of a non-uniform t-intersecting family $\mathcal{F} \subseteq 2^{[n]}$. The extremal family consists of all sets of size at least $\frac{n+t}{2}$ if $n+t$ is even, while if $n+t$ is odd, then extremal family consists of all sets of size at least $\lceil \frac{n+t}{2} \rceil$ together with $\binom{[n-1]}{\lfloor \frac{n+t}{2} \rfloor}$. One would hope to see a similar result for non-uniform s-sum t-intersecting families. That is extremal families are expected to consist of vectors of large rank. This is not going to hold as for two such vectors $\mathbf{x}, \mathbf{y}$ there might be coordinates where they s-sum 'intersect very much' (i.e. $x_i + y_i$ is much larger than s), but do not intersect anywhere else.

To remedy this situation, we can define the size of the *multi-s-*sum intersection of two vectors $\mathbf{x}, \mathbf{y} \in Q^n$ as $|\mathbf{x} \cap_{m,s} \mathbf{y}| = \sum_{i=1}^{n} (x_i + y_i - s + 1)^+$, where for any real z we define $z^+ := \max\{0, z\}$. A family $\mathcal{F} \subseteq Q^n$ is *s-multisum t-intersecting* if for any $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ we have $|\mathbf{x} \cap_{m,s} \mathbf{y}| \ge t$. Below, we show the first step towards such intersection theorems. Katona's tool was his intersecting shadow theorem and we will need a similar result.

We need to define the well-known shifting operation $\tau_{i,j}$ for the vector setting. For a vector $\mathbf{x}$ of length n and integers $1 \le i \ne j \le n$ we let $\tau_{i,j}(\mathbf{x})$ be the vector obtained from $\mathbf{x}$ by exchanging its ith and jth coordinates if $x_i < x_j$ and we let $\tau_{i,j}(\mathbf{x}) = \mathbf{x}$ otherwise. For a family $\mathcal{F}$ of vectors we define $\tau_{i,j}(\mathcal{F}) = \{\tau_{i,j}(\mathbf{x}) : \mathbf{x} \in \mathcal{F}, \tau_{i,j}(\mathbf{x}) \notin \mathcal{F}\} \cup \{\mathbf{x} \in \mathcal{F} : \tau_{i,j}(\mathbf{x}) \in \mathcal{F}\}$.

The next lemma shows 2 basic properties of the shifting operating that are well-known for set systems.

Lemma 4.1. *For any $\mathcal{F} \subseteq Q(n,r)$ and $1 \le i,j \le n$ we have $|\Delta(\tau_{i,j}(\mathcal{F}))| \le |\Delta(\mathcal{F})|$. Furthermore, if $\mathcal{F}$ is s-multisum t-intersecting, then so is $\tau_{i,j}(\mathcal{F})$.*

Proof. Let us start with the proof of the claim concerning t-intersection. Suppose for $\mathbf{x},\mathbf{y} \in \tau_{i,j}(\mathcal{F})$ we have $|\mathbf{x} \cap_{m,s} \mathbf{y}| < t$. We cannot have $\mathbf{x},\mathbf{y} \in \mathcal{F}$, as it is impossible by the s-multisum t-intersecting property of $\mathcal{F}$. If $\mathbf{x},\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$, then $\tau_{j,i}(\mathbf{x}),\tau_{j,i}(\mathbf{y}) \in \mathcal{F}$ and $t > |\mathbf{x} \cap_{m,s} \mathbf{y}| = |\tau_{j,i}(\mathbf{x}) \cap_{m,s} \tau_{j,i}(\mathbf{y})|$ contradicts the s-multisum t-intersecting property of $\mathcal{F}$. Finally, if $\mathbf{x} \in \mathcal{F}$ and $\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$, then $\mathbf{y}' := \tau_{j,i}(\mathbf{y}) \in \mathcal{F} \setminus \tau_{i,j}(\mathcal{F})$. So if $\mathbf{x} = \tau_{i,j}(\mathbf{x})$, then $t > |\mathbf{x} \cap_{m,s} \mathbf{y}| = |\mathbf{x} \cap_{m,s} \mathbf{y}'|$ contradicts the s-multisum t-intersecting property of $\mathcal{F}$. If $\mathbf{x}' := \tau_{i,j}(\mathbf{x}) \neq \mathbf{x}$, then as $\mathbf{x} \in \tau_{i,j}(\mathcal{F})$, we must have $\mathbf{x}' \in \mathcal{F}$, and thus $t > |\mathbf{x} \cap_{m,s} \mathbf{y}| = |\mathbf{x}' \cap_{m,s} \mathbf{y}'|$ contradicts the s-multisum t-intersecting property of $\mathcal{F}$. This finishes the proof that shifting preserves multisum intersecting properties.

To see $|\Delta(\tau_{i,j}(\mathcal{F}))| \le |\Delta(\mathcal{F})|$ we define an injection $i : \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F}) \to \Delta(\mathcal{F}) \setminus \Delta(\tau_{i,j}(\mathcal{F}))$ by letting $i(\mathbf{x})$ be the vector obtained from $\mathbf{x}$ by interchanging its ith and jth coordinate. This is clearly an injection, all we need to verify is that every image belongs to $\Delta(\mathcal{F}) \setminus \Delta(\tau_{i,j}(\mathcal{F}))$. So let $\mathbf{x} \in \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F})$ be arbitrary. Then there exists $\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$ with $\mathbf{x} \in \Delta(\mathbf{y})$ and $\mathbf{y}' := \tau_{j,i}(\mathbf{y}) \in \mathcal{F} \setminus \tau_{i,j}(\mathcal{F})$. Clearly, $i(\mathbf{x}) \in \Delta(\mathbf{y}') \subset \Delta(\mathcal{F})$. It remains to show $i(\mathbf{x}) \notin \Delta(\tau_{i,j}(\mathcal{F}))$. First we claim $x_i > x_j$. Indeed, as $\mathbf{y} \in \tau_{i,j}(\mathcal{F}) \setminus \mathcal{F}$, we have $y_i > y_j$ showing $x_i \ge x_j$, and $x_i = x_j$ would mean $\mathbf{x} \in \Delta(\mathbf{y}')$ and $\mathbf{x} \in \Delta(\mathcal{F})$ contradicting $\mathbf{x} \in \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F})$. Now, $x_i > x_j$ implies $i(\mathbf{x})_i < i(\mathbf{x})_j$. Assume for a contradiction that there exists $\mathbf{y}^* \in \tau_{i,j}(\mathcal{F})$ with $i(\mathbf{x}) \in \Delta(\mathbf{y}^*)$. Then we must have $y^*_i \le y^*_j$. This is only possible if $\tau_{i,j}(\mathbf{y}^*) \in \mathcal{F}$. But then $\mathbf{x} \in \Delta(\tau_{i,j}(\mathbf{y}^*)) \subset \Delta(\mathcal{F})$ contradicting $\mathbf{x} \in \Delta(\tau_{i,j}(\mathcal{F})) \setminus \Delta(\mathcal{F})$. This finishes the proof. $\square$

Note that Lemma 4.1 is not valid for s-sum t-intersection instead of s-multisum t-intersection in the case of general t as, say, the family $\{(3,2),(1,3)\}$ is 4-sum 2-intersecting, while its $(1,2)$-shift $\{(3,2),(3,1)\}$ is only 4-sum 1-intersecting.

We say that $\mathcal{F}$ is *left-shifted* if $\tau_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $i < j$. Whenever $\tau_{i,j}(\mathcal{F}) \neq \mathcal{F}$ for some $i < j$, then $w(\mathcal{F}) = \sum_{\mathbf{x} \in \mathcal{F}} \sum_{i=1}^{n} ix_i$ strictly decreases, so starting from any family $\mathcal{F}$, after a finite number of shift operations one obtains a left-shifted family. Furthermore, by Lemma 4.1, the size of the shadow does not increase and intersection properties are preserved. Therefore, when proving a lower bound on the size of shadows, one can assume that $\mathcal{F}$ is left-shifted. We use the notation $2(n,r)$ for the set of vectors of rank r in $\{0,1,2\}^n$.

Theorem 4.2. *If $\mathcal{F} \subseteq 2(n,r)$ is 3-sum intersecting, then $|\Delta(\mathcal{F})| \ge |\mathcal{F}|$.*

Proof. We proceed by induction on n. If $n < r$, then for any $\mathbf{x} \in 2(n,r)$ we have $|\{i : x_i = 2\}| > |\{j : x_j = 0\}|$. Therefore for any $\mathcal{F} \subseteq 2(n,r)$ in the auxiliary bipartite graph B with parts $\mathcal{F}$ and $\Delta(\mathcal{F})$ and edges between pairs $\mathbf{y} \in \Delta(\mathbf{x})$, we have that the degrees of any vector $\mathbf{x}$ in $\mathcal{F}$ is at least as large as the degree of any of its neighbors $\mathbf{y} \in \Delta(\mathbf{x})$. Consequently, $|\Delta(\mathcal{F})| \ge |\mathcal{F}|$ as claimed.

If $n \ge r$, then, by Lemma 2.4, we can assume that $\mathcal{F}$ is left-shifted. For $a = 0,1,2$ we introduce $\mathcal{F}_a := \{\mathbf{x} \in \mathcal{F} : x_n = a\}$ and $\mathcal{F}^-_a := \{\mathbf{x}^- : \mathbf{x} \in \mathcal{F}_a\}$, where $\mathbf{x}^-$ is the vector obtained from $\mathbf{x}$ by omitting its last coordinate. Observe that if $\mathbf{y} \in \Delta(\mathcal{F}^-_a)$, then $\mathbf{y}^{+a} \in \Delta(\mathcal{F})$, where

$\mathbf{y}^{+a}$ is the vector obtained from $\mathbf{y}$ by concatenating a as a last coordinate. So, by induction, $|\Delta(\mathcal{F})| \geq \sum_{a=0}^{2} |\Delta(\mathcal{F}_a^-)| \geq \sum_{a=0}^{2} |\mathcal{F}_a^-| = |\mathcal{F}|$ if we can prove for the second inequality that $\mathcal{F}_a^-$ is 3-sum intersecting for all $a = 0, 1, 2$. This is clear for $a = 0, 1$ as vectors in $\mathcal{F}_a$ 3-sum intersect but as their last coordinate is 0 or 1, they must 3-sum intersect among the first $n-1$ coordinates.

Finally, consider $\mathcal{F}_2^-$. Suppose for a contradiction that $\mathbf{x}^-, \mathbf{y}^- \in \mathcal{F}_2^-$ with $|\mathbf{x}^- \cap_3 \mathbf{y}^-| = 0$. If for some $i \in [n-1]$ we have $x_i = 0$, $y_i \leq 1$, then $|\tau_{i,n}(\mathbf{x}) \cap_3 \mathbf{y}| = 0$ contradicting the 3-sum intersecting property of $\mathcal{F}_2$. We derive the same contradiction if $x_i \leq 1$, $y_i = 0$. But $\mathbf{x}^-, \mathbf{y}^- \in 2(n-1, r-2)$, so there are at most $r-2$ coordinates from $i \in [n-1]$ with $x_i, y_i \geq 1$, hence there exists at least one coordinate i for which we get the desired contradiction. $\square$

Theorem 4.3. *If $\mathcal{F} \subseteq 2^n$ is 3-multisum 2-intersecting, then $|\mathcal{F}| \leq |\cup_{r=n+1}^{2n} 2(n,r)|$.*

Proof. Let $\mathcal{F}$ be a 3-multisum 2-intersecting family of maximum size. Clearly, $\mathcal{F}$ is upward closed, i.e. $\mathbf{y} > \mathbf{x} \in \mathcal{F}$ implies $\mathbf{y} \in \mathcal{F}$. Observe that writing $\nabla(\mathbf{x}) = \{\mathbf{y} > \mathbf{x} : r(\mathbf{y}) = r(\mathbf{x})+1\}$, we have that for any $\mathbf{x} \in \mathcal{F}$ the shade $\nabla(\overline{\mathbf{x}})$ is disjoint with $\mathcal{F}$. Let r be the rank of a smallest ranked vector in $\mathcal{F}$ and consider $\mathcal{F}_r = \{\mathbf{x} \in \mathcal{F} : r(\mathbf{x}) = r\}$. Observe that $\mathcal{F}' := (\mathcal{F} \setminus \mathcal{F}_r) \cup \nabla(\overline{\mathcal{F}_r})$ is 3-multisum 2-intersecting. Indeed, vectors from $\mathcal{F}' \setminus \mathcal{F}$ are all of rank $2n - r + 1$ and vectors from $\mathcal{F} \cap \mathcal{F}'$ are all of rank at least $r + 1$, so they must 3-multisum 2-intersect. As $|\nabla(\overline{\mathcal{F}})| = |\Delta(\mathcal{F})|$, by Theorem 4.2, $|\mathcal{F}'| \geq |\mathcal{F}|$ and we can repeat this procedure as long as $r \leq n$ and thus $2n - r + 1 > r$. We obtain that $|\mathcal{F}| \leq |\cup_{r=n+1}^{2n} 2(n,r)|$. $\square$

Theorem 4.4. *If $\mathcal{F} \subseteq 2^n$ is 3-multisum 3-intersecting, then $|\mathcal{F}| \leq |\cup_{r=n+2}^{2n} 2(n,r)| + M(n)$, where $M(n)$ denotes the maximum size of a 3-multisum 3-intersecting family in $2(n, n+1)$.*

Proof. The proof is almost identical to that of Theorem 4.3. Let $\mathcal{F}$ be a 3-multisum 3-intersecting family of maximum size. Observe that writing $\nabla_2(\mathbf{x}) = \{\mathbf{y} > \mathbf{x} : r(\mathbf{y}) = r(\mathbf{x})+2\}$, we have that for any $x \in \mathcal{F}$ the 2-shade $\nabla_2(\mathbf{x})$ is disjoint with $\mathcal{F}$. Let r be the rank of a smallest ranked vector in $\mathcal{F}$ and consider $\mathcal{F}_r = \{\mathbf{x} \in \mathcal{F} : r(\mathbf{x}) = r\}$. Observe that $\mathcal{F}' := (\mathcal{F} \setminus \mathcal{F}_r) \cup \nabla_2(\overline{\mathcal{F}_r})$ is 3-multisum 2-intersecting. Indeed, vectors fro $\mathcal{F}' \setminus \mathcal{F}$ are all of rank $2n - r + 2$ and vectors from $\mathcal{F} \cap \mathcal{F}'$ are all of rank at least $r + 1$, so they must 3-multisum 3-intersect. Note that if $\mathcal{G}$ is s-multisum t-intersecting, then $\Delta(\mathcal{G})$ is s-multisum $(t-2)$-intersecting. So applying Theorem 4.2 twice and $|\nabla_2(\overline{\mathcal{F}})| = |\Delta(\Delta(\mathcal{F}))|$, we obtain $|\mathcal{F}'| \geq |\mathcal{F}|$ and we can repeat this procedure as long as $r \leq n$ and thus $2n - r + 2 > r$. We obtain that there exist a maximum-sized 3-multisum 2-intersecting family $\mathcal{F}$ consisting only of vectors of rank at least $n + 1$. $\square$

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