# PIERCING THE CHESSBOARD

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ABSTRACT. We consider the minimum number of lines $h_n$ and $p_n$ needed to intersect or pierce, respectively, all the cells of the $n \times n$ chessboard. Determining these values can also be interpreted as a strengthening of the classical plank problem for integer points. Using the symmetric plank theorem of K. Ball, we prove that $h_n = \lceil \frac{n}{2} \rceil$ for each $n \geq 1$. Studying the piercing problem, we show that $0.7n \leq p_n \leq n - 1$ for $n \geq 3$, where the upper bound is conjectured to be sharp. The lower bound is proven by using the linear programming method, whose limitations are also demonstrated.

## 1. CELLS AND LINES

How many lines are needed to pierce each cell of the $n \times n$ chessboard? Likewise, what is the minimum number of lines required to intersect every cell? These innocent-looking questions serve as targets of the present note.

The question of determining $p_n$ was raised by Bárány and Frankl in [BF21, BF21+]. It turns out that the question has a close connection with the classical plank problem in the plane [B51, T32]. In particular, the celebrated symmetric plank theorem of K. Ball [B91] implies that given any set of $n-1$ lines, there always exists a point $(x, y)$ in $[0, n-1]^2$ such that the interior of the cell $[x, x+1] \times [y, y+1]$ is not intersected by any of the lines. The present question boils down to the following (see Conjecture 5): does there exist an *integer point* with the same property? If true, this would provide a significant strengthening of the plank theorem in this special case, and could also initiate the study of plank problems for lattice points. Even though we are not able to give a complete answer to the above question, we show nontrivial bounds on $p_n$ (see Theorem 4).

To start with, we introduce some notations. For $n \geq 1$, let $Q_n$ denote the $n \times n$ chessboard embedded in $[-1, 1]^2$. Its *cells* are the closed squares

$$(1) \qquad c_{ij} = \left[ -1 + (i-1) \cdot \frac{2}{n}, -1 + i \cdot \frac{2}{n} \right] \times \left[ -1 + (j-1) \cdot \frac{2}{n}, -1 + j \cdot \frac{2}{n} \right]$$

with $i, j \in [n]$, where $[n] = \{1, \ldots, n\}$. A line $\ell \subset \mathbb{R}^2$ is said to *hit* or *intersect* a cell $c_{ij}$ if $\ell \cap c_{ij} \neq \emptyset$, and it *pierces* $c_{ij}$ if $\ell \cap \operatorname{int} c_{ij} \neq \emptyset$. Let $h_n$ and $p_n$ be the minimal number of lines needed to hit or pierce, respectively, each cell of $Q_n$.

Our first observations are trivial. Clearly, $h_n \leq p_n$ holds for each $n$. Piercing each column of $Q_n$ with a vertical line shows that $p_n \leq n$. More generally, all the cells of $Q_n$ can be pierced by $n$ parallel lines in any given direction which are at distance $\frac{2}{n}$ from each other in the $\ell_1$ distance and do not go through any grid points. On a similar note, selecting every second vertical boundary line between the cells of $Q_n$ yields that $h_n \leq \lceil \frac{n}{2} \rceil$.

It is easy to give a sharp upper bound for the number of cells pierced by an arbitrary line. The following simple statement is part of the mathematical folklore (see also [B83]):

**Proposition 1.** *Every line pierces at most $2n - 1$ cells of $Q_n$.*

This readily implies that $p_n \geq \frac{n}{2}$. We note that higher dimensional versions of this estimate have been recently studied by Bárány and Frankl [BF21, BF21+].

Note that the analogue of Proposition 1 does not hold for hitting: diagonals of the square $Q_n$ intersect $3n$ cells. Nevertheless, using the symmetric plank theorem of K. Ball, we prove that the upper bound on $h_n$ given above is sharp:

**Theorem 2.** *For each $n \geq 1$, $h_n = \lceil \frac{n}{2} \rceil$ holds.*

Determining $p_n$ proves to be more difficult. Surprisingly, and somewhat counter-intuitively, the upper bound $p_n \leq n$ can be improved: there exist several configurations of $n - 1$ lines piercing all the cells of $Q_n$. This is also the subject of a mathematics puzzle [BS19] that appeared in The Guardian.

**Theorem 3.** *$p_2 = 2$ and for each $n \geq 3$, $p_n \leq n - 1$.*

The lower bound $p_n \geq \frac{n}{2}$ is not sharp either: using linear programming methods, we asymptotically strengthen it. Here comes our main result.

**Theorem 4.** *If $n$ is sufficiently large, then $p_n > 0.7n$.*

The gap between the upper and lower estimates for $p_n$ is large, and there is certainly room for improvement. A computer search was carried out in order to find configurations of $n-2$ lines piercing all cells of $Q_n$ when $n \leq 15$, to no avail. Based on this computational evidence, we venture to formulate the following conjecture.

**Conjecture 5.** *For all $n \geq 3$, $p_n = n - 1$.*

Note that the Corollary of [B91] implies that given a set of $n - 1$ lines, there always exists a *translate* of a cell contained within $Q_n$ which is not pierced by any of these lines. Conjecture 5 is the analogue of this statement for lattice cells.

Even though the linear programming method used for proving Theorem 3 may be strengthened, in Section 5 we demonstrate its limitations. Theorem 9 states that this method cannot give a lower bound larger than $0.925\,n$. The proof Conjecture 5 will require novel ideas.

## 2. HITTING

In this section we prove Theorem 2. We use the symmetric plank theorem of K. Ball.

**Lemma 6** (Ball [B91])**.** *If $(\varphi_i)_1^m$ is a sequence of unit functionals in a finite-dimensional normed space $X$, $(t_i)_1^m$ is a sequence of reals and $(w_i)_1^m$ is a sequence of positive numbers with $\sum_1^m w_i = 1$ then there is a point in the unit ball of $X$ for which*

$$|\varphi_i(x) - t_i| \geq w_i$$

*for every $i \in [m]$.*

*Proof of Theorem 2.* By the remark preceding Proposition 1 it suffices to prove $h_n \geq \lceil \frac{n}{2} \rceil$. Assume on the contrary that there exists a set of $m := \lceil \frac{n}{2} \rceil - 1$ lines, $\{\ell_1, \ldots, \ell_m\}$, intersecting each cell of $Q_n$. Let $u_i$ denote the (Euclidean) unit normal of $\ell_i$. Let $X$ be the space $\mathbb{R}^2$ endowed with the $\ell_\infty$-norm, that is, with unit ball $[-1, 1]^2$. To each planar vector $u \neq 0$, assign the linear functional

$$(2) \qquad\qquad\qquad\qquad \varphi_u : \ \mathbf{x} \mapsto \frac{\langle \mathbf{x}, u \rangle}{\|u\|_1}$$

which has norm 1 in $X^*$. Set $\varphi_i := \varphi_{u_i}$ for each $i \in [m]$. Then

(3) $$\ell_i = \{\mathbf{x} \in \mathbb{R}^2 : \varphi_i(\mathbf{x}) = t_i\}$$

holds for every $i \in [m]$ with some $t_i \in \mathbb{R}$. Note that the set of points $\mathbf{x} \in \mathbb{R}^2$ for which $|\varphi_i(\mathbf{x}) - t_i| \leq w$ holds equals to the union of all (closed) squares of edge length $2w$ whose center lies on $\ell_i$.

Set $\varepsilon > 0$ so that $(\frac{2}{n} + \varepsilon)(\lceil \frac{n}{2} \rceil - 1) = 1$, and let $w_i = \frac{2}{n} + \varepsilon$ for every $i$. Lemma 6 implies the existence of $\mathbf{x} \in [-1, 1]^2$ for which

$$|\varphi_i(\mathbf{x}) - t_i| \geq \frac{2}{n} + \varepsilon$$

holds for each $i$. This is equivalent to the fact that the open square

$$\mathbf{x} + \left( -\frac{2}{n} - \varepsilon, \frac{2}{n} + \varepsilon \right)^2$$

is not met by any of the lines $\ell_i$. We finally observe that any open square of side length strictly greater than $\frac{4}{n}$ centered in $[-1, 1]^2$ contains a cell of $Q_n$. $\square$

## 3. A PIERCING CONSTRUCTION

To each line $\ell$ we assign the corresponding "snake" $\sigma(\ell)$ of cells of $Q_n$ which are pierced by $\ell$:

(4) $$\sigma(\ell) = \{c_{ij} : i, j \in [n] \text{ and } \ell \text{ pierces } c_{ij}\}$$

(see the shaded region on Figure 1.a)).

We continue with a nontrivial upper bound on $p_n$.

*Proof of Theorem 3.* We will construct a set of $n - 1$ lines which pierces every cell of $Q_n$. Define the line $\ell$ by the equation

$$y = (1 - \varepsilon)x$$

where $\varepsilon > 0$ is a small positive number, for instance $\varepsilon = \frac{1}{n^2}$ will do. Thus, $\ell$ is obtained by a small clockwise rotation of the line $y = x$ about the origin, see Figure 1a).

Let now

$$\ell_i = \ell + \left( 1 - \frac{2i+1}{n}, -1 + \frac{2i+1}{n} \right)$$

for each $i \in [n-2]$, see Figure 1b). Then $\ell_i$ passes through the center of the cell $c_{(n-i)\,i}$ and intersects its boundary on its two vertical sides (because its slope is slightly less than one). Choose the value of $\varepsilon$ so that $\sigma(\ell_i)$ contains exactly three cells in row $i$, at most 2 cells in any other row of $Q_n$, and $\ell_i$ does not pass through any vertex of a cell in $Q_n$. Then along with any cell of $\sigma(\ell_i)$, it also contains a horizontal or vertical neighbour of the cell. Since $\ell_{i+1}$ is obtained from $\ell_i$ by a translation with $(-\frac{2}{n}, \frac{2}{n})$, this implies that $\bigcup_{i=1}^{n-2} \sigma(\ell_i)$ covers each cell of $Q_n$ in between $\ell_1$ and $\ell_{n-2}$ (simply check the off-diagonal chains of cells). Therefore, the lines $\ell_1, \ldots, \ell_{n-2}$ pierce all the cells of $Q_n$ except $c_{1n}, c_{2n}$ and $c_{(n-1)\,1} c_{n\,1}$. Clearly these four cells can be pierced by one line $\ell_0$, which leads to a set of $n - 1$ lines piercing every cell of $Q_n$. $\square$

We note that the above construction is not unique. Indeed, the same pattern works using translates of a line which pierces exactly 3 consecutive cells in some row and does not pass through any grid point. Thus, for any $\alpha \in (\frac{1}{3}, 1)$ there exists a set of $n - 2$ lines of slope $\alpha$, along with a line of slope $-\frac{n-1}{n-2}$ which pierce $Q_n$.
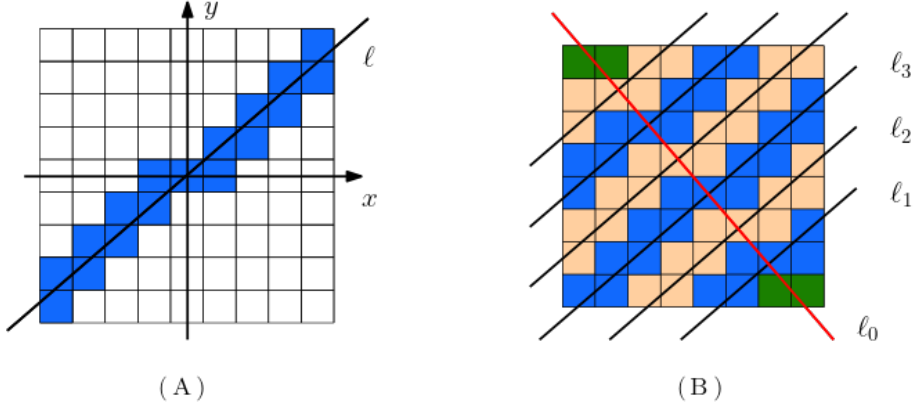
FIGURE 1. Piercing $Q_n$ with $n-1$ lines.

Going even further, we challenge the dedicated reader to find a piercing configuration which consists of $k$ parallel lines of an approximately diagonal direction and $n-1-k$ lines in the orthogonal direction, for each $k \in [n-2]$. (The two families of lines have to be positioned in a cross-like pattern, so that every boundary cell is pierced by them.)

## 4. FEW LINES DO NOT PIERCE

*Proof of Proposition 1.* Assume the slope of $\ell$ is non-negative. We move a point on $\ell$ from left to right and order the cells in $\sigma(\ell)$ (cf. (4)) in the order the point enters them. If $c_{ij} \in \sigma(\ell)$, then the next cell in this order is either $c_{(i+1)j}$ or $c_{i(j+1)}$. The cells in $\sigma(\ell)$ thus form a zig-zag going left or up at each step. $\qquad\square$

The lower bound $p_n \geq \frac{n}{2}$ follows from Proposition 1. We are going to improve this lower bound by the linear programming method that yields the following statement. Below, $\mathbf{x} = (x, y)$, and $\mathrm{d}\|\mathbf{x}\|_1$ stands for $\mathrm{d}(|x| + |y|)$.

**Lemma 7.** *Assume that $\mu : [-1,1]^2 \to \mathbb{R}_{\geq 0}$ is a Lipschitz continuous density function such that for each line $\ell$ in the plane,*

$$(5) \qquad \fint_{\ell \cap [-1,1]^2} \mu(\mathbf{x})\, \mathrm{d}\|\mathbf{x}\|_1 \leq 1.$$

*Then for any $\varepsilon > 0$,*

$$p_n > \left( \frac{1}{2} \fint_{[-1,1]^2} \mu(\mathbf{x})\, \mathrm{d}\mathbf{x} - \varepsilon \right) n$$

*holds if $n$ is sufficiently large.*

*Proof.* Denote by $\mathcal{L}$ the set of all lines intersecting $[-1,1]^2$, and set $\mathcal{S}_n = \{\sigma(\ell) : \ell \in \mathcal{L}\}$. So $\mathcal{S}_n$ is the set of all snakes. Clearly, $\mathcal{S}_n$ is a finite set, and for every element of $\mathcal{S}_n$ there exists a line which pierces the cells therein. Determining $p_n$ is equivalent to finding the optimal value of the following integer linear program (LP$_i$):

$$\text{Minimize } \sum_{\sigma \in \mathcal{S}_n} \rho(\sigma) \text{ subject to}$$

(LP$_i$) $\qquad \rho(\sigma) \in \{0, 1\} \text{ for all } \sigma \in \mathcal{S}_n, \text{ and}$

$$\sum_{\sigma \in \mathcal{S}_n: \, c_{ij} \in \sigma} \rho(\sigma) \geq 1 \text{ for every } i, j \in [n].$$

Therefore, the optimal value of the following *continuous* linear program $(\mathrm{LP}_c)$ gives a lower bound on $p_n$:

$$(\mathrm{LP}_c) \quad \begin{aligned} &\text{Minimize } \sum_{\sigma \in \mathcal{S}_n} \rho(\sigma) \text{ subject to}\\ &\rho(\sigma) \geq 0 \text{ for all } \sigma \in \mathcal{S}_n, \text{ and}\\ &\sum_{\sigma \in \mathcal{S}_n:\ c_{ij} \in \sigma} \rho(\sigma) \geq 1 \text{ for every } i, j \in [n]. \end{aligned}$$

Taking the dual program of $(\mathrm{LP}_c)$ leads to the following setup. Let $w : [n] \times [n] \to [0,1]$ be a weight function, and use the notation $w_{ij} = w(i,j)$. Let $M$ be the solution of the following continuous linear program:

$$(\mathrm{LP}_d) \quad \begin{aligned} &\text{Maximize } \sum_{i,j=1}^{n} w_{ij} \text{ subject to}\\ &w_{ij} \geq 0 \text{ for every } i, j = [n] \text{ and}\\ &\sum_{i,j\in[n]:\ c_{ij} \in \sigma(\ell)} w_{ij} \leq 1 \text{ for every line } \ell. \end{aligned}$$

By weak linear programming duality, the optimal value $M$ of $(\mathrm{LP}_d)$ gives a lower bound on that of $(\mathrm{LP}_c)$, which in turn gives a lower bound on the solution of $(\mathrm{LP}_i)$. Thus,

$$(6) \qquad p_n \geq M.$$

(Note that this fact also follows elementarily, without referring to LP duality. The above linear programs depend on $n$ but we suppress this dependence.)

Thus, we face the problem: How to solve $(\mathrm{LP}_d)$? Since there are only finitely many snakes in $\mathcal{S}_n$, the number of constraints in $(\mathrm{LP}_d)$ is finite. Therefore, $(\mathrm{LP}_d)$ can be solved computationally, at least for small values of $n$. In the case $n = 30$, the optimal weight distribution on $Q_n$ found by computational methods is plotted on Figure 2. This yields the estimate $p_n \geq 0.7205n$ for $n = 30$.

However, for large values of $n$ the computational approach brakes down; solving the $n = 30$ case already required several days of computing. We are going to replace $(\mathrm{LP}_d)$ by its continuous approximation when $n \to \infty$.

Let $\mu : [-1,1]^2 \to \mathbb{R}_{\geq 0}$ be a density function which is Lipschitz continuous with respect to the Euclidean distance, with constant $\lambda$. To each cell $c_{ij}$ of $Q_n$ we assign the weight

$$(7) \qquad w_{ij} = \frac{n}{2} \int_{c_{ij}} \mu(\mathbf{x})\, \mathrm{d}\mathbf{x}$$

where $\mathrm{d}\mathbf{x}$ stands for the standard Lebesgue measure. Note that

$$(8) \qquad \sum_{i,j=1}^{n} w_{ij} = \frac{n}{2} \int_{[-1,1]^2} \mu(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

On the other hand, for each line $\ell$ defined by the equation $\varphi_u(\mathbf{x}) = t$ (cf. (2)), we introduce the corresponding plank

$$(9) \qquad P(\ell) = \left\{ \mathbf{x} \in \left[ -1 - \frac{2}{n}, 1 + \frac{2}{n} \right]^2 : \ |\varphi_u(\mathbf{x}) - t| \leq \frac{1}{n} \right\}.$$
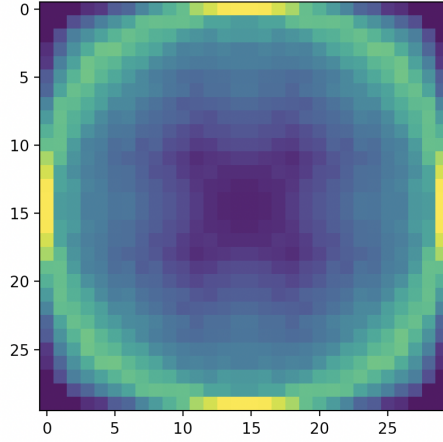
FIGURE 2. Optimal weight distribution on $Q_n$ with $n = 30$, where darker (blue) colors represent values close to 0.

Then $P(\ell)$ is the intersection of $[-1 - \frac{2}{n}, 1 + \frac{2}{n}]^2$ with the union of the squares of edge length $\frac{2}{n}$ centered on $\ell$. Note that $P(\ell)$ is not contained in $Q_n$, a small part of it is outside.
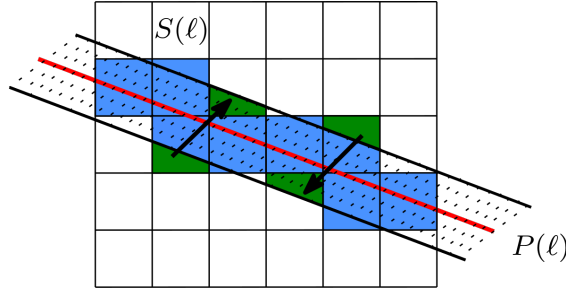


FIGURE 3. Comparison of the regions $P(\ell)$ and $S(\ell)$

**Claim 8.** *Using the above notations, for each line $\ell \in \mathcal{L}$,*

$$\sum_{i,j,\in[n]: \ c_{ij} \in \sigma(\ell)} w_{ij} \leq \frac{n}{2} \int_{P(\ell)} \mu(\mathbf{x}) \, d\mathbf{x} + \frac{34\lambda}{n} \ .$$

*Proof.* We may assume that the slope of $\ell$ is non-positive. Then, the upper boundary line of $P(\ell)$ is obtained from the lower boundary line by a translation with $v = (\frac{2}{n}, \frac{2}{n})$.

Let

$$S(\ell) = \bigcup_{\sigma(\ell)} c_{ij}.$$

Notice that $S(\ell) \cap (S(\ell) + v) = \emptyset$ and $S(\ell) \cap (S(\ell) - v) = \emptyset$. Observe that if a cell $c_{ij} \in \sigma(\ell)$ reaches below $P(\ell)$, then $(c_{ij} \setminus P(\ell)) + v \subset P(\ell)$, and similarly, if $c_{ij} \in \sigma(\ell)$ reaches above $P(\ell)$, then $(c_{ij} \setminus P(\ell)) - v \subset P(\ell)$, unless $c_{ij}$ is at the boundary of $Q'_n$ (see Figure 3). Thus, the parts of $S(\ell)$ which are not contained in $P(\ell)$ may be moved into $P(\ell)$ by a translation with either $v$ or $-v$, without creating overlaps (note that this is the reason for defining $P(\ell)$

on $[-1 - \frac{2}{n}, 1 + \frac{2}{n}]^2$ instead of $[-1,1]^2$). Therefore, by the Lipschitz property of $\mu$,

$$\sum_{i,j,\in[n]:\, c_{ij}\in\sigma(\ell)} w_{ij} = \frac{n}{2}\int_{S(\ell)} \mu(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \frac{n}{2}\int_{P(\ell)}\left(\mu(\mathbf{x}) + \frac{2\sqrt{2}}{n}\lambda\right)\mathrm{d}\mathbf{x}$$

$$\le \frac{n}{2}\int_{P(\ell)}\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} + \sqrt{2}\lambda\,\mathrm{Area}(P(\ell))$$

$$\le \frac{n}{2}\int_{P(\ell)}\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} + \sqrt{2}\lambda \cdot \frac{2}{n}\cdot 4\left(1 + \frac{2}{n}\right)$$

$$\le \frac{n}{2}\int_{P(\ell)}\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} + \frac{34\lambda}{n}. \qquad \square$$

Now, the proof of the Lemma is easy to complete. The Lipschitz property of $\mu$ ensures that as $n \to \infty$,

$$\frac{n}{2}\int_{P(\ell)}\mu(\mathbf{x})\,\mathrm{d}\mathbf{x} + \frac{34\lambda}{n} \to \int_{\ell\cap[-1,1]^2}\mu(\mathbf{x})\,\mathrm{d}\|\mathbf{x}\|_1.$$

Thus, (5) guarantees that the weights $w_{ij}$ given by (7) satisfy the criterion of $(\mathrm{LP}_d)$, and hence by (6) and (8), $\frac{n}{2}\int_{[-1,1]^2}\mu(\mathbf{x})\,\mathrm{d}\mathbf{x}$ provides a lower bound on $p_n$. $\qquad \square$

In view of Lemma 7, the next task is to find a suitable density function on $[-1,1]^2$.

*Proof of Theorem 4.* In order to illustrate the method, we will first consider the following density function which leads to a slightly weaker estimate:

$$(10) \qquad\qquad \mu_1(\mathbf{x}) = \frac{3}{4}(x^2 - 2x^2y^2 + y^2)$$

(see Figure 4a). The function $\mu_1(\mathbf{x})$ is clearly Lipschitz. We will show that (5) holds for $\mu_1$.

Assume that the line $\ell$ is defined by the equation $y = ax + b$. By the symmetries of $\mu(\mathbf{x})$, for proving (5) we may assume that $a \in [0,1]$ and $b \ge 0$. Also, if $\ell$ hits $[-1,1]^2$, then $b \le 1 + a$ must hold. Let $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$, $x_1 \le x_2$ be the points where $\ell$ hits the boundary of $[-1,1]^2$ (these may coincide when $\ell$ hits only a corner). By the assumptions on $a$ and $b$, we have that $x_1 = -1$ and $y_1 \in [-1,1]$. Depending on the magnitude of $b$, $\mathbf{x}_2$ may lie on the upper or the right side of $[-1,1]^2$:

- if $0 \le b \le 1 - a$, then $x_2 = 1$ and $y_2 = a + b$;
- if $1 - a \le b \le 1 + a$, then $x_2 = \frac{1-b}{a}$ and $y_2 = 1$.

Accordingly, (5) is equivalent to

$$(11) \qquad \max_{a\in[0,1],b\in[0,1-a]} (1+a)\int_{-1}^{1}(x^2 - 2x^2(ax+b)^2 + (ax+b)^2)\,\mathrm{d}x \le \frac{4}{3}$$

and

$$(12) \qquad \max_{a\in[0,1],b\in[1-a,1+a]} (1+a)\int_{-1}^{\frac{1-b}{a}}(x^2 - 2x^2(ax+b)^2 + (ax+b)^2)\,\mathrm{d}x \le \frac{4}{3}.$$

By evaluating the above integrals, (11) reads as

$$\max_{a\in[0,1],b\in[0,1-a]} (1+a)\left(\frac{2}{3} - \frac{2a^2}{15} + \frac{2b^2}{3}\right) \le \frac{4}{3}.$$

It is simple to check that the above function attains its maximum on the given domain at $a = 0, b = 1$ with the maximum value being exactly $\frac{4}{3}$. Similarly, (12) amounts to

$$\max_{a\in[0,1],b\in[1-a,1+a]} \frac{(1+a)(-1+5a^2+5a^3-a^5+5b^2+5a^3b^2-5b^3-5a^2b^3+b^5)}{15a^3} \leq \frac{4}{3}.$$

By analyzing the function above, one obtains that for any given $a \in [0,1]$, the above function is maximized at $b = (-1 - a + \sqrt{9 - 6a + 9a^2})/2$ on the interval $b \in [1-a, 1+a]$. Substituting this value leads to a function of $a$ which is decreasing on $(0,1]$.

Thus, we obtain that the maximum of $\int_{\ell\cap[-1,1]^2} \mu_1(\mathbf{x})\,\mathrm{d}\|\mathbf{x}\|_1$ is attained for lines $\ell$ which contain a side of $[-1,1]^2$, with the extreme value being 1. Since

$$\int_{[-1,1]^2} \mu_1(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{4}{3},$$

Lemma 7 guarantees that for any $\varepsilon > 0$,

$$p_n > \left(\frac{2}{3} - \varepsilon\right)n$$

if $n$ is sufficiently large.

The stronger estimate $p_n > 0.7n$ of Theorem 4 can be shown by considering the density function

$$\mu_2(\mathbf{x}) = 0.3(|x| + |y|) + 0.43(|x|^3 + |y|^3) - 0.585(|x|^3|y| + |y|^3|x|) - 0.16x^2y^2$$

(see Figure 4b), which has been found by numerical optimization. Clearly, $\mu_2$ is Lipschitz on $[-1,1]^2$, and

$$\int_{[-1,1]^2} \mu_2(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx 1.4039.$$

That (5) is satisfied for $\mu_2$ is again checked by elementary calculus, although the calculations are more tedious than in the previous case because of the absolute values in the definition of $\mu_2(\mathbf{x})$. We only note that, using the notation above, the maximum line integral is taken at $a = 0.5612, b = 0.5612$ with the maximum value being approximately 0.9971. Thus, because of the symmetries taken into account, lines of maximal weight go close to a corner of $[-1,1]^2$, and they are of slope around 0.56, $-0.56$, 1.78 or $-1.78$. Further calculations may be completed by the aid of a computer algebra software.          $\square$

## 5. Limitations of the linear programming method

Along the lines of the previous section, one may increase the lower bound on $p_n$ by including higher order terms in the density function, although at the price of increased computational difficulty. However, in this section we prove that the conjectured value $p_n = n - 1$ can not be proved by applying the continuous linear programming method. In order to show that, note that any feasible solution of the linear program $(\mathrm{LP}_c)$ yields an upper bound on the optimal value of $(\mathrm{LP}_c)$, hence, by linear programming duality, on that of $(\mathrm{LP}_d)$ as well. Therefore, in order to demonstrate the confinedness of the LP method, it suffices to provide a weight distribution $\rho(\sigma)$ on the set $S$ of snakes which satisfies the constraints of $(\mathrm{LP}_c)$ and for which $\sum_{\sigma\in S} \rho(\sigma) < n - 1$. We will, in fact, prove a much stronger bound.

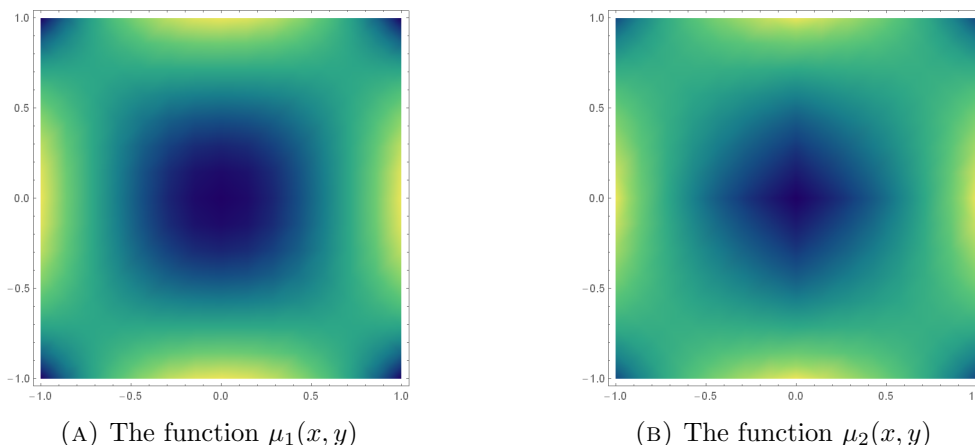(A) The function $\mu_1(x, y)$          (B) The function $\mu_2(x, y)$

FIGURE 4. The two density functions used in the proof of Theorem 4. Darker (blue) colors represent values close to 0, while lighter (yellow) stands for values close to 0.75.

**Theorem 9.** *For every sufficiently large $n$, there exists a weight distribution $\rho(\sigma)$ on $\mathcal{S}_n$ satisfying the conditions of $(LP_c)$ for which*

$$\sum_{\sigma \in \mathcal{S}_n} \rho(\sigma) < 0.925\, n.$$

Recall that $\mathcal{L}$ is the set of lines intersecting $[-1, 1]^2$. We parameterize $\mathcal{L}$ as follows: for each $s \in [-1, 1]$, introduce the vector

$$u_s = (s, 1 - |s|)$$

and for each $s, t \in [-1, 1]$, define

$$(13) \qquad \ell(s, t) = \{\mathbf{x} \in \mathbb{R}^2 : \ \varphi_{u_s} = t\}$$

(cf. (2)). That is, $\ell(s, t)$ is the line with normal $u_s$ defined by $sx + (1 - |s|)y = t$. It is easy to check that $\ell \in \mathcal{L}$ if and only if it may be expressed in the form (13) with $s, t \in [-1, 1]$.

Reminiscent of Lemma 7, we will construct the sought-after weight distribution on $\mathcal{S}$ by applying a continuous approximation.

**Lemma 10.** *Assume that $\nu : [-1, 1]^2 \to \mathbb{R}_{\geq 0}$ is a Lipschitz continuous density function which satisfies that for each $(x_0, y_0) \in [-1, 1]^2$,*

$$(14) \qquad \int_{-1}^{0} \nu(s, y_0 + s(x_0 + y_0))\, \mathrm{d}s + \int_{0}^{1} \nu(s, y_0 + s(x_0 - y_0))\, \mathrm{d}s \geq 1.$$

*Then for any $\varepsilon > 0$, the optimal value of $(\mathrm{LP}_c)$ is bounded from above by*

$$\left( \frac{1}{2} \int_{[-1,1]^2} \nu(s, t)\, \mathrm{d}s\, \mathrm{d}t + \varepsilon \right) n$$

*if $n$ is sufficiently large.*

*Proof.* To each snake $\sigma \in \mathcal{S}_n$, assign the weight

$$(15) \qquad \rho(\sigma) = \frac{n}{2} \int_{(s,t) \in [-1,1]^2 : \ \sigma(\ell(s,t)) = \sigma} \nu(s, t)\, \mathrm{d}s\, \mathrm{d}t.$$

Clearly,

$$\sum_{\sigma \in \mathcal{S}_n} \rho(\sigma) = \frac{n}{2} \int_{[-1,1]^2} \nu(s,t) \, \mathrm{d}s \, \mathrm{d}t. \tag{16}$$

In order for the weights $\rho(\sigma)$ to satisfy the conditions of $(\mathrm{LP}_c)$, the following inequality must hold for all $i, j \in [n]$:

$$\frac{n}{2} \int_{(s,t) \in [-1,1]^2: \ c_{ij} \in \sigma(\ell(s,t))} \nu(s,t) \, \mathrm{d}s \, \mathrm{d}t \geq 1. \tag{17}$$

The above integration goes over the set of parameters $(s,t)$ whose corresponding lines pierce $c_{ij}$. This region can be determined as follows. Let $(x_0, y_0)$ be the center of $c_{ij}$. Recall that the side-length of $c_{ij}$ equals to $\frac{2}{n}$. First, assume that $s \leq 0$, equivalently, that the slope of $\ell(s,t)$ is non-positive. Then $\ell(s,t)$ pierces $c_{ij}$ if and only if it contains a point of the form $(x_0 + a, y_0 - a)$ with $|a| < \frac{1}{n}$. That is equivalent to the condition

$$|s(x_0 + y_0) - t + y_0| < \frac{1}{n}.$$

Similarly, if $s \geq 0$, then $\ell(s,t)$ intersects $c_{ij}$ iff it goes through a point of the form $(x_0 + b, y_0 + b)$ with $|b| < \frac{1}{n}$, which is equivalent to

$$|s(x_0 - y_0) - t + y_0| < \frac{1}{n}.$$

Thus, we derive that the set of lines in $\mathcal{L}$ which pierce $c_{ij}$ is represented on the $(s,t)$-plane by the region $R_{ij}$ which is the vertical parallel neighborhood of radius $\frac{1}{n}$ of the union of two segments, connecting the points $(-1, -x_0)$ and $(0, y_0)$, and $(0, y_0)$ and $(1, x_0)$, respectively. In particular, all the cross-sections of $R_{ij}$ parallel to the $t$-axis are of length $\frac{2}{n}$ (see Figure 5).
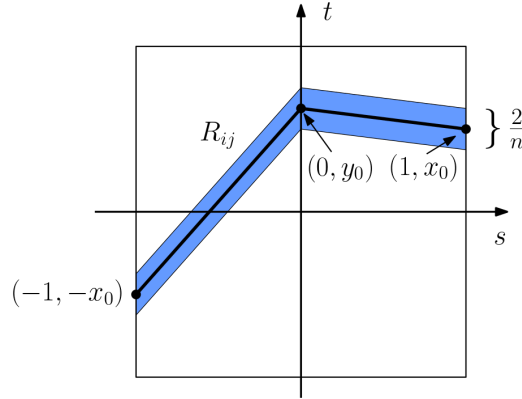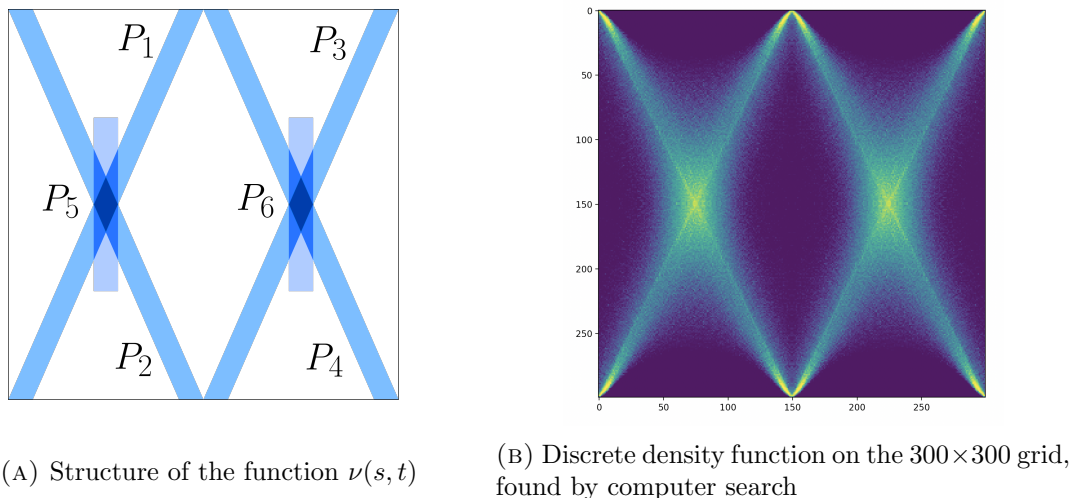


FIGURE 5. The region $R_{ij}$ on the $(s,t)$-plane which represents the set of lines intersecting $c_{ij}$

By the Lipschitz property of $\nu$, the integrals in (17) converge uniformly as $n \to \infty$:

$$\frac{n}{2} \int_{(s,t) \in [-1,1]^2: \ c_{ij} \in \sigma(\ell(s,t))} \nu(s,t) \, \mathrm{d}s \, \mathrm{d}t \to \int_{-1}^{0} \nu(s, y_0 + s(x_0 + y_0)) \, \mathrm{d}s + \int_{0}^{1} \nu(s, y_0 + s(x_0 - y_0)) \, \mathrm{d}s.$$

Thus, (14) and (17) guarantee that the conditions of $(\mathrm{LP}_c)$ hold for a suitably scaled copy of $\rho$ provided by (15), and the statement of Lemma 10 follows from (16).   $\square$

We are left with the task of finding a suitable density function on the $(s,t)$-plane.

(A) Structure of the function $\nu(s, t)$

(B) Discrete density function on the $300 \times 300$ grid, found by computer search

FIGURE 6. Dual density functions on the $(s, t)$-plane

*Proof of Theorem 9.* We will construct a density function $\nu(s, t)$ which is not Lipschitz continuous, but satisfies (14) as well as

$$(18) \qquad \int_{[-1,1]^2} \nu(s, t) \, ds \, dt < 1.849.$$

Lemma 10 may then be applied to a suitably fine Lipschitz continuous approximation of $\nu$, yielding the estimate of Theorem 9.

The density $\nu$ is defined using a parameter $\gamma \in [0, 1]$ whose value we will set later. Let $\varepsilon > 0$ be a small positive number. Define $P_1$ to be the parallelogram

$$P_1 = \text{conv}\{(0, 1), (-\varepsilon, 1), (-1, -1), (-1 + \varepsilon, -1)\}$$

of area $2\varepsilon$. Let $P_2 = \{(s, t) : (s, -t) \in P_1\}$ be the mirror image of $P_1$ with respect to the $s$-axis, and set $P_3 = -P_1$ and $P_4 = -P_2$. Define $P_5$ to be the rectangle

$$P_5 = \text{conv}\left\{\left(-\frac{1-\varepsilon}{2}, h\right), \left(-\frac{1+\varepsilon}{2}, h\right), \left(-\frac{1+\varepsilon}{2}, -h\right), \left(-\frac{1-\varepsilon}{2}, -h\right)\right\}$$

where $h = \frac{\tan \gamma}{2} + \varepsilon$. Note that if $\varepsilon < 1 - \frac{\tan 1}{2}$, then $h \in [0, 1)$. Finally, let $P_6 = -P_5$ (see Figure 6a).

We will say that a line $\ell$ is of *angle* $\alpha$ if its slope is $\tan \alpha$. Let $\frac{\pi}{2} - \beta$ be the angle of the line connecting the points $(-1 + \varepsilon/2, -1)$ and $(-\varepsilon/2, 1)$. Then $\tan \beta = \frac{1-\varepsilon}{2}$.

Now, let $\ell$ be a line of angle $\alpha$ which goes through two points $(-1, t_1)$ and $(0, t_2)$ with $t_1, t_2 \in [-1, 1]$. A simple calculation shows (see Figure 7a) that the horizontal projection of $\ell \cap P_1$ has length

$$\frac{\varepsilon \cos \alpha \cos \beta}{\cos(\alpha + \beta)}.$$

By symmetry, the horizontal projection of $\ell \cap P_2$ has length $\frac{\varepsilon \cos \alpha \cos \beta}{\cos(\alpha - \beta)}$. Let

$$(19) \qquad \phi(\alpha) = \frac{\cos \alpha \cos \beta}{\cos(\alpha + \beta)} + \frac{\varepsilon \cos \alpha \cos \beta}{\cos(\alpha - \beta)}.$$

It is easy to check that $\phi(\alpha)$ is symmetric, convex on $(-\arctan 2, \arctan 2)$, and it attains its minimum on this interval at $0$ with $\phi(0) = 2$. In particular, $\phi$ is increasing on $[0, \arctan 2]$.

(A) Scheme of $\ell \cap P_1$

(B) The case $|\alpha_1| + |\alpha_2| < \gamma$
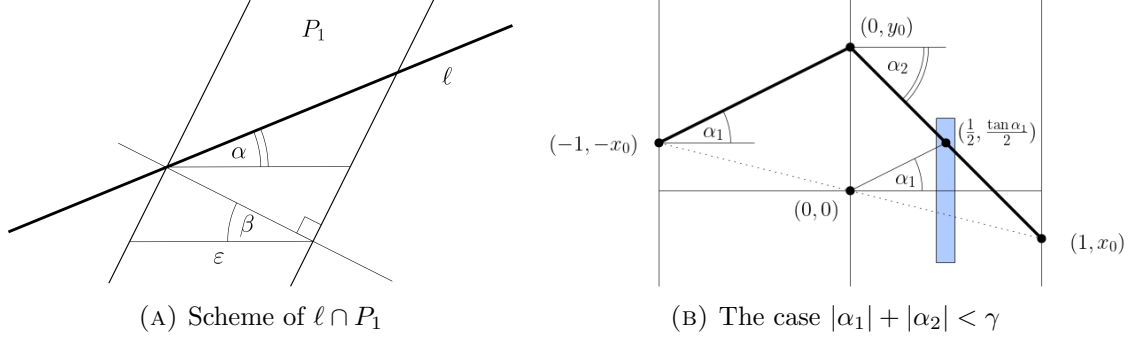
FIGURE 7

Next, we define the density function $\nu(s,t)$. Set $w = \frac{1}{2\phi(\gamma)}$. Denote by $\chi_A$ the indicator function of the set $A$, and let

$$\nu_1(s,t) = \frac{w}{\varepsilon} \sum_{i=1}^{4} \chi_{P_i}(s,t).$$

Introduce

$$\psi(t) = \left(1 - \frac{\phi(\arctan 2t)}{\phi(\gamma)}\right).$$

Then $\psi(t)$ is convex on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with the maximum value taken at $t = 0$, and $\psi(t) > 0$ for $|t| \leq h$. Define

$$\nu_2(s,t) = \begin{cases} \frac{1}{\varepsilon}\psi(t) & \text{for } (s,t) \in P_5 \cup P_6 \\ 0 & \text{otherwise,} \end{cases}$$

and let $\nu(s,t) = \nu_1(s,t) + \nu_2(s,t)$. We will show that $\nu$ satisfies the condition (14).

Let $x_0, y_0 \in [-1,1]$ be arbitrary, and denote by $\alpha_1$ and $\alpha_2$ the angle of the lines through the points $(-1, -x_0)$ and $(0, y_0)$, and $(0, y_0)$ and $(1, x_0)$, respectively. Then, since $\phi$ is symmetric and convex,

(20)
$$\int_{-1}^{0} \nu_1(s, y_0 + s(x_0 + y_0))\,\mathrm{d}s + \int_{0}^{1} \nu_1(s, y_0 + s(x_0 - y_0))\,\mathrm{d}s = w(\phi(\alpha_1) + \phi(\alpha_2))$$

$$\geq 2w\phi\left(\frac{|\alpha_1| + |\alpha_2|}{2}\right) = \frac{\phi\left(\frac{|\alpha_1|+|\alpha_2|}{2}\right)}{\phi(\gamma)}.$$

By the monotonicity of $\phi$, the value of the above integral is at least 1 whenever $|\alpha_1| + |\alpha_2| \geq 2\gamma$. Since $\nu(s,t) \geq \nu_1(s,t)$, (14) is satisfied for such pairs $x_0, y_0$.

Assume now that $|\alpha_1| + |\alpha_2| < 2\gamma$, and $|\alpha_1| \leq |\alpha_2|$. Note that $y_0 = -x_0 \pm \tan \alpha_1$, and therefore the line containing $(0, y_0)$ and $(1, x_0)$ passes through $(\frac{1}{2}, \frac{\tan \alpha_1}{2})$ or $(\frac{1}{2}, -\frac{\tan \alpha_1}{2})$ (see Figure 7b). Thus, symmetry and convexity of $\psi(t)$ implies that

$$\int_{0}^{1} \nu_2(s, y_0 + s(x_0 - y_0))\,\mathrm{d}s \geq \psi\left(\frac{\tan \alpha_1}{2}\right) = 1 - \frac{\phi(|\alpha_1|)}{\phi(\gamma)}.$$

Accordingly, by (20),

$$\int_{-1}^{0} \nu(s, y_0 + s(x_0 + y_0)) \, \mathrm{d}s + \int_{0}^{1} \nu(s, y_0 + s(x_0 - y_0)) \, \mathrm{d}s$$

$$\geq 2w\phi(|\alpha_1|) + 1 - \frac{\phi(|\alpha_1|)}{\phi(\gamma)} = 1.$$

Thus, $\nu(s,t)$ satisfies (14), and we must show (18):

$$\int_{[-1,1]^2} \nu(s,t) \, \mathrm{d}s \, \mathrm{d}t = \int_{[-1,1]^2} \nu_1(s,t) \, \mathrm{d}s \, \mathrm{d}t + \int_{[-1,1]^2} \nu_2(s,t) \, \mathrm{d}s \, \mathrm{d}t$$

$$= \frac{4}{\phi(\gamma)} + 2 \int_{-h}^{h} \psi(t) \, \mathrm{d}t$$

$$= \frac{4}{\phi(\gamma)} + 2 \tan \gamma - \frac{8}{\phi(\gamma)} \operatorname{arctanh}\left( \frac{\tan \gamma}{2} + \varepsilon \right)$$

$$\approx \frac{4}{\phi(\gamma)} + 2 \tan \gamma - \frac{8}{\phi(\gamma)} \operatorname{arctanh}\left( \frac{\tan \gamma}{2} \right).$$

On the interval $[0,1]$, the above quantity is minimal at $\gamma = 0.746$ with the attained value 1.8485. Therefore, by setting this value for $\gamma$, for sufficiently small $\varepsilon$, the integral of $\nu(s,t)$ on $[-1,1]^2$ is less than 1.849. $\qquad\square$

Our goal above was to demonstrate that the linear programming method cannot yield a proof for Conjecture 5, and we did not set off to minimize $\int_{[-1,1]^2} \nu(s,t) \, \mathrm{d}s \, \mathrm{d}t$ among suitable density functions. By refining the construction, the factor 0.925 of Theorem 9 can be improved. For example, approximating $[-1,1]^2$ with a $300\times300$ grid, the discrete density function found by computer search (see Figure 6b) yields the upper estimate $0.7915\,n$.

## 6. Higher dimensions

The analogous questions may be formulated in higher dimensions as well, when we would like to hit or cut (pierce) the cells of the $d$-dimensional box $Q_n^d$ of size $n \times n \times \ldots \times n$ with as few hyperplanes as possible. Higher dimensional analogues of Proposition 1 were studied in [BF21] and [BF21+]. The authors proved that in the 3-dimensional case, any given plane cuts at most $\frac{9}{4}n^2 + 2n + 1$ cells, while the upper bound in the $d$-dimensional case is $v_d n^{d-1}(1 + o(1))$ where $v_d \approx \sqrt{6d/\pi}$ is a well-defined constant. Thus, we derive that the minimum number of hyperplanes needed to cut each cell is at least $\frac{4}{9}n(1+o(1))$ when $d = 3$ and at least $\frac{1}{\sqrt{2d}}n(1 + o(1))$ when $d \geq 4$. On the other hand, $n$ parallel hyperplanes clearly suffice.

Turning to the hitting problem, the situation is different: the proof of Theorem 2 extends to higher dimensions with no difficulty. Therefore, we obtain that the minimal number of hyperplanes needed to hit each cell of $Q_n^d$ is exactly $\lceil \frac{n}{2} \rceil$.

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