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On discretizing integral norms of exponential sums

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ABSTRACT

In this paper we study L_p Marcinkiewicz-Zygmund type inequalities

$$c_1 \sum_{1\le j\le N} w_j |g(\mathbf{x}_j)|^p \le \|g\|^p_{L_p(K)} \le c_2 \sum_{1\le j\le N} w_j |g(\mathbf{x}_j)|^p$$

for general exponential sums of the form $g(\mathbf{x}) = \sum_{1\le j\le n} a_j e^{\langle \lambda_j, \mathbf{x}\rangle}$, $\mathbf{x}, \lambda_j \in \mathbb{R}^d$, $a_j \in \mathbb{R}$. One of the main results of the paper asserts that when $1 \le p < \infty, K = [a,b]$ and the exponents satisfy relations $\lambda_{j+1} - \lambda_j \ge \epsilon, 1 \le j \le n-1$, $\max_{1\le j\le n} |\lambda_j| \le \Lambda$ then Marcinkiewicz-Zygmund type inequalities hold for certain point sets of cardinality

$$N \le cn \ln^{\frac{1}{p}+1} \frac{\Lambda}{\epsilon}.$$

Since the dependence of cardinality on the parameters ϵ, Λ appears only in the logarithmic term, this bound is "almost" degree and separation independent. Moreover, the discrete meshes x_j and weights $w_j = x_{j+1} - x_j$ are given explicitly and they are *universal* in the sense that they work for any exponential sum as above. This result will rely on some new *degree independent* Bernstein-Markov type inequality for exponential sums. Moreover we will extend our considerations to multivariate exponential sums. In addition, it will be shown that much stronger results hold for exponential sums with nonnegative coefficients.

1. Introduction

Consider the space $L_p(K), 1 \le p \le \infty$ endowed with some probability measure on the compact set $K \subset \mathbb{R}^d$. Then given a subspace $U \subset L_p(K)$ the Marcinkiewicz-Zygmund type problem for $1 \le p < \infty$ consists

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in finding discrete point sets $Y_N = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset K$ and corresponding positive weights $w_j > 0, 1 \leq j \leq N$ such that for any $g \in U$ we have

$$c_1 \sum_{1 \leq j \leq N} w_j |g(\mathbf{x}_j)|^p \leq \|g\|^p_{L_p(K)} \leq c_2 \sum_{1 \leq j \leq N} w_j |g(\mathbf{x}_j)|^p \tag{1}$$

with some constants $c_1, c_2 > 0$ depending only on p, d and K. In case when $K = [a, b] \subset \mathbb{R}$ is an interval on the real line a natural choice of the weights associated with the points $a < x_1 < x_2 < ... < x_N < b$ is given by $w_j = x_{j+1} - x_j, 1 \leq j \leq N - 1$. In this respect the classical Marcinkiewicz-Zygmund inequality for trigonometric polynomials states that (1) holds for the space $U = T_n$ of trigonometric polynomials of degree $\leq n$ and uniformly distributed points on the period with $w_j = \frac{1}{n}$ and $N = 2n + 1$. Clearly the cardinality of the discrete point set $N = 2n + 1$ is optimal here. This equivalence relation turned out to be an effective tool used for the discretization of the L_p norms of trigonometric polynomials which is widely applied in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, see for instance [9] for a survey on the univariate Marcinkiewicz-Zygmund type inequalities. An important generalization of the classical Marcinkiewicz-Zygmund inequality for the so called *doubling weights* was given by Mastroianni and Totik [10]. Various extensions of the Marcinkiewicz-Zygmund inequality for multivariate algebraic and trigonometric polynomials can be found in [1], [4] and [2].

In this paper we will consider problem (1) for general real exponential polynomials of the form

$$g(\mathbf{x}) = \sum_{1 \leq j \leq n} a_j e^{\langle \lambda_j, \mathbf{x} \rangle}, \;\; a_j \in \mathbb{R}, \; \mathbf{x} \in \mathbb{R}^d \tag{2}$$

with arbitrary given $\lambda_j \in \mathbb{R}^d$. In general, results related to these exponential sums depend on the number n of terms in the sums and the size of λ_j-s. The "degree" of these exponential polynomials is given by $\max_{1 \leq j \leq n} |\lambda_j|$. Here and throughout the paper $|\cdot|$ denotes the usual Euclidian norm of vectors in $\mathbb{R}^d$. Naturally one should try to aim for discrete meshes with possibly smallest cardinality. A particularly interesting problem consists in obtaining discrete meshes of cardinality depending only on the dimension n of the exponential sums (2), that is independent of their degree. For the trigonometric exponential sums with λ_j-s being of the form $\lambda_j = in_j, n_j \in \mathbb{Z}$ this question is discussed in detail in the survey paper [2]. In particular in [12] Theorem 1.1 it is shown that for general trigonometric exponential sums one can get meshes of cardinality $\sim n$. In trigonometric case the basic property which is the main tool of investigation is the pair wise orthogonality of exponentials. In the general algebraic case the exponents in (2) do not have this crucial feature. So instead of orthogonality our considerations will rely heavily on Bernstein-Markov type inequalities for the size of derivatives of exponential sums. We will verify in this paper some new degree independent Bernstein-Markov type inequalities for exponential sums. This will allow us to extend Marcinkiewicz-Zygmund type inequalities to general real exponential sums. In particular, it will be shown (see Theorem 1 below) that there exist meshes $Y_N = \{x_1, ..., x_N\} \subset [a, b] \subset \mathbb{R}$ of cardinality

$$N \sim n \ln^{\frac{1}{p}+1} \frac{\Lambda}{\epsilon} \tag{3}$$

so that (1) holds with weights $w_j := x_{j+1} - x_j$ for every $1 \leq p < \infty$ and every exponential polynomial (2) with any $\lambda_j \in \mathbb{R}$ satisfying $\lambda_{j+1} - \lambda_j \geq \epsilon > 0, 1 \leq j \leq n - 1$ and $\max_{1 \leq j \leq n} |\lambda_j| \leq \Lambda$. An important feature of this result is the fact that the cardinality of the discrete mesh is of optimal order n, while their degree Λ and the separation parameter ϵ of the exponents essentially appear only in the logarithmic term. So in this sense our bound is "almost" *degree and exponent independent*. The explicitly given discrete sets used in Theorem 1 are *universal* in the sense that they work for all exponential sums as above. We will also include similar results for multivariate exponential sums. Finally, we will present some new dimension and degree independent Bernstein-Markov type inequalities for multivariate exponential sums with *nonnegative coefficients*. This will lead to a considerable improvement of the corresponding Marcinkiewicz-Zygmund type inequalities for multivariate exponential sums with nonnegative coefficients. It should be also noted that all discrete meshes are constructed in this paper explicitly.

2. Marcinkiewicz-Zygmund type inequalities for general exponential sums

The next theorem which is one of the main results of this paper presents a Marcinkiewicz-Zygmund type inequality for general real univariate exponential sums based on point sets of cardinality $N \leq cn\ln^{\frac{1}{p}+1}\frac{\Lambda}{\epsilon}$.

Theorem 1. *Let $1 \leq p < \infty, [a,b] \subset \mathbb{R}, 0 < \epsilon \leq 1, n \in \mathbb{N}, \Lambda > 1$. Then there exist discrete points sets $Y_N = \{x_1 < ... < x_N\} \subset (a,b)$ of cardinality*

$$N \leq cpn\ln^{\frac{1}{p}+1}\frac{\Lambda}{\epsilon}$$

where $c > 0$ is an absolute constant, so that for every exponential sum (2) with arbitrary $\lambda_j \in \mathbb{R}$ satisfying

$$\lambda_{j+1} - \lambda_j \geq \frac{\epsilon}{b-a}, 1 \leq j \leq n-1, \quad \max_{1\leq j\leq n} |\lambda_j| \leq \Lambda$$

we have

$$\frac{1}{2}\sum_{1\leq j\leq N-1}(x_{j+1}-x_j)|g(x_j)|^p \leq \|g\|^p_{L_p([a,b])} \leq 2\sum_{1\leq j\leq N-1}(x_{j+1}-x_j)|g(x_j)|^p. \tag{4}$$

Remark 1. One should note the fact that the discrete nodes provided by Theorem 1 are rather universal, they work for all exponential sums satisfying the separation condition $\lambda_{j+1} - \lambda_j \geq \frac{\epsilon}{b-a}, 1 \leq j \leq n-1$ and the upper bound $\max_{0\leq j\leq n} |\lambda_j| \leq \Lambda$. In addition the dependence of cardinality on the parameters ϵ, Λ appears only in the logarithmic term, so the bound on cardinality is "almost" degree and separation independent. It should be also mentioned that the constants $\frac{1}{2}$ and 2 in (4) can be replaced by $1-\xi$ and $1+\xi$, respectively, with an arbitrarily small $0 < \xi < 1$. Of course, one would have to pay a price for this in the sense that this will make the constant in the upper bound for the cardinality of Y_N dependent on ξ. The proof of Theorem 1 given below indicates that constants of order $-\frac{\ln\xi}{\xi}$ can be used for the upper bound of cardinality. In a recent paper [8] the precise dependence of this constant on parameter $\xi > 0$ was given in the classical univariate trigonometric case.

The proof of the above Marcinkiewicz-Zygmund inequality will be based on a new degree independent L_p Bernstein type inequality for derivatives of exponential sums (2). We will also use the following $L_p, 1 \leq p < \infty$ Bernstein type inequality for derivatives of univariate exponential sums $f_n(x) = \sum_{1\leq j\leq n} c_j e^{\lambda_j x}$ given in [5], Theorem 3.4

$$\|f_n'\|_{L_p[-1+\delta,1-\delta]} \leq \frac{2n-1}{\delta}\|f_n\|_{L_p[-1,1]}, \ \ 0 < \delta < 1. \tag{5}$$

This elegant result provides exponent independent upper bounds for L_p norms of derivatives of exponential sums inside the interval. The drawback of the above estimate is the appearance of the term $\frac{1}{\delta}$ in the upper estimate which leads to larger than required discretization sets in L_p Marcinkiewicz-Zygmund type inequalities. Our next lemma shows that introducing a weight $1-x^2$ into the L_p norms of derivatives of exponential sums allows to replace $\frac{1}{\delta}$ by a substantially smaller term $\ln\frac{2}{\delta}$. This improvement will be subsequently used in order to verify near optimal discretization meshes.

Lemma 1. *Let* $1 \le p < \infty, 0 < \delta < 1, n \in \mathbb{N}$. *Then for any distinct real numbers* $\lambda_1, ..., \lambda_n \in \mathbb{R}$ *and any exponential sum* $f_n(x) = \sum_{1\le j\le n} c_j e^{\lambda_j x}, \forall c_j \in \mathbb{R}$ *we have*

$$\|(1-x^2)f_n'(x)\|_{L_p[-1+\delta,1-\delta]} \le 9n \ln^{\frac{1}{p}} \frac{2}{\delta} \|f_n\|_{L_p[-1,1]}. \tag{6}$$

Proof. Using (5) with any $0 < t < 1$ and multiplying the p-th power of this estimate by t^{p-1} yields

$$t^{p-1} \int_{-1+t}^{1-t} |f_n'(x)|^p dx \le \frac{(2n-1)^p}{t} \int_{-1}^{1} |f_n(x)|^p dx, \quad 0 < t < 1.$$

Integrating above inequality with respect to $t \in [\frac{\delta}{2}, 1]$ we have

$$\int_{\frac{\delta}{2}}^{1} \int_{-1+t}^{1-t} t^{p-1} |f_n'(x)|^p dx dt \le (2n-1)^p \int_{\frac{\delta}{2}}^{1} \frac{dt}{t} \int_{-1}^{1} |f_n(x)|^p dx = (2n-1)^p \ln \frac{2}{\delta} \int_{-1}^{1} |f_n(x)|^p dx.$$

Furthermore applying Fubini theorem for the integral on the left hand side of above estimate implies

$$\int_{\frac{\delta}{2}}^{1} \int_{-1+t}^{1-t} t^{p-1} |f_n'(x)|^p dx dt = \int_{-1+\frac{\delta}{2}}^{1-\frac{\delta}{2}} \int_{\frac{\delta}{2}}^{1-|x|} t^{p-1} |f_n'(x)|^p dt dx = \frac{1}{p} \int_{-1+\frac{\delta}{2}}^{1-\frac{\delta}{2}} |f_n'(x)|^p ((1-|x|)^p - (\frac{\delta}{2})^p) dx$$

$$\ge \frac{1}{p} \int_{-1+\delta}^{1-\delta} |f_n'(x)|^p ((1-|x|)^p - (\frac{\delta}{2})^p) dx \ge \frac{1}{p} \int_{-1+\delta}^{1-\delta} |f_n'(x)|^p ((1-|x|)^p - ((1-|x|)/2)^p) dx$$

$$\ge \frac{1}{p 2^{p+1}} \int_{-1+\delta}^{1-\delta} |f_n'(x)|^p (1-x^2)^p dx.$$

Thus combining the last two estimates above we arrive at

$$\int_{-1+\delta}^{1-\delta} |f_n'(x)|^p (1-x^2)^p dx \le p 2^{2p+1} n^p \ln \frac{2}{\delta} \int_{-1}^{1} |f_n(x)|^p dx.$$

Finally taking the p-th root above yields the required estimate with a constant $c \le 4e^{\frac{2}{e}} \le 9$.

By a standard linear transformation Lemma 1 can be extended to any interval $[a, b] \subset \mathbb{R}$.

Corollary 1. *Let* $1 \le p < \infty, [a, b] \subset \mathbb{R}, 0 < \delta < \frac{b-a}{2}, n \in \mathbb{N}$. *Then for any distinct real numbers* $\lambda_1, ..., \lambda_n \in \mathbb{R}$ *and any exponential sum* $f_n(x) = \sum_{1\le j\le n} c_j e^{\lambda_j x}, \forall c_j \in \mathbb{R}$ *we have*

$$\|(b-x)(x-a)f_n'(x)\|_{L_p[a+\delta,b-\delta]} \le \frac{9(b-a)}{2} \left(\ln \frac{b-a}{\delta}\right)^{\frac{1}{p}} n \|f_n\|_{L_p[a,b]}. \tag{7}$$

An important feature of Lemma 1 consists in the fact that it provides an estimate for the derivatives of exponential sums $\sum_{1\le j\le n} a_j e^{\lambda_j x}, \;\; x, a_j \in \mathbb{R}$ which is independent of the exponents λ_j. In fact the size of the derivatives essentially depends just on n, the number of terms in the exponential sum. What makes this happen is the presence of the weight $(b-x)(x-a)$ in the norm of the derivatives. This weight enforces a linear decline at the endpoints of the interval. A natural question arises in this respect: can an upper bound similar to Lemma 1 hold with a weight $(b-x)(x-a)^{1-\varepsilon}$ for some $\varepsilon > 0$? The next simple example shows that in general, the answer to this question is negative, i.e., linear decline of the weight is crucial here.

Example. Let $n = 1$ and consider the exponent $g(x) = e^{\lambda x}, \lambda > 4$. Clearly with any $0 < \varepsilon < 1$ and $0 < \delta < 1/3$ we have

$$\|x(1-x)^{1-\varepsilon} g'(x)\|^2_{L_2[\delta,1-\delta]} \geq \int\limits_{\frac{1}{2}}^{1-\delta} x^2(1-x)^{2-2\varepsilon}\lambda^2 e^{2\lambda x}dx \geq \frac{\lambda\delta^{2-2\varepsilon}}{8}(e^{2\lambda(1-\delta)} - e^{\lambda}).$$

Thus setting $\delta = \frac{1}{\lambda}$ implies when $\lambda > 4$

$$\|x(1-x)^{1-\varepsilon} g'(x)\|^2_{L_2[\delta,1-\delta]} \geq \frac{\lambda^{2\varepsilon-1}e^{2\lambda}}{8}(e^{-2} - e^{-\lambda}) \geq ce^{2\lambda}\lambda^{2\varepsilon-1}.$$

Since in addition,

$$\|g(x)\|^2_{L_2[0,1]} \leq \frac{e^{2\lambda}}{\lambda}$$

we arrive at

$$\|x(1-x)^{1-\varepsilon} g'(x)\|^2_{L_2[\frac{1}{\lambda},1-\frac{1}{\lambda}]} \geq ce^{2\lambda}\lambda^{2\varepsilon-1} \geq c\lambda^{2\varepsilon}\|g(x)\|^2_{L_2[0,1]}.$$

Note that when $n = 1$ and $\delta = \frac{1}{\lambda}$ we get an upper bound of magnitude $O(\log\lambda)$ in the Bernstein type estimate (6) which means that no matter how small is $\varepsilon > 0$ an upper bound similar to (6) is impossible for the weight $(b-x)(x-a)^{1-\varepsilon}$. Obviously the same conclusion holds for the weight $(b-x)^{1-\varepsilon}(x-a)$, as well.

The next lemma provides an explicit construction of nodes used for the discretization of the L_p norms of the exponential sums. Essentially these nodes are chosen to be equidistributed with respect to the measure

$$\mu_1(E) := \int\limits_E \frac{dx}{x(1-x)}, \quad E \subset (0,1)$$

appearing in the Bernstein type inequality (6).

Lemma 2. *For any $0 < h < 1/2$ and $m \geq 1$ set*

$$x_{j,m} := \frac{1}{1 + \frac{1-h}{h}e^{-(j-1)/m}}, \quad 1 \leq j \leq N = N_m := [2m\ln\frac{1-h}{h}] + 2.$$

Then $1 - h \leq x_{N_m,m} \leq 1 - h/e$ and

$$x_{j+1,m} - x_{j,m} \leq \frac{4x(1-x)}{m}, \;\; x \in (x_{j,m}, x_{j+1,m}), \; 1 \leq j \leq N_m - 1. \tag{8}$$

Proof. Clearly

$$2\ln\frac{1-h}{h} \leq \frac{N_m - 1}{m} \leq 2\ln\frac{1-h}{h} + 1$$

and thus we obtain

$$1-h=\frac{1}{1+\frac{1-h}{h}e^{-2\ln\frac{1-h}{h}}}\le x_{N_m,m}\le\frac{1}{1+\frac{h}{1-h}e^{-1}}\le 1-\frac{h}{e}.$$

Set $h_j:=\frac{1-h}{h}e^{-(j-1)/m}$. Then evidently

$$\frac{x_{j+1,m}-x_{j,m}}{1-x_{j+1,m}}=\frac{h_j-h_{j+1}}{h_{j+1}(1+h_j)}\le\frac{h_j}{h_{j+1}}-1=e^{\frac{1}{m}}-1\le\frac{2}{m}.$$

Hence

$$x_{j+1,m}-x_{j,m}\le\frac{2}{m}(1-x),\ \ x\in(x_{j,m},x_{j+1,m}).$$

Similarly,

$$\frac{x_{j+1,m}-x_{j,m}}{x_{j,m}}=\frac{1+h_j}{1+h_{j+1}}-1\le\frac{h_j}{h_{j+1}}-1=e^{\frac{1}{m}}-1\le\frac{2}{m},$$

yielding that

$$x_{j+1,m}-x_{j,m}\le\frac{2}{m}x_{j,m}\le\frac{2}{m}x,\ \ x\in(x_{j,m},x_{j+1,m}).$$

Thus for any $x\in(x_{j,m},x_{j+1,m})$ we have that

$$x_{j+1,m}-x_{j,m}\le\frac{2}{m}\min\{1-x,x\}\le\frac{4x(1-x)}{m}.\quad\square$$

We will also need below an L_∞ Markov type inequality verified in [7] for multivariate exponential sums on convex bodies. Let us denote by ∇g the gradient of a differentiable function g, $|\nabla g|$ stands for its Euclidian norm which is used in defining $\|\nabla g\|_{L_\infty(K)}:=\sup_{x\in K}|\nabla g(x)|$. In addition, c will denote possibly distinct positive absolute constants.

Lemma 3. *Let $K\subset\mathbb{R}^d, d\ge 1$ be a convex body with r_K being the radius of its largest inscribed ball. Then for every exponential sum $g(\mathbf{w})=\sum_{1\le j\le n}c_je^{\langle\lambda_j,\mathbf{w}\rangle}, \mathbf{w}\in\mathbb{R}^d$ with $\lambda_j\in\mathbb{R}^d$ satisfying $\max_{1\le j\le n}|\lambda_j|\le\Lambda,\ \ |\lambda_{j+1}-\lambda_j|\ge\frac{\epsilon}{r_K}, j\ne k, 0<\epsilon\le 1$*

$$\|\nabla g\|_{L_\infty(K)}\le\frac{cd^3n^3\Lambda}{\epsilon}\|g\|_{L_\infty(K)}.$$

In addition, when $d=1$ the stronger inequality

$$\|g'\|_{L_\infty[a,b]}\le\frac{cn\Lambda}{\epsilon}\|g\|_{L_\infty[a,b]}$$

holds.

A standard consequence of an L_∞ Markov type inequality is a corresponding Nikolskii type upper bound for sup norms via the L_p norms. Namely we can easily deduce from Lemma 3 the next

Corollary 2. *For any convex body $K \subset \mathbb{R}^d, d \geq 1$ and every exponential sum $g(\mathbf{w}) = \sum_{1 \leq j \leq n} c_j e^{\langle \lambda_j, \mathbf{w} \rangle}$, $\mathbf{w} \in \mathbb{R}^d$ with $\lambda_j \in \mathbb{R}^d$ satisfying $|\lambda_{j+1} - \lambda_j| \geq \frac{\epsilon}{r_K}, j \neq k, 0 < \epsilon \leq 1,\quad \max_{1 \leq j \leq n} |\lambda_j| \leq \Lambda$ we have for any $1 \leq p < \infty$ and some constant $c(K,d) > 0$ depending on K, d*

$$\|g\|_{L_\infty(K)} \leq c(K,d) \left(\frac{n^3 \Lambda}{\epsilon} \right)^{\frac{d}{p}} \|g\|_{L_p(K)}. \tag{9}$$

When $d = 1$ we have the stronger estimate

$$\|g\|_{L_\infty[a,b]} \leq c(a,b) \left(\frac{n \Lambda}{\epsilon} \right)^{\frac{1}{p}} \|g\|_{L_p[a,b]}. \tag{10}$$

Proof. First we need to observe that when $K \subset \mathbb{R}^d$ is a convex body then for any d dimensional ball $B_r \subset \mathbb{R}^d$ of radius r centered in K its intersection with K will have Lebesgue measure at least $c(K,d) r^d$ with some fixed constant depending only on K and d. Furthermore, assuming that $\|g\|_{L_\infty(K)} = 1 = g(\mathbf{y}), \mathbf{y} \in K$ we have by Lemma 3 that $g(\mathbf{x}) \geq \frac{1}{2}$ whenever $\mathbf{x} \in K \cap B_r$ with B_r being the ball centered at $\mathbf{y}$ and radius $r := \frac{\epsilon}{2cd^3 n^3 \Lambda}$. Hence

$$\|g\|^p_{L_p(K)} \geq \int\limits_{K \cap B_r} |g|^p \geq 2^{-p} c(K,d) r^d \geq 2^{-p} c(K,d) \left(\frac{\epsilon}{n^3 \Lambda} \right)^d$$

with a proper $c(K,d) > 0$. This clearly implies (9). The second claim follows analogously.

Now we have all the tools needed in order to prove Theorem 1.

Proof of Theorem 1. Clearly, it suffices to verify the theorem when $[a,b] = [0,1]$, the general case will then follow by a standard linear transformation. We will use the elementary estimate

$$\left| \int_a^b g(x) dx - (b-a) g(a) \right| \leq (b-a) \int_a^b |g'(x)| dx$$

applied to the function $g(x) = |f(x)|^p$ on the interval $[x_{j,m}, x_{j+1,m}]$ where $x_{j,m}, 1 \leq j \leq N_m - 1$ is the discrete point set specified in Lemma 2. Then it follows that

$$\left| \int_{x_{j,m}}^{x_{j+1,m}} |f(x)|^p dx - (x_{j+1,m} - x_{j,m}) |f(x_{j,m})|^p \right| \leq (x_{j+1,m} - x_{j,m}) \int_{x_{j,m}}^{x_{j+1,m}} p |f(x)|^{p-1} |f'(x)| dx.$$

Applying estimate (8) of Lemma 2 we have

$$\left| \int_{x_{j,m}}^{x_{j+1,m}} |f(x)|^p dx - (x_{j+1,m} - x_{j,m}) |f(x_{j,m})|^p \right| \leq \frac{4p}{m} \int_{x_{j,m}}^{x_{j+1,m}} x(1-x) |f(x)|^{p-1} |f'(x)| dx, \quad 1 \leq j \leq N_m - 1.$$

Summing up above upper bounds for $1 \leq j \leq N_m - 1$ we arrive at

$$\left| \int_{x_{1,m}}^{x_{N_m}} |f(x)|^p dx - \sum_{1 \leq j \leq N_m - 1} (x_{j+1,m} - x_{j,m}) |f(x_{j,m})|^p \right| \leq \frac{4p}{m} \int_{x_{1,m}}^{x_{N_m}} x(1-x) |f(x)|^{p-1} |f'(x)| dx.$$

Moreover, applying the Hölder inequality to the integral on the right hand side it follows that

$$\left|\int\limits_{x_{1,m}}^{x_{N_m}} |f(x)|^p dx - \sum_{1\le j\le N_m-1} (x_{j+1,m}-x_{j,m})|f(x_{j,m})|^p\right| \le \frac{4p}{m}\|f\|^{p-1}_{L_p[0,1]}\|x(1-x)f'(x)\|_{L_p[x_{1,m},x_{N_m}]}.$$

Thus

$$\left|\|f\|^p_{L_p[0,1]} - \sum_{1\le j\le N_m-1} (x_{j+1,m}-x_{j,m})|f(x_{j,m})|^p\right| \le \frac{4p}{m}\|f\|^{p-1}_{L_p[0,1]}\|x(1-x)f'(x)\|_{L_p[x_{1,m},x_{N_m}]} + \int\limits_0^{x_{1,m}} |f(x)|^p dx + \int\limits_{x_{N_m}}^1 |f(x)|^p dx. \tag{11}$$

Now we need to estimate the three terms on the right hand side of (11).

In order to estimate the first term recall that by Lemma 2 we have $1-h \le x_{N_m,m} \le 1-h/e$ and $x_{1,m} = h$. Therefore

$$\|x(1-x)f'(x)\|_{L_p[x_{1,m},x_{N_m}]} \le \|x(1-x)f'(x)\|_{L_p[h/e,1-h/e]}.$$

Furthermore, applying estimate (7) with $[a,b] = [0,1]$ and $\delta := \frac{h}{e}$ yields

$$\|x(1-x)f'(x)\|_{L_p[h/e,1-h/e]} \le \frac{9}{2}(1-\ln h)^{\frac{1}{p}} n\|f\|_{L_p[0,1]}.$$

Using the last upper bound for the first term on the right hand side of (11) we obtain

$$\frac{4p}{m}\|f\|^{p-1}_{L_p[0,1]}\|x(1-x)f'(x)\|_{L_p[x_{1,m},x_{N_m}]} \le \frac{18p}{m}(1-\ln h)^{\frac{1}{p}} n\|f\|^p_{L_p[0,1]}.$$

For the second and third terms on the right hand side of (11) we can proceed using (10) as follows

$$\int\limits_0^{x_{1,m}} |f(x)|^p dx + \int\limits_{x_{N_m,m}}^1 |f(x)|^p dx \le 2h\|f\|^p_{L_\infty[0,1]} \le c_1 h \left(\frac{n\Lambda}{\epsilon}\right)^{\frac{1}{p}} \|f\|^p_{L_p[0,1]}.$$

Substituting the last two bounds into (11) we arrive at

$$\left|\|f\|^p_{L_p[0,1]} - \sum_{1\le j\le N_m-1} (x_{j+1,m}-x_{j,m})|f(x_{j,m})|^p\right| \le \frac{18p}{m}(1-\ln h)^{\frac{1}{p}} n\|f\|^p_{L_p[0,1]} + c_1 h \left(\frac{n\Lambda}{\epsilon}\right)^{\frac{1}{p}} \|f\|^p_{L_p[0,1]}. \tag{12}$$

Now setting $h := \xi\left(\frac{\epsilon}{n\Lambda}\right)^{\frac{1}{p}}$ and $m := \frac{pn(1-\ln h)^{\frac{1}{p}}}{\xi}$ evidently yields

$$\left|\|f\|^p_{L_p[0,1]} - \sum_{1\le j\le N_m-1} (x_{j+1,m}-x_{j,m})|f(x_{j,m})|^p\right| \le (18+c_1)\xi\|f\|^p_{L_p[0,1]}.$$

Since $0 < \xi < 1$ can be chosen arbitrarily the last estimate obviously implies the required Marcinkiewicz-Zygmund type inequality after proper choice of the constant ξ.

It remains now to check the cardinality $N_m = [2m \ln \frac{1-h}{h}] + 2$ of the discrete point set. Clearly

$$N_m \le cm \ln \frac{1}{h} \le cpn \ln^{\frac{1}{p}+1} \frac{1}{h} \le cpn \ln^{\frac{1}{p}+1} \frac{n\Lambda}{\epsilon}.$$

Recalling that $|\lambda_{j+1} - \lambda_j| \ge \epsilon, j \ne k$ and $\max_{1\le j \le n} |\lambda_j| \le \Lambda$ it follows that

$$(n-1)\epsilon \le 2\Lambda \tag{13}$$

i.e., $\ln \frac{n\Lambda}{\epsilon} \le c \ln \frac{\Lambda}{\epsilon}$. This clearly yields that $N_m \le cpn \ln^{\frac{1}{p}+1} \frac{\Lambda}{\epsilon}$. □

Remark 2. Theorem 1 provides explicit discrete meshes x_j of cardinality $\sim n \ln^{\frac{1}{p}+1} \frac{\Lambda}{\epsilon}$ for the weighted Marcinkiewicz-Zygmund type inequalities with weights $x_{j+1} - x_j$. The nodes x_j are equidistributed with respect to the measure $\mu_1(E) = \int_E \frac{dx}{x(1-x)}$, $\quad E \subset (0,1)$. It should be noted that Theorem 1 remains valid for any sequence of nodes satisfying (8) together with the proper restrictions on the first and last nodes. In addition, the restrictions of the exponents imposed in Theorem 1 were used only for the first and last intervals of the partition. This means that "near" discretization of the norm $\|f\|_{L_p[\delta,1-\delta]}$ is possible using order of $n \log \frac{1}{\delta}$ nodes without the separation and the degree requirements on the exponents.

The arguments given in the proof of Theorem 1 can be used to derive a Marcinkiewicz-Zygmund type result with equal weights 1 for uniformly distributed discrete meshes $x_0 := h, x_j := h + \frac{1-2h}{m} j, 0 \le j \le m$ leading to the next

Proposition 1. *Let $1 \le p < \infty, 0 < \epsilon \le 1, n \in \mathbb{N}, \Lambda > 1$. Then given equidistributed discrete points sets $x_0 := h, x_j := h + \frac{1-2h}{m} j, 0 \le j \le m,\ h = c\left(\frac{\epsilon}{n\Lambda}\right)^{\frac{1}{p}}$ of cardinality*

$$m \le c_1 n \left(\frac{\Lambda}{\epsilon}\right)^{\frac{2}{p}}$$

where $c, c_1 > 0$ are proper absolute constants, we have that

$$\frac{1}{2m} \sum_{1\le j\le m-1} |g(x_j)|^p \le \|g\|^p_{L_p[0,1]} \le \frac{2}{m} \sum_{1\le j \le m-1} |g(x_j)|^p \tag{14}$$

for every exponential sum (2) satisfying $\lambda_{j+1} - \lambda_j \ge \epsilon, 1 \le j \le n-1$ and $\max_{1\le j\le n} |\lambda_j| \le \Lambda$.

Proof. Since the proof of the proposition uses the same technique as Theorem 1 we give a brief outline. The Bernstein type inequality of Lemma 1 can be replaced by estimate (5). Then similarly to (12) we will arrive at

$$\left| \|f\|^p_{L_p[0,1]} - \frac{1}{m} \sum_{0\le j\le m-1} |f(x_{j,m})|^p \right| \le \frac{cpn}{mh} \|f\|^p_{L_p[0,1]} + c_1 h \left(\frac{n\Lambda}{\epsilon}\right)^{\frac{1}{p}} \|f\|^p_{L_p[0,1]}.$$

Now setting $h := \xi\left(\frac{\epsilon}{n\Lambda}\right)^{\frac{1}{p}}$ and $m := \frac{pn}{\xi h}$ with a proper $0 < \xi < 1$ will lead to a Marcinkiewicz-Zygmund type result with equal weights 1 and uniformly distributed mesh of cardinality

$$m \le cn \left(\frac{n\Lambda}{\epsilon}\right)^{\frac{1}{p}} \le cn \left(\frac{\Lambda}{\epsilon}\right)^{\frac{2}{p}}.$$

Note that the last estimate above follows from the inequality (13).

Remark 3. As noted at the end of the proof of Theorem 1 under given assumptions on exponents λ_j we have that $(n-1)\epsilon \le 2\Lambda$. Thus the quantity $\frac{\Lambda}{\epsilon}$ appearing both in Theorem 1 and Proposition 1 ideally is of order n for proper distributions of exponents λ_j. This results in discrete meshes of cardinality $\sim n\ln^{\frac{1}{p}+1} n,\ 1 \le p < \infty$ in Theorem 1. Similarly in Proposition 1 when all weights equal 1 the cardinality will be of order $\sim n^{1+\frac{2}{p}}$ for every $1 \le p < \infty$. In a recent paper [3], Theorem 2.3 the authors verified a weighted Marcinkiewicz-Zygmund type inequality for $L^p, 1 \le p \le 2$ norm with meshes of cardinality $\sim n\ln^3 n$ and arbitrary n-dimensional subspaces of L^∞. This is a rather general beautiful result, but in contrast to Theorem 1 the weights and meshes are not given explicitly. In addition in [3], Theorem 2.2 a Marcinkiewicz-Zygmund type result with weights 1 is verified for $L^p, 1 \le p \le 2$ norms with the cardinality of the discrete mesh being effected by the quantity appearing in a required Nikolskii type inequality. On the other hand it is known that for general exponential sums the magnitude of the factors in a Nikolskii type inequality depends on the exponents λ_j and it can be very large, see [6], Theorem 1. Hence even though Theorem 1 and Corollary 3 are less general in comparison to [3] in terms of subspaces considered for the exponential sums our results provide a compensation in terms of explicit nature of meshes and weights, their universality, smaller cardinality of discrete meshes and their validity for every $1 \le p < \infty$.

We can extend Marcinkiewicz-Zygmund type inequalities for general multivariate exponential sums, as well. Similarly to the univariate case this can be done by applying the Bernstein-Markov type estimates verified earlier. In addition, we will have to overcome a technical difficulty related to the separation condition $|\lambda_{j+1} - \lambda_j| \ge \delta$ which needs to be imposed on the exponents. For this end the next auxiliary proposition, see [7], Lemma 5 will be required.

Lemma 4. *Let $\lambda_j \in \mathbb{R}^d, 1 \le j \le n, d \ge 2$ satisfy the separation condition $|\lambda_j - \lambda_k| \ge \delta > 0, j \ne k$. Then for any $\mathbf{w} \in \mathbb{R}^d, \mathbf{w} \ne \mathbf{0}$ any every $\epsilon > 0$ there exist $\mathbf{u} \in \mathbb{R}^d, |\mathbf{w} - \mathbf{u}| \le \epsilon,\ |\mathbf{w}| = |\mathbf{u}|$ so that with some $c_d > 0$ depending only on d we have*

$$|\langle \lambda_j - \lambda_k, \mathbf{u}\rangle| \ge \frac{c_d \delta \epsilon^{d-1}}{|\mathbf{w}|^{d-2} n^2}, \quad \forall j \ne k. \tag{15}$$

The above lemma shows that the separation condition $|\lambda_j - \lambda_k| \ge \delta > 0, j \ne k$ is preserved in a certain form when the exponents are restricted to small perturbations of arbitrarily chosen lines. Clearly the orthogonal projections of λ_j-s into lines $\{t\mathbf{u} : t \in \mathbb{R}\}, \mathbf{u} \in S^{d-1}$ are given by $\langle \lambda_j, \mathbf{u}\rangle \mathbf{u}$, where these projections are separated by quantities $|\langle \lambda_j - \lambda_k, \mathbf{u}\rangle|$.

Consider the unit cube in $\mathbb{R}^d$ given by $I^d := [0,1]^d$. Using Bernstein-Markov type estimates verified above and separation Lemma 4 we can prove the next Marcinkiewicz-Zygmund type inequality for general multivariate exponential sums on the cube. We shall use below the notation $A \sim B$ in order to indicate that $c_1(d,p)A \le B \le c_2(d,p)A$ with some positive constants $c_1(d,p), c_2(d,p)$ depending only on p, d.

Theorem 2. *Let $1 \le p < \infty, d, n \in \mathbb{N}, \Lambda > 1, 0 < \delta < 1$, and consider any $\lambda_j \in \mathbb{R}^d, 1 \le j \le n$ satisfying*

$$|\lambda_j - \lambda_k| \ge \delta > 0, j \ne k, \quad \max_{1 \le j \le n} |\lambda_j| \le \Lambda.$$

Then there exist positive weights $b_1, ..., b_N$ and discrete point sets $Y_N = \{\mathbf{w}_1, ..., \mathbf{w}_N\} \subset I^d$ of cardinality

$$N \le c(d,p) n^d \ln^{\frac{d}{p}+d} \frac{\Lambda}{\delta},$$

so that for every exponential sum $g(\mathbf{w}) = \sum_{1 \le j \le n} c_j e^{\langle \lambda_j, \mathbf{w}\rangle}, \mathbf{w} \in \mathbb{R}^d$ we have

$$\|g\|_{L_p(I^d)}^p \sim \sum_{1\le i\le N} b_i|g(\mathbf{w}_i)|^p. \tag{16}$$

Proof. The proof of the theorem would be a straightforward product type argument if the separation condition $|\lambda_j - \lambda_k| \ge \delta > 0, j \ne k$ was true for the orthogonal projections of λ_j-s to every coordinate axis in $\mathbb{R}^d$. Indeed, if this was the case then successive integration plus standard induction would accomplish the proof. Thus if $|\langle \lambda_j - \lambda_k, \mathbf{e}_s\rangle| \ge \delta^* > 0, j \ne k$ with some $\delta^* > 0$ and $\mathbf{e}_s := (\delta_{s,i})_{1\le i\le d} \in \mathbb{R}^d, 1 \le s \le d$ being the standard basis in $\mathbb{R}^d$ then for certain positive weights $b_1, ..., b_N$ and a discrete point set $Y_N = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset I^d$ of cardinality $N \le c(d,p)n^d \ln^{\frac{d}{p}+d} \frac{\Lambda}{\delta^*}$

$$\|g\|_{L_p(I^d)}^p \sim \sum_{1\le i\le N} b_i|g(\mathbf{x}_i)|^p, \quad g(\mathbf{x}) = \sum_{1\le j\le n} c_j e^{\langle \lambda_j, \mathbf{x}\rangle}. \tag{17}$$

Now we will apply Lemma 4 to certain properly chosen vectors. For any $\epsilon < \frac{1}{4d}$ set

$$\mathbf{e}_s^* := (1-\epsilon)\mathbf{e}_s + \epsilon \sum_{k\ne s} \mathbf{e}_k, \ \ \frac{1}{2} < |\mathbf{e}_s^*| < 2, \ \ 1 \le s \le d.$$

Note that each coordinate of $\mathbf{e}_s^*$ is not smaller than ϵ. By Lemma 4 for every $1 \le s \le d$ there exist $\mathbf{u}_s \in \mathbb{R}^d, |\mathbf{e}_s^* - \mathbf{u}_s| \le \epsilon, |\mathbf{e}_s^*| = |\mathbf{u}_s|$ so that

$$|\langle \lambda_j - \lambda_k, \mathbf{u}_s\rangle| \ge \frac{c_d \delta \epsilon^{d-1}}{n^2}, \ \ \forall j \ne k. \tag{18}$$

Since all coordinates of $\mathbf{e}_s^*$ are $\ge \epsilon$ the condition $|\mathbf{e}_s^* - \mathbf{u}_s| \le \epsilon$ implies that

$$\mathbf{u}_s \in \mathbb{R}_+^d := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : x_j \ge 0, |\mathbf{x}| = 1\}.$$

Consider now the parallelepiped in $\mathbb{R}^d$ given by

$$J^d := \{t_1\mathbf{u}_1 + ... + t_d\mathbf{u}_d, \ 0 \le t_s \le \frac{1}{1+2d\epsilon}, \ 1 \le s \le d\}.$$

Since $\mathbf{u}_s \in \mathbb{R}_+^d$ and $|\mathbf{e}_s^* - \mathbf{u}_s| \le \epsilon, 1 \le s \le d$ it is easy to check that $J^d \subset I^d$.

Denote by $T : \mathbb{R}^d \to \mathbb{R}^d$ the regular matrix transformation defined by $T(\mathbf{e}_s) = \frac{\mathbf{u}_s}{1+2d\epsilon}, 1 \le s \le d$. Then evidently, $T(I^d) = J^d$. Furthermore, note that

$$\left|\frac{\mathbf{u}_s}{1+2d\epsilon} - \mathbf{e}_s\right| \le \left|\frac{\mathbf{e}_s^*}{1+2d\epsilon} - \mathbf{e}_s\right| + \epsilon \le \frac{|\mathbf{e}_s^* - \mathbf{e}_s|}{1+2d\epsilon} + (2d+1)\epsilon \le (3d+1)\epsilon, \ \ 1 \le s \le d.$$

Evidently, this means that for every $\mathbf{x} = \sum_{1\le s\le d} t_s\mathbf{e}_s \in I^d, 0 \le t_s \le 1$ we have

$$|\mathbf{x} - T(\mathbf{x})| = \left|\sum_{1\le s\le d} t_s\mathbf{e}_s - \sum_{1\le s\le d} t_s\frac{\mathbf{u}_s}{1+2d\epsilon}\right| = \left|\sum_{1\le s\le d} t_s\left(\mathbf{e}_s - \frac{\mathbf{u}_s}{1+2d\epsilon}\right)\right| \le d(3d+1)\epsilon.$$

Hence for any $\mathbf{x} \in I^d \setminus J^d$ we have that $T(\mathbf{x}) \in J^d$ and $|\mathbf{x} - T(\mathbf{x})| \le d(3d+1)\epsilon$. Obviously this means that I^d is contained in a $1 + d(3d+1)\epsilon$ dilation of J^d about the origin which in turn implies that $\mu(I^d \setminus J^d) \le c_d\epsilon$ where μ stands for the Lebesgue measure.

Consider now any exponential sum $g(\mathbf{w}) = \sum_{1\le j\le n} c_j e^{\langle \lambda_j, \mathbf{w}\rangle}, \mathbf{w} \in \mathbb{R}^d$ with $|\lambda_j - \lambda_k| \ge \delta > 0, j \ne k$. Since $J^d \subset I^d$ and $\mu(I^d \setminus J^d) \le c_d\epsilon$ we have by (9)

$$\left|\int_{I^d}|g|^p-\int_{J^d}|g|^p\right|\le c_d\epsilon\|g\|^p_{L_\infty(I^d)}\le c(p,d)\epsilon\left(\frac{n^3\Lambda}{\delta}\right)^d\int_{I^d}|g|^p.$$

Thus setting $\epsilon:=\frac{1}{2c(p,d)}\left(\frac{\delta}{n^3\Lambda}\right)^d$ in the above upper bound we obtain that

$$\frac{1}{2}\int_{I^d}|g|^p\le\int_{J^d}|g|^p\le\frac{3}{2}\int_{I^d}|g|^p. \tag{19}$$

Moreover denoting by T^* the transpose of T and substituting $\mathbf{w}=T(\mathbf{x}),\mathbf{x}\in I^d$ we have

$$\int_{J^d}|g(\mathbf{w})|^pd\mathbf{w}=|\mathrm{det}T|\int_{I^d}|\sum_{1\le j\le n}c_je^{\langle\lambda_j,T(\mathbf{x})\rangle}|^pd\mathbf{x}=|\mathrm{det}T|\int_{I^d}|\sum_{1\le j\le n}c_je^{\langle T^*(\lambda_j),\mathbf{x}\rangle}|^pd\mathbf{x}, \tag{20}$$

where by (18) we have

$$|\langle T^*(\lambda_j-\lambda_k),\mathbf{e}_s\rangle|=|\langle\lambda_j-\lambda_k,\frac{\mathbf{u}_s}{1+2d\epsilon}\rangle|\ge\frac{c_d\delta\epsilon^{d-1}}{n^2},\ \ \forall j\ne k.$$

The last relation means that exponents $T^*(\lambda_j)$ satisfy the coordinate wise separation condition needed for (17) with

$$\delta^*:=\frac{c_d\delta\epsilon^{d-1}}{n^2}\ge c(p,d)\left(\frac{\delta}{n^3\Lambda}\right)^{d^2}.$$

In addition,

$$\max_{1\le j\le n}|T^*(\lambda_j)|\le c(d)\max_{1\le j\le n}|\lambda_j|\le c(d)\Lambda.$$

Hence setting $\mathbf{w}_i:=T(\mathbf{x}_i),1\le i\le N$ we have by (17)

$$\int_{I^d}|\sum_{1\le j\le n}c_je^{\langle T^*(\lambda_j),\mathbf{x}\rangle}|^pd\mathbf{x}\sim\sum_{1\le i\le N}b_i|g(\mathbf{w}_i)|^p \tag{21}$$

with

$$N\le c(d)n^d\ln^{\frac{d}{p}+d}\frac{\Lambda}{\delta^*}\le c(p,d)n^d\ln^{\frac{d}{p}+d}\frac{n\Lambda}{\delta}.$$

Recall that $|\lambda_{j+1}-\lambda_j|\ge\delta,j\ne k$ and $\max_{1\le j\le n}|\lambda_j|\le\Lambda$. Hence all open balls of radius δ centered at λ_j-s are disjoint while all of them are contained in a ball of radius $\Lambda+\delta$ yielding that $n\le c(d)\left(\frac{\Lambda}{\delta}\right)^d$. Using this upper bound in the last estimate for cardinality N provides the required upper bound $N\le c(d,p)n^d\ln^{\frac{d}{p}+d}\frac{\Lambda}{\delta}$. Finally, combining relations (19)-(21) and using that $\mathrm{det}T\sim 1$ we have

$$\int_{I^d}|g|^p\sim\int_{J^d}|g|^p\sim\int_{I^d}|\sum_{1\le j\le n}c_je^{\langle T^*(\lambda_j),\mathbf{x}\rangle}|^pd\mathbf{x}\sim\sum_{1\le i\le N}b_i|g(\mathbf{w}_i)|^p.\quad\square$$

3. Marcinkiewicz-Zygmund type inequalities for exponential sums with nonnegative coefficients

We can get a substantial improvement of the Marcinkiewicz-Zygmund type inequalities proved in the previous section when exponential sums $g(\mathbf{x}) = \sum_{1\leq j\leq n} a_j e^{\langle\lambda_j,\mathbf{x}\rangle}$, $\mathbf{x}\in\mathbb{R}^d$ have *nonnegative* coefficients, that is $a_j \geq 0, 1\leq j\leq n$. This is due to the fact that we are able to verify a much stronger upper bound in the Bernstein type inequality for univariate exponential sums with nonnegative coefficients which are independent of n and λ_j-s. Similarly to Lemma 1 this will be accomplished below by measuring the L_p norm of the derivative of exponential sums with weight $1-x^2$. In addition, the L_p norm considered is also endowed with the Jacobi type weight $\phi(x) := (1-x)^{\alpha/p}(1+x)^{\beta/p}, \alpha,\beta > 0$. This Jacobi type weight will make it possible to extend the univariate results to multivariate convex polytopes.

Lemma 5. *Let $p\in\mathbb{N}$, $\phi(x) := (1-x)^{\alpha/p}(1+x)^{\beta/p}, 0<\alpha<\beta$. Then for any $n\in\mathbb{N}$ and any distinct real numbers $\lambda_j\in\mathbb{R}, 1\leq j\leq n$ and arbitrary exponential sum $f_n(x) = \sum_{1\leq j\leq n} a_j e^{\lambda_j x}, a_j\geq 0$ with nonnegative coefficients we have*

$$\|(1-x^2)\phi(x)f_n'(x)\|_{L_p[-1,1]} \leq 4(1+\beta/p)\|\phi(x)f_n(x)\|_{L_p[-1,1]}. \tag{22}$$

Proof. Using notation $\psi(x) := \phi^p(x) = (1-x)^\alpha(1+x)^\beta$ it is easy to see that

$$\|\phi(x)f_n\|^p_{L_p[-1,1]} = \int_{-1}^{1}\psi(x)(\sum_{1\leq j\leq n} a_j e^{\lambda_j x})^p dx = \sum_{1\leq s_1\leq n}\ldots\sum_{1\leq s_p\leq n}\prod_{k=1}^{p} a_{s_k}\int_{-1}^{1}\psi(x)e^{(\lambda_{s_1}+\ldots+\lambda_{s_p})x}dx; \tag{23}$$

$$\|(1-x^2)\phi(x)f_n'(x)\|^p_{L_p[-1,1]} \leq \int_{-1}^{1}(1-x^2)^p\psi(x)(\sum_{1\leq j\leq n} a_j|\lambda_j|e^{\lambda_j x})^p dx =$$
$$\sum_{1\leq s_1\leq n}\ldots\sum_{1\leq s_p\leq n}\prod_{k=1}^{p} a_{s_k}|\lambda_{s_k}|\int_{-1}^{1}(1-x^2)^p\psi(x)e^{(\lambda_{s_1}+\ldots+\lambda_{s_p})x}dx. \tag{24}$$

Evidently, integrating by parts p times yields

$$\int_{-1}^{1}(1-x^2)^p\psi(x)e^{(\lambda_{s_1}+\ldots+\lambda_{s_p})x}dx = \frac{(-1)^p}{(\lambda_{s_1}+\ldots+\lambda_{s_p})^p}\int_{-1}^{1}G_p^{\alpha,\beta}(x)e^{(\lambda_{s_1}+\ldots+\lambda_{s_p})x}dx$$

where $G_p^{\alpha,\beta}(x) := \frac{d^p}{dx^p}(1-x)^{p+\alpha}(1+x)^{p+\beta}$ is a scalar multiple of the classical Jacobi polynomial $J_p^{\alpha,\beta}(x)$ of degree p. Namely using relations given in [11], pp. 67-68 we have that

$$G_p^{\alpha,\beta}(x) = (-1)^p 2^p p!(1-x)^\alpha(1+x)^\beta J_p^{\alpha,\beta}(x) =$$
$$(-1)^p 2^p p!(1-x)^\alpha(1+x)^\beta\sum_{0\leq k\leq p}\binom{p+\alpha}{p-k}\binom{p+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \quad x\in[-1,1].$$

Hence we clearly get that

$$|G_p^{\alpha,\beta}(x)| \leq 2^p p!(1-x)^\alpha(1+x)^\beta\sum_{0\leq k\leq p}\frac{(p+\alpha)^{p-k}}{(p-k)!}\frac{(p+\beta)^k}{k!} \leq 4^p(p+\beta)^p\psi(x), \quad x\in[-1,1].$$

Thus applying this upper bound in the above integral and using also the arithmetic geometric mean inequality we obtain

$$\prod_{k=1}^{p}|\lambda_{s_k}|\int_{-1}^{1}(1-x^2)^p\psi(x)e^{(\lambda_{s_1}+...+\lambda_{s_p})x}dx=\left|\frac{\lambda_{s_1}...\lambda_{s_p}}{(\lambda_{s_1}+...+\lambda_{s_p})^p}\int_{-1}^{1}G_p^{\alpha,\beta}(x)e^{(\lambda_{s_1}+...+\lambda_{s_p})x}dx\right|$$

$$\leq 4^p(p+\beta)^p p^{-p}\int_{-1}^{1}\psi(x)e^{(\lambda_{s_1}+...+\lambda_{s_p})x}dx\leq(4+4\beta/p)^p\int_{-1}^{1}\psi(x)e^{(\lambda_{s_1}+...+\lambda_{s_p})x}dx.$$

Combining the last estimate with relations (24) and (23) clearly yields

$$\|(1-x^2)\phi(x)f_n'(x)\|^p_{L_p[-1,1]}\leq(4+4\beta/p)^p\|\phi(x)f_n(x)\|^p_{L_p[-1,1]}.$$

Finally, taking the p-th root above completes the proof. □

Note that for a general interval $[a,b]$ estimate (22) appears in the form

$$\|(b-x)(x-a)\chi(x)f_n'(x)\|_{L_p[a,b]}\leq 2(b-a)(1+\beta/p)\|\chi(x)f_n\|_{L_p[a,b]}, \tag{25}$$

where $\chi(x):=(2/(b-a))^{(\alpha+\beta)/p}(b-x)^{\alpha/p}(x-a)^{\beta/p},\alpha,\beta>0$.

Now we can use Lemma 5 and the method applied in the proof of Theorem 1 to exponential sums $g(x)=\sum_{1\leq j\leq n}a_je^{\lambda_jx},\ a_j\geq 0$ with nonnegative coefficients in order to derive a corresponding Marcinkiewicz-Zygmund type inequality. Note that we have $|g'(x)|\leq\lambda_n^*g(x),x\in\mathbb{R}$ for any such exponential sum.

Theorem 3. *Let $p\in\mathbb{N},\Lambda>1$. Set $w(x):=(1-x)^\alpha x^\beta,0<\alpha<\beta$. There exist discrete points sets $Y_N=\{x_1,...,x_N\}\subset[0,1]$ of cardinality*

$$N\leq c(p+\beta)(\ln p+\ln\Lambda+\alpha+1)$$

so that for any distinct real numbers $\lambda_j\in\mathbb{R},1\leq j\leq n$ with $\max_{1\leq j\leq n}|\lambda_j|\leq\Lambda$ and for every exponential sum $f_n(x)=\sum_{1\leq j\leq n}a_je^{\lambda_jx},a_j\geq 0$ with nonnegative coefficients we have

$$\frac{1}{2}\sum_{1\leq j\leq N}(x_{j+1}-x_j)w(x_j)f_n^p(x_j)\leq\int_0^1 w(x)f_n^p(x)dx\leq 2\sum_{1\leq j\leq N}(x_{j+1}-x_j)w(x_j)f_n^p(x_j). \tag{26}$$

Proof. Similarly to the proof of Theorem 1 we use the discrete point set of Lemma 2 to derive

$$\left|\int_0^1 w(x)f_n^p(x)dx-\sum_{1\leq j\leq N_m-1}(x_{j+1,m}-x_{j,m})w(x_j)f_n^p(x_{j,m})\right|$$

$$\leq\frac{4}{m}\int_{x_{1,m}}^{x_{N_m}}x(1-x)\,|(w(x)f_n^p(x))'|\,dx+\int_0^{x_{1,m}}w(x)f_n^p(x)dx+\int_{x_{N_m}}^1 w(x)f_n^p(x)dx\leq$$

$$\frac{4}{m}\int_{x_{1,m}}^{x_{N_m}}x(1-x)|w'|f_n^pdx+\frac{4p}{m}\int_{x_{1,m}}^{x_{N_m}}x(1-x)wf_n^{p-1}|f_n'|dx+\int_0^{x_{1,m}}wf_n^pdx+\int_{x_{N_m}}^1 wf_n^pdx \tag{27}$$

Now we need to estimate the four terms on the right hand side of (27). Using that $x(1-x)|w'|\leq\beta w,x\in[0,1]$ we obtain for the first integral

$$\frac{4}{m}\int_{x_{1,m}}^{x_{N_m}} x(1-x)|w'|f_n^p dx \le \frac{4\beta}{m}\int_{x_{1,m}}^{x_{N_m}} w(x)f_n^p(x)dx \le \frac{4\beta}{m}\int_0^1 w(x)f_n^p(x)dx.$$

For the second term on the right hand side of (27) we will need the Bernstein type inequality (25) transformed to the interval $[0,1]$ which gives with $w(x) := (1-x)^\alpha x^\beta$

$$\int_0^1 w(x)(x(1-x)|f_n'(x)|)^p dx \le (2+2\beta/p)^p \int_0^1 w(x)f_n^p(x)dx.$$

Thus using the Hölder inequality and last upper bound we obtain for the second term on the right hand side of (27)

$$\frac{4p}{m}\int_{x_{1,m}}^{x_{N_m}} x(1-x)wf_n^{p-1}|f_n'|dx \le \frac{4p}{m}\left(\int_0^1 w(x)f_n^p(x)dx\right)^{\frac{p-1}{p}}\left(\int_0^1 w(x)(x(1-x)|f_n'(x)|)^p dx\right)^{\frac{1}{p}}$$

$$\le \frac{8(p+\beta)}{m}\int_0^1 w(x)f_n^p(x)dx.$$

The estimate of the third and fourth terms in (27) is analogous and it uses relations $x_{1,m} = h, 1-h \le x_{N_m}$ (see Lemma 2) and the following upper bound

$$|(f_n^p(x))'| \le p\lambda_n^* f_n^p(x) \le p\Lambda f_n^p(x), \quad x \in \mathbb{R}. \tag{28}$$

Furthermore, with arbitrary $h < \frac{1}{2p\Lambda} < \frac{1}{2}$ set $\gamma := \|f_n^p\|_{L_\infty[h,1/2]} = f_n^p(y), \mu := \|f_n^p\|_{L_\infty[0,1/2]} = f_n^p(z)$. Then using (28) we clearly have

$$\int_h^{\frac{1}{2}} f_n^p(x)dx \ge \int_{|x-y|\le\frac{1}{2p\Lambda}} f_n^p(x)dx \ge \frac{\gamma}{4p\Lambda}.$$

In addition, let us show that $\mu \le 2\gamma$. Indeed, assuming that $\mu > \gamma$ it follows that $z \in [0,h]$ and hence applying again (28)

$$\mu - \gamma \le f_n^p(z) - f_n^p(h) \le hp\Lambda\mu \le \frac{\mu}{2}, \quad \text{i.e. } \mu \le 2\gamma.$$

Using the last two upper bounds we obtain for the third term on the right hand side of (27)

$$\int_0^{x_{1,m}} wf_n^p dx = \int_0^h (1-x)^\alpha x^\beta f_n^p dx \le h^{\beta+1}\mu \le 2h^{\beta+1}\gamma \le 8p\Lambda h^{\beta+1}\int_h^{\frac{1}{2}} f_n^p(x)dx \le 8p2^\alpha\Lambda h\int_h^{\frac{1}{2}} w(x)f_n^p(x)dx.$$

Since the same estimate holds for the fourth term on the right hand side of (27) we can now collect all above upper bounds yielding that the right hand side of (27) can be estimated from above by

$$\left(\frac{8p+12\beta}{m} + 16p2^\alpha\Lambda h\right)\int_0^1 w(x)f_n^p(x)dx$$

Finally, choosing $h := \frac{1}{64p2^\alpha\Lambda}$, $m := 32p + 48\beta$ yields a constant $\frac{1}{2}$ in front of the integral above. Evidently this implies the required Marcinkiewicz-Zygmund type inequality. It remains to note that with this choice of parameters h, m we have that

$$N_m := [2m \ln \frac{1-h}{h}] + 2 \le c(p+\beta) \ln \frac{1}{h} + 2 \le c(p+\beta)(\ln p + \ln \Lambda + \alpha + 1). \quad \square$$

It is remarkable, that above theorem provides a dimension and exponent independent L_p Marcinkiewicz-Zygmund type inequality for exponential sums with nonnegative coefficients in case when $p \in \mathbb{N}$ is an integer. A slight modification leads to a similar result in case of any $p \ge 1$.

Corollary 3. *Let $p \ge 1, n \in \mathbb{N}$. There exist discrete points sets $Y_N = \{x_1, ..., x_N\} \subset [0,1]$ of cardinality*

$$N \le c(p+\beta)(\ln p + \ln \Lambda + \alpha)$$

so that for any distinct real numbers $\lambda_j \in \mathbb{R}, 1 \le j \le n$ with $\max_{1\le j\le n} |\lambda_j| \le \Lambda, \Lambda > 1$ and for every exponential sum $f_n(x) = \sum_{1\le j\le n} a_j e^{\lambda_j x}, a_j \ge 0$ with nonnegative coefficients we have

$$\frac{1}{2} \sum_{1\le j\le N} (x_{j+1} - x_j)|f_n(x_j)|^{[p]+1} \le \|f_n\|^{[p]+1}_{L_p[0,1]} \le 2 \sum_{1\le j\le N} (x_{j+1} - x_j)|f_n(x_j)|^{[p]+1}. \tag{29}$$

Proof. Clearly for every $p \ge 1$

$$\|f_n\|_{L_{[p]}[0,1]} \le \|f_n\|_{L_p[0,1]} \le \|f_n\|_{L_{[p]+1}[0,1]}.$$

Now using (26) with $[p]+1$ and $[p]$ and $w = 1$ we can estimate $\|f_n\|^{[p]+1}_{L_{[p]+1}[0,1]}$ and from above, and $\|f_n\|^{[p]}_{L_{[p]}[0,1]}$ from below by $\sum_{1\le j\le N}(x_{j+1} - x_j)|f_n(x_j)|^{[p]+1}$ and by $\sum_{1\le j\le N}(x_{j+1} - x_j)|f_n(x_j)|^{[p]}$, respectively. It should be noted that as in the proof of Theorem 3 in both upper and lower bounds we can use the same discrete point set $x_j, 1 \le j \le N$ given by Lemma 2 as long as $h < \frac{1}{64([p]+1)\Lambda}$ and $m > 32([p] + 1)$. Then relations (29) easily follow since the l_p norm is monotone decreasing in p. With the above choice of parameters h, m we have again the same bound on the cardinality of the discrete mesh. $\square$

The weighted univariate result given by Theorem 3 leads to an extension of Marcinkiewicz-Zygmund type inequality for multivariate exponential sums with nonnegative coefficients to convex polytopes. Namely we have the next

Theorem 4. *Let $d, p \in \mathbb{N}, \Lambda > 1$. Consider any convex polytope $K \subset \mathbb{R}^d$. There exist discrete points sets $Y_N = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset K$ of cardinality*

$$N \le c(K, d)(p+d)^d(\ln p + \ln(\Lambda + 1))^d$$

and positive weights $b_1, ..., b_N$ so that for any distinct $\lambda_j \in \mathbb{R}^d, 1 \le j \le n$ with $\max_{1\le j\le n} |\lambda_j| \le \Lambda$ and for every exponential sum $g(\mathbf{x}) = \sum_{1\le j\le n} c_j e^{\langle\lambda_j, \mathbf{x}\rangle}, \mathbf{x} \in \mathbb{R}^d, c_j \ge 0$ with nonnegative coefficients we have

$$\|g\|^p_{L_p(K)} \sim \sum_{1\le i\le N} b_i g(\mathbf{x}_i)^p. \tag{30}$$

Proof. Clearly if $\max_{1\le j\le n} |\lambda_j| \le \Lambda$ then for any affine mapping T of $\mathbb{R}^d$ we have $\max_{1\le j\le n} |T(\lambda_j)| \le c\Lambda$ with some $c > 1$. In addition, affine maps preserve the property of the exponential sum to have nonnegative coefficients. Evidently, replacing Λ by $c\Lambda$ in the estimate for cardinality of the mesh in the above theorem will only alter the size of $c(K,d)$. Therefore since every convex polytope $K \subset \mathbb{R}^d$ decomposes into a finite union of simplices it suffices to verify the theorem for any of the standard simplices

$$\Delta^d := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : x_j \geq 0, 1 \leq j \leq d, x_1 + ... + x_{d-1} \leq x_d \leq 1\},$$
$$\Omega^d := \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : x_j \geq 0, 1 \leq j \leq d, x_1 + ... + x_d \leq 1\}.$$

We will verify the theorem for the simplex by induction on the dimension d. The case $d = 1$ is covered by Theorem 3. Let us assume now that the statement of Theorem 4 holds for the $d-1$ dimensional simplex Δ^{d-1} (or Ω^{d-1}). Now for given $g(\mathbf{x}) = \sum_{1\leq j\leq n} c_j e^{\langle \lambda_j, \mathbf{x}\rangle}, \mathbf{x} \in \mathbb{R}^d, c_j \geq 0$ consider the integral

$$\|g\|^p_{L_p(\Delta^d)} = \int\limits_{\Delta^d} g(\mathbf{x})^p d\mathbf{x} = \int\limits_0^1 \int\limits_{x_1+...+x_{d-1}\leq x_d} g(x_1, ..., x_d)^p dx_1...dx_{d-1}dx_d.$$

For any a fixed $x_d \in [0,1]$ the change of the variable $\mathbf{x} = x_d(\mathbf{y}+\mathbf{e}_d), \mathbf{y} = (y_1, ..., y_{d-1}, 0), \mathbf{e}_d := (0, ..., 0, 1) \in \mathbb{R}^d$ establishes a bijection between $\mathbf{x} \in \Delta^d$ and $\mathbf{y} \in \Omega^{d-1}$ where the Jacobian of this transformation has its determinant equal to x_d^{d-1}. So performing this transformation for the above integral yields

$$\int\limits_{\Delta^d} g(\mathbf{x})^p d\mathbf{x} = \int\limits_0^1 \int\limits_{x_1+...+x_{d-1}\leq x_d} g(x_1, ..., x_d)^p dx_1...dx_{d-1}dx_d = \int\limits_0^1 x_d^{d-1} \int\limits_{\Omega^{d-1}} g(x_d(\mathbf{y}+\mathbf{e}_d))^p d\mathbf{y}dx_d. \quad (31)$$

Now using the induction hypothesis for the integral on the $d-1$ dimensional simplex Ω^{d-1} it follows that there exist discrete points sets $Y_m = \{\mathbf{y}_1, ..., \mathbf{y}_m\} \subset \Omega^{d-1}$ of cardinality

$$m \leq (c(p+d-1)(\ln p + \ln \Lambda + 1))^{d-1} \quad (32)$$

and positive weights $a_1, ..., a_m$ so that for every fixed $x_d \in [0,1]$ we have for the exponential sum with nonnegative coefficients $g(x_d(\mathbf{y}+\mathbf{e}_d))$ of variable $\mathbf{y} \in \Omega^{d-1}$

$$\int\limits_{\Omega^{d-1}} g(x_d(\mathbf{y}+\mathbf{e}_d))^p d\mathbf{y} \sim \sum_{1\leq i\leq m} a_i g(x_d(\mathbf{y}_i+\mathbf{e}_d))^p.$$

Substituting this relation into (31) we obtain

$$\int\limits_{\Delta^d} g(\mathbf{x})^p d\mathbf{x} \sim \sum_{1\leq i\leq m} a_i \int\limits_0^1 x_d^{d-1} g(x_d(\mathbf{y}_i+\mathbf{e}_d))^p dx_d. \quad (33)$$

Now note that for every $1 \leq i \leq m$ the function $g(x_d(\mathbf{y}_i+\mathbf{e}_d))$ is a univariate exponential sum with nonnegative coefficients of variable $x_d \in [0,1]$ with the maximal size of exponents not exceeding $c\Lambda$. So we can apply to $g(x_d(\mathbf{y}_i+\mathbf{e}_d)), 1 \leq i \leq m$ Theorem 3 with weight x_d^{d-1} that is $\alpha = 0, \beta = d-1$. Hence there exist discrete meshes $Y_{N_i} = \{x_{1,i}, ..., x_{N_i,i}\} \subset [0,1]$ of cardinality

$$N_i \leq c(p+d-1)(\ln p + \ln \Lambda + 1), \quad 1 \leq i \leq m \quad (34)$$

so that

$$\int\limits_0^1 x_d^{d-1} g(x_d(\mathbf{y}_i+\mathbf{e}_d))^p dx_d \sim \sum_{1\leq j\leq N_i} (x_{j+1,i} - x_{j,i}) x_{j,i}^{d-1} g(x_{j,i}(\mathbf{y}_i+\mathbf{e}_d))^p, \quad 1 \leq i \leq m.$$

Thus using the last relation together with (33) we arrive at

$$\int_{\Delta^d} g(\mathbf{x})^p d\mathbf{x} \sim \sum_{1\le i\le m} \sum_{1\le j\le N_i} a_i(x_{j+1,i} - x_{j,i}) x_{j,i}^{d-1} g(x_{j,i}(\mathbf{y}_i + \mathbf{e}_d))^p.$$

Clearly this last relation provides a needed Marcinkiewicz-Zygmund type inequality with positive weights $a_i(x_{j+1,i} - x_{j,i})x_{j,i}^{d-1}$ and discrete mesh $x_{j,i}(\mathbf{y}_i + \mathbf{e}_d), 1 \le j \le N_i, 1 \le i \le m$. Finally, by (32) and (34) the cardinality $N := \sum_{1\le i\le m} N_i$ of this point set satisfies upper bound

$$N \le m \max_{1\le i\le m} N_i \le (c(p+d-1)(\ln p + \ln\Lambda))^{d-1}(p+d-1)(\ln p + \ln \Lambda) \le (c(p+d)(\ln p + \ln\Lambda))^d. \quad \square$$

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