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Vieta–Fibonacci-like polynomials and some identities*

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Abstract

In this paper, we introduce a new type of the Vieta polynomial, which is Vieta–Fibonacci-like polynomial. After that, we establish the Binet formula, the generating function, the well-known identities, and the sum formula of this polynomial. Finally, we present the relationship between this polynomial and the previous well-known Vieta polynomials.

Keywords: Vieta–Fibonacci polynomial, Vieta–Lucas polynomial, Vieta–Fibonacci-like polynomial

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1. Introduction

In 2002, Horadam [1] introduced the new types of second order recursive sequences of polynomials which are called Vieta–Fibonacci and Vieta–Lucas polynomials respectively. The definition of Vieta–Fibonacci and Vieta–Lucas polynomials are defined as follows:

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Definition 1.1 ([1]). For any natural number n the Vieta–Fibonacci polynomials sequence $\{V_n(x)\}_{n=0}^{\infty}$ and the Vieta–Lucas polynomials sequence $\{v_n(x)\}_{n=0}^{\infty}$ are defined by

$$\begin{aligned} V_n(x) &= xV_{n-1}(x) - V_{n-2}(x), \quad \text{for } n \geq 2, \\ v_n(x) &= xv_{n-1}(x) - v_{n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

respectively, where $V_0(x) = 0$, $V_1(x) = 1$ and $v_0(x) = 2$, $v_1(x) = x$.

The first few terms of the Vieta–Fibonacci polynomials sequence are $0, 1, x, x^2 - 1, x^3 - 2x, x^4 - 3x^2 + 1$ and the first few terms of the Vieta–Lucas polynomials sequence are $2, x, x^2 - 2, x^3 - 3x, x^4 - 4x^2 + 2, x^5 - 5x^3 + 5x$. The Binet formulas of the Vieta–Fibonacci and Vieta–Lucas polynomials are given by

$$\begin{aligned} V_n(x) &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \\ v_n(x) &= \alpha^n(x) + \beta^n(x), \end{aligned}$$

respectively. Where $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$ are the roots the characteristic equation $r^2 - xr + 1 = 0$. We also note that $\alpha(x) + \beta(x) = x$, $\alpha(x)\beta(x) = 1$, and $\alpha(x) - \beta(x) = \sqrt{x^2 - 4}$.

Recall that the Chebyshev polynomials are a sequence of orthogonal polynomials which can be defined recursively. The n^{th} Chebyshev polynomials of the first and second kinds are denoted by $\{T_n(x)\}_{n=0}^{\infty}$ and $\{U_n(x)\}_{n=0}^{\infty}$ and are defined respectively by $T_0(x) = 1$, $T_1(x) = x, T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, for $n \geq 2$, and $U_0(x) = 1$, $U_1(x) = 2x, U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, for $n \geq 2$. These polynomials are of great importance in many areas of mathematics, particularly approximation theory. It is well known that the Chebyshev polynomials of the first kind and second kind are closely related to Vieta–Fibonacci and Vieta–Lucas polynomials. So, in [4] Vitula and Slota redefined Vieta polynomials as modified Chebyshev polynomials. The related features of Vieta and Chebyshev polynomials are given as $V_n(x) = U_n\left(\frac{1}{2}x\right)$ and $v_n(x) = 2T_n\left(\frac{1}{2}x\right)$ (see [1, 2, 5]).

In 2013, Tasci and Yalcin [6] introduced the recurrence relation of Vieta–Pell and Vieta–Pell–Lucas polynomials as follows:

Definition 1.2 ([6]). For $|x| > 1$ and for any natural number n the Vieta–Pell polynomials sequence $\{t_n(x)\}_{n=0}^{\infty}$ and the Vieta–Pell–Lucas polynomials sequence $\{s_n(x)\}_{n=0}^{\infty}$ are defined by

$$\begin{aligned} t_n(x) &= 2xt_{n-1}(x) - t_{n-2}(x), \quad \text{for } n \geq 2, \\ s_n(x) &= 2xs_{n-1}(x) - s_{n-2}(x), \quad \text{for } n \geq 2. \end{aligned}$$

respectively, where $t_0(x) = 0$, $t_1(x) = 1$ and $s_0(x) = 2$, $s_1(x) = 2x$.

The $t_n(x)$ and $s_n(x)$ are called the n^{th} Vieta–Pell polynomial and the n^{th} Vieta–Pell–Lucas polynomial respectively. Tasci and Yalcin [6] obtained the Binet form and generating functions of Vieta–Pell and Vieta–Pell–Lucas polynomials. Also, they obtained some differentiation rules and the finite summation formulas. Moreover, the following relations are obtained

$$s_n(x) = 2T_n(x), \quad \text{and} \quad t_{n+1}(x) = U_n(x).$$

In 2015, Yalcin et al. [8], introduced and studied the Vieta–Jacobsthal and Vieta–Jacobsthal–Lucas polynomials which defined as follows:

Definition 1.3 ([8])**.** For any natural number n the Vieta–Jacobsthal polynomials sequence $\{G_n(x)\}_{n=0}^{\infty}$ and the Vieta–Jacobsthal-Lucas polynomials sequence $\{g_n(x)\}_{n=0}^{\infty}$ are defined by

$$\begin{aligned} G_n(x) &= G_{n-1}(x) - 2xG_{n-2}(x), \quad \text{for } n \geq 2, \\ g_n(x) &= g_{n-1}(x) - 2xg_{n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

respectively, where $G_0(x) = 0$, $G_1(x) = 1$ and $g_0(x) = 2$, $g_1(x) = 1$.

Moreover, for any nonnegative integer k with $1-2^{k+2}x \neq 0$, Yalcin et al. [8] also considered the generalized Vieta–Jacobsthal polynomials sequences $\{G_{k,n}(x)\}_{n=0}^{\infty}$ and Vieta–Jacobsthal–Lucas polynomials sequences $\{g_{k,n}(x)\}_{n=0}^{\infty}$ by the following recurrence relations

$$\begin{aligned} G_{k,n}(x) &= G_{k,n-1}(x) - 2^k xG_{k,n-2}(x), \quad \text{for } n \geq 2, \\ g_{k,n}(x) &= g_{k,n-1}(x) - 2^k xg_{k,n-2}(x), \quad \text{for } n \geq 2, \end{aligned}$$

respectively, where $G_{k,0}(x) = 0$, $G_{k,1}(x) = 1$ and $g_{k,0}(x) = 2$, $g_{k,1}(x) = 1$. If $k = 1$, then $G_{1,n}(x) = G_n(x)$ and $g_{1,n}(x) = g_n(x)$. In [8], the Binet form and generating functions for these polynomials are derived. Furthermore, some special cases of the results are presented.

Recently, the generalization of Vieta–Fibonacci, Vieta–Lucas, Vieta–Pell, Vieta–Pell–Lucas, Vieta–Jacobsthal, and Vieta–Jacobsthal–Lucas polynomials have been studied by many authors.

In 2016 Kocer [3], considered the bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials which are generalized of Vieta–Fibonacci, Vieta–Lucas, Vieta–Pell, Vieta–Pell–Lucas polynomials. She also gave some properties. Afterward, she obtained some identities for the bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials by using the known properties of bivariate Vieta–Fibonacci and bivariate Vieta–Lucas polynomials.

In 2020 Uygun et al. [7], introduced the generalized Vieta–Pell and Vieta–Pell–Lucas polynomial sequences. They also gave the Binet formula, generating functions, sum formulas, differentiation rules, and some important properties for these sequences. And then they generated a matrix whose elements are of generalized Vieta–Pell terms. By using this matrix they derived some properties for generalized Vieta–Pell and generalized Vieta–Pell–Lucas polynomial sequences.

Inspired by the research going on in this direction, in this paper, we introduce a new type of Vieta polynomial, which is called Vieta–Fibonacci-like polynomial.

We also give the Binet form, the generating function, the well-known identities, and the sum formula for this polynomial. Furthermore, the relationship between this polynomial and the previous well-known Vieta polynomials are given in this study.

2. Vieta–Fibonacci-like polynomials

In this section, we introduce a new type of Vieta polynomial, called the Vieta–Fibonacci-like polynomials, as the following definition.

Definition 2.1. For any natural number n the Vieta–Fibonacci-like polynomials sequence $\{S_n(x)\}_{n=0}^{\infty}$ is defined by

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x), \quad \text{for } n \geq 2, \tag{2.1}$$

with the initial conditions $S_0(x) = 2$ and $S_1(x) = 2x$.

The first few terms of $\{S_n(x)\}_{n=0}^{\infty}$ are 2, $2x$, $2x^2 - 2$, $2x^3 - 4x$, $2x^4 - 6x^2 + 2$, $2x^5 - 8x^3 + 6x$, $2x^6 - 10x^4 + 12x^2 - 2$, $2x^7 - 12x^5 + 20x^3 - 8x$ and so on. The n^{th} terms of this sequence are called Vieta–Fibonacci-like polynomials.

First, we give the generating function for the Vieta–Fibonacci-like polynomials as follows.

Theorem 2.2 (The generating function)**.** *The generating function of the Vieta–Fibonacci-like polynomials sequence is given by*

$$g(x,t) = \frac{2}{1 - xt + t^2}.$$

Proof. The generating function $g(x,t)$ can be written as $g(x,t) = \sum_{n=0}^{\infty} S_n(x)t^n$. Consider,

$$g(x,t) = \sum_{n=0}^{\infty} S_n(x)t^n = S_0(x) + S_1(x)t + S_2(x)t^2 + \cdots + S_n(x)t^n + \ldots .$$

Then, we get

$$\begin{aligned} -xtg(x,t) &= -xS_0(x)t - xS_1(x)t^2 - xS_2(x)t^3 - \cdots - xS_{n-1}(x)t^n - \ldots \\ t^2g(x,t) &= S_0(x)t^2 + S_1(x)t^3 + S_2(x)t^4 + \cdots + S_{n-2}(x)t^n + \ldots . \end{aligned}$$

Thus,

$$\begin{aligned} g(x,t)(1 - xt + t^2) &= S_0(x) + (S_1(x) - xS_0(x))t \\ &\quad + \sum_{n=2}^{\infty}(S_n(x) - xS_{n-1}(x) + S_{n-2}(x))t^n \\ &= 2, \end{aligned}$$

$$g(x,t) = \frac{2}{1 - xt + t^2}.$$

This completes the proof. □

Next, we give the explicit formula for the n^{th} Vieta–Fibonacci-like polynomials.

Theorem 2.3 (Binet's formula). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials, then*

$$S_n(x) = A\alpha^n(x) + B\beta^n(x), \tag{2.2}$$

where $A = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}$, $B = \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}$ and $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$, $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$ are the roots of the characteristic equation $r^2 - xr + 1 = 0$.

Proof. The characteristic equation of the recurrence relation (2.1) is $r^2 - xr + 1 = 0$ and the roots of this equation are $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2-4}}{2}$.
It follows that

$$S_n(x) = d_1\alpha^n(x) + d_2\beta^n(x),$$

for some real numbers d_1 and d_2. Putting $n = 0$, $n = 1$, and then solving the system of linear equations, we obtain that

$$S_n(x) = \frac{2(x - \beta(x))}{\alpha(x) - \beta(x)}\alpha^n(x) + \frac{2(\alpha(x) - x)}{\alpha(x) - \beta(x)}\beta^n(x).$$

Setting $A = \frac{2(x-\beta(x))}{\alpha(x)-\beta(x)}$ and $B = \frac{2(\alpha(x)-x)}{\alpha(x)-\beta(x)}$, we get

$$S_n(x) = A\alpha^n(x) + B\beta^n(x).$$

This completes the proof. □

We note that $A + B = 2$, $AB = -\frac{4}{(\alpha(x)-\beta(x))^2}$, and $A\beta(x) + B\alpha(x) = 0$.

The other explicit forms of Vieta–Fibonacci-like polynomials are given in the following two theorems.

Theorem 2.4 (Explicit form). *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n(x) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n-i}{i}x^{n-2i}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 2.2, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} S_n(x)t^n &= \frac{2}{1 - (xt - t^2)} \\ &= 2\sum_{n=0}^{\infty}(xt - t^2)^n\end{aligned}$$

$$= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(xt)^{n-i}(-t^2)^i$$
$$= 2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-1)^i x^{n-i}t^{n+i}$$
$$= \sum_{n=0}^{\infty}\left[2\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^i\binom{n-i}{i}x^{n-2i}\right]t^n.$$

From the equality of both sides, the desired result is obtained. This complete the proof. □

Theorem 2.5 (Explicit form)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n(x) = 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^i\binom{n+1}{2i+1}x^{n-2i}(x^2-4)^i, \quad \textit{for } n \geq 1.$$

Proof. Consider,

$$\begin{aligned}\alpha^{n+1}(x) - \beta^{n+1}(x) &= 2^{-(n+1)}[(x+\sqrt{x^2-4})^{n+1} - (x-\sqrt{x^2-4})^{n+1}]\\ &= 2^{-(n+1)}\left[\sum_{i=0}^{n+1}\binom{n+1}{i}x^{n-i+1}(\sqrt{x^2-4})^i\right.\\ &\quad\left.-\sum_{i=0}^{n+1}\binom{n+1}{i}x^{n-i+1}(-\sqrt{x^2-4})^i\right]\\ &= 2^{-n}\left[\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2i+1}x^{n-2i}(\sqrt{x^2-4})^{2i+1}\right].\end{aligned}$$

Thus,

$$\begin{aligned}S_n(x) &= A\alpha^n(x) + B\beta^n(x)\\ &= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)}\\ &= 2\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\sqrt{x^2-4}}\\ &= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2i+1}x^{n-2i}(x^2-4)^i.\end{aligned}$$

This completes the proof. □

Theorem 2.6 (Sum formula)**.** *Let* $\{S_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$\sum_{k=0}^{n-1} S_k(x) = \frac{2 - S_n(x) + S_{n-1}(x)}{2-x}, \quad \text{for } n \geq 1.$$

Proof. By using Binet formula (2.2), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} S_k(x) &= \sum_{k=0}^{n-1} \left(A\alpha^k(x) + B\beta^k(x)\right) \\
&= A\frac{1-\alpha^n(x)}{1-\alpha(x)} + B\frac{1-\beta^n(x)}{1-\beta(x)} \\
&= \frac{A + B - (A\beta(x) + B\alpha(x)) - (A\alpha^n(x) + B\beta^n(x))}{1 - x + 1} \\
&\quad + \frac{A\alpha^{n-1}(x) + B\beta^{n-1}(x)}{1-x+1} \\
&= \frac{2 - S_n(x) + S_{n-1}(x)}{2-x}.
\end{aligned}$$

This completes the proof. □

Since the derivative of the polynomials is always exists, we can give the following formula.

Theorem 2.7 (Differentiation formula)**.** *The derivative of* $S_n(x)$ *is obtained as the follows.*

$$\frac{\mathrm{d}}{\mathrm{d}x} S_n(x) = \frac{(n+1)v_{n+1}(x) - xV_{n+1}(x)}{2(x^2-4)},$$

where $V_n(x)$ *and* $v_n(x)$ *are the* n^{th} *Vieta–Fibonacci and Vieta–Lucas polynomials, respectively.*

Proof. The result is obtained by using Binet formula (2.2). □

Again, by using Binet formula (2.2), we obtain some well-known identities as follows.

Theorem 2.8 (Catalan's identity or Simson identities)**.** *Let* $\{S_n(x)\}_{n=0}^{\infty}$ *be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n^2(x) - S_{n+r}(x)S_{n-r}(x) = S_{r-1}^2(x), \quad \text{for } n \geq r \geq 1. \tag{2.3}$$

Proof. By using Binet formula (2.2), we obtain

$$\begin{aligned}
&S_n^2(x) - S_{n+r}(x)S_{n-r}(x) \\
&\qquad = (A\alpha^n(x) + B\beta^n(x))^2 - \left(A\alpha^{n+r}(x) + B\beta^{n+r}(x)\right)\left(A\alpha^{n-r}(x) + B\beta^{n-r}(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -AB\left(\alpha(x)\beta(x)\right)^{n-r}\left(\alpha^{2r}(x) - 2\left(\alpha(x)\beta(x)\right)^{r} + \beta^{2r}(x)\right) \\
&= \frac{4}{(\alpha(x)-\beta(x))^2}\left(\alpha^{r}(x) - \beta^{r}(x)\right)^2 \\
&= (A\alpha^{r-1}(x) + B\beta^{r-1}(x))^2 \\
&= S_{r-1}^2(x).
\end{aligned}
$$

Thus,

$$S_n^2(x) - S_{n+r}(x)S_{n-r}(x) = S_{r-1}^2(x).$$

This completes the proof. □

Take $r = 1$ in Catalan's identity (2.3), then we get the following corollary.

Corollary 2.9 (Cassini's identity)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_n^2(x) - S_{n+1}(x)S_{n-1}(x) = 4, \quad \textit{for } n \geq 1.$$

Theorem 2.10 (d' Ocagne's identity)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_m(x)S_{n+1}(x) - S_{m+1}(x)S_n(x) = 2S_{m-n-1}(x), \quad \textit{for } m \geq n \geq 1. \tag{2.4}$$

Proof. We will prove d' Ocagne's identity (2.4) by using Binet formula (2.2). Consider,

$$
\begin{aligned}
&S_m(x)S_{n+1}(x) - S_{m+1}(x)S_n(x) \\
&\qquad = (A\alpha^{m}(x) + B\beta^{m}(x))\left(A\alpha^{n+1}(x) + B\beta^{n+1}(x)\right) \\
&\qquad\quad - \left(A\alpha^{m+1}(x) + B\beta^{m+1}(x)\right)(A\alpha^{n}(x) + B\beta^{n}(x)) \\
&\qquad = -AB\left(\alpha(x)\beta(x)\right)^{n}\left(\alpha(x) - \beta(x)\right)\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right) \\
&\qquad = \frac{4}{\left(\alpha(x)-\beta(x)\right)^2}\left(\alpha(x) - \beta(x)\right)\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right) \\
&\qquad = 2(A\alpha^{m-n-1}(x) + B\beta^{m-n-1}(x)) \\
&\qquad = 2S_{m-n-1}(x).
\end{aligned}
$$

This completes the proof. □

Theorem 2.11 (Honsberger identity)**.** *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then*

$$S_{m+1}(x)S_{n+1}(x) + S_m(x)S_n(x) = \frac{4xv_{m+n+3}(x) - 8v_{m-n}(x)}{x^2-4}, \quad \textit{for } m \geq n \geq 1,$$

where $v_n(x)$ is the n^{th} Vieta–Lucas polynomials.

Proof. By using Binet formula (2.2), we obtain

$$\begin{aligned}
S_{m+1}(x)S_{n+1}(x) &+ S_m(x)S_n(x)\\
&= \left(A\alpha^{m+1}(x)+B\beta^{m+1}(x)\right)\left(A\alpha^{n+1}(x)+B\beta^{n+1}(x)\right)\\
&\quad + \left(A\alpha^{m}(x)+B\beta^{m}(x)\right)\left(A\alpha^{n}(x)+B\beta^{n}(x)\right)\\
&= xA^2\alpha^{m+n+1}(x)+xB^2\beta^{m+n+1}(x)+2AB(\alpha^{m-n}(x)+\beta^{m-n}(x))\\
&= \frac{4x(\alpha^{m+n+3}(x)+\beta^{m+n+3}(x))-8(\alpha^{m-n}(x)+\beta^{m-n}(x))}{(\alpha(x)-\beta(x))^2}\\
&= \frac{4xv_{m+n+3}(x)-8v_{m-n}(x)}{x^2-4}.
\end{aligned}$$

This completes the proof. □

In the next theorem, we obtain the relation between the Vieta–Fibonacci-like, Vieta–Fibonacci and the Vieta–Lucas polynomials by using Binet formula (2.2).

Theorem 2.12. *Let $\{S_n(x)\}_{n=0}^{\infty}$, $\{V_n(x)\}_{n=0}^{\infty}$ and $\{v_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Fibonacci-like, Vieta–Fibonacci and Vieta–Lucas polynomials, respectively. Then*

(1) $S_n(x) = 2V_{n+1}(x)$, *for* $n \geq 0$,

(2) $S_n(x) = v_n(x) + xV_n(x)$, *for* $n \geq 0$,

(3) $S_n(x)v_{n+1}(x) = 2V_{2n+2}(x)$, *for* $n \geq 0$,

(4) $S_{n+1}(x) + S_{n-1}(x) = 2xV_{n+1}(x)$, *for* $n \geq 1$,

(5) $S_{n+1}(x) - S_{n-1}(x) = 2v_{n+1}(x)$, *for* $n \geq 1$,

(6) $S^2_{n+2}(x) - S^2_{n-1}(x) = 4xV_{2n+2}(x)$, *for* $n \geq 1$,

(7) $2S_n(x) - xS_{n-1}(x) = 2v_n(x)$, *for* $n \geq 1$,

(8) $S_{n+2}(x) + S_{n-2}(x) = (2x^2-4)V_{n+1}(x)$, *for* $n \geq 2$,

(9) $S^2_{n+2}(x) - S^2_{n-2}(x) = 4x(x^2-2)V_{2n+2}(x)$, *for* $n \geq 2$,

(10) $v_{n+1}(x) - v_n(x) = \frac{1}{2}(x^2-4)S_{n-1}(x)$, *for* $n \geq 1$,

(11) $2v_{n+1}(x) - xv_n(x) = \frac{1}{2}(x^2-4)S_{n-1}(x)$, *for* $n \geq 1$,

(12) $4v^2_n(x) + (x^2-4)S^2_{n-1}(x) = 8v_n(x)$, *for* $n \geq 1$,

(13) $4v^2_n(x) - (x^2-4)S^2_{n-1}(x) = 16$, *for* $n \geq 1$.

Proof. The results (1)–(13) are easily obtained by using Binet formula (2.2). □

3. Matrix Form of Vieta–Fibonacci-like polynomials

In this section, we establish some identities of Vieta–Fibonacci-like and Vieta–Fibonacci polynomials by using elementary matrix methods.

Let Q_s be 2×2 matrix defined by

$$Q_S = \begin{bmatrix} 2x^2 - 2 & 2x \\ -2x & -2 \end{bmatrix}. \tag{3.1}$$

Then by using this matrix we can deduce some identities of Vieta–Fibonacci-like and Vieta–Fibonacci polynomials.

Theorem 3.1. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials and Q_s be 2×2 matrix defined by (3.1). Then*

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Proof. For the proof, mathematical induction method is used. It obvious that the statement is true for $n = 1$. Suppose that the result is true for any positive integer k, then we also have the result is true for $k + 1$. Because

$$\begin{aligned} Q_S^{k+1} &= Q_S^k \cdot Q_S \\ &= 2^{k-1} \begin{bmatrix} S_{2k}(x) & S_{2k-1}(x) \\ -S_{2k-1}(x) & -S_{2k-2}(x) \end{bmatrix} \begin{bmatrix} 2x^2 - 2 & 2x \\ -2x & -2 \end{bmatrix} \\ &= 2^{(k+1)-1} \begin{bmatrix} S_{2(k+1)}(x) & S_{2(k+1)-1}(x) \\ -S_{2(k+1)-1}(x) & -S_{2(k+1)-2}(x) \end{bmatrix}. \end{aligned}$$

By Mathematical induction, we have that the result is true for each $n \in \mathbb{N}$, that is

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \qquad \square$$

Theorem 3.2. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci-like polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $2S_{2(m+n)}(x) = S_{2m}(x)S_{2n}(x) - S_{2m-1}(x)S_{2n-1}(x)$,

(2) $2S_{2(m+n)-1}(x) = S_{2m}(x)S_{2n-1}(x) - S_{2m-1}(x)S_{2n-2}(x)$,

(3) $2S_{2(m+n)-1}(x) = S_{2m-1}(x)S_{2n}(x) - S_{2m-2}(x)S_{2n-1}(x)$,

(4) $2S_{2(m+n)-2}(x) = S_{2m-1}(x)S_{2n-1}(x) - S_{2m-2}(x)S_{2n-2}(x)$.

Proof. By Theorem 3.1 and the property of power matrix $Q_s^{m+n} = Q_s^m \cdot Q_s^n$, then we obtained the results. $\square$

By Theorem 3.1 and $S_n(x) = 2V_{n+1}(x)$, we get the following Corollary.

Corollary 3.3. *Let $\{V_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci polynomials and Q_s be 2×2 matrix defined by* (3.1)*. Then*

$$Q_S^n = 2^n \begin{bmatrix} V_{2n+1}(x) & V_{2n}(x) \\ -V_{2n}(x) & -V_{2n-1}(x) \end{bmatrix}, \quad \textit{for } n \geq 1.$$

Proof. From Theorem 3.1, we get

$$Q_S^n = 2^{n-1} \begin{bmatrix} S_{2n}(x) & S_{2n-1}(x) \\ -S_{2n-1}(x) & -S_{2n-2}(x) \end{bmatrix}, \quad \text{for } n \geq 1.$$

Since $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} Q_S^n &= 2^{n-1} \begin{bmatrix} 2V_{2n+1}(x) & 2V_{2n}(x) \\ -2V_{2n}(x) & -2V_{2n-1}(x) \end{bmatrix} \\ &= 2^n \begin{bmatrix} V_{2n+1}(x) & V_{2n}(x) \\ -V_{2n}(x) & -V_{2n-1}(x) \end{bmatrix}, \quad \text{for } n \geq 1. \end{aligned}$$

This completes the proof. □

By Theorem 3.2 and $S_n(x) = 2V_{n+1}(x)$, we get the following Corollary.

Corollary 3.4. *Let $\{V_n(x)\}_{n=0}^{\infty}$ be the sequence of Vieta–Fibonacci polynomials. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $V_{2(m+n)+1}(x) = V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)$,

(2) $V_{2(m+n)}(x) = V_{2m+1}(x)V_{2n}(x) - V_{2m}(x)V_{2n-1}(x)$,

(3) $V_{2(m+n)}(x) = V_{2m}(x)V_{2n+1}(x) - V_{2m-1}(x)V_{2n}(x)$,

(4) $V_{2(m+n)-1}(x) = V_{2m}(x)V_{2n}(x) - V_{2m-1}(x)V_{2n-1}(x)$.

Proof. From Theorem 3.2 and $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} V_{2(m+n)+1}(x) &= \frac{1}{2} S_{2(m+n)}(x) \\ &= \frac{1}{4} \left(S_{2m}(x)S_{2n}(x) - S_{2m-1}(x)S_{2n-1}(x)\right) \\ &= \frac{1}{4} \left(2V_{2m+1}(x)2V_{2n+1}(x) - 2V_{2m}(x)2V_{2n}(x)\right) \\ &= V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x). \end{aligned}$$

Thus, we get that (1) holds. By the same argument as above, we get that (2), (3), and (4) holds. This completes the proof. □

By Corollary 3.4 and $S_n(x) = 2V_{n+1}(x)$, we get the following corollary.

Corollary 3.5. *Let $\{S_n(x)\}_{n=0}^{\infty}$ and $\{V_n(x)\}_{n=0}^{\infty}$ be the sequences of Vieta–Fibonacci-like polynomials and Vieta–Fibonacci polynomials, respectively. Then for all integers $m \geq 1$, $n \geq 1$, the following statements hold.*

(1) $S_{2(m+n)}(x) = 2\left(V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)\right),$

(2) $S_{2(m+n)-1}(x) = 2\left(V_{2m+1}(x)V_{2n}(x) - V_{2m}(x)V_{2n-1}(x)\right),$

(3) $S_{2(m+n)-1}(x) = 2\left(V_{2m}(x)V_{2n+1}(x) + V_{2m-1}(x)V_{2n}(x)\right),$

(4) $S_{2(m+n)-2}(x) = 2\left(V_{2m}(x)V_{2n}(x) + V_{2m-1}(x)V_{2n-1}(x)\right).$

Proof. From Corollary 3.4 and $S_n(x) = 2V_{n+1}(x)$, we get that

$$\begin{aligned} S_{2(m+n)}(x) &= 2V_{2(m+n)+1}(x) \\ &= 2\left(V_{2m+1}(x)V_{2n+1}(x) - V_{2m}(x)V_{2n}(x)\right). \end{aligned}$$

Thus, we get that (1) holds. By the same argument as above, we get that (2), (3), and (4) holds. This completes the proof. □

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