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The Diophantine equation $x^2 + 3^a \cdot 5^b \cdot 11^c \cdot 19^d = 4y^n$

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Abstract

We investigate the Diophantine equation $x^2 + 3^a \cdot 5^b \cdot 11^c \cdot 19^d = 4y^n$ with $n \geq 3$, $x, y, a, b, c, d \in \mathbb{N}$, $x, y > 0$, and $\gcd(x, y) = 1$.

Keywords: Diophantine equations, Lesbegue–Ramanujan–Nagell equations, primitive divisors of Lucas numbers

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1. Introduction

Let D be a positive integer. The equation

$$x^2 + D = 4y^n \tag{1.1}$$

is called a Lesbgue-Ramanujan-Nagell equation. It has been studied by several authors. Luca, Tengely, and Togbé [7] studied (1.1) when $1 \leq D \leq 100$ and $D \not\equiv 1 \pmod 4$, $D = 7^a \cdot 11^b$, or $D = 7^a \cdot 13^b$, where $a, b \in \mathbb{N}$. Bhatter, Hoque, and Sharma [1] studied (1.1) when $D = 19^{2k+1}$, where $k \in \mathbb{N}$. Chakraborty, Hoque, and Sharma [4] studied (1.1) when $D = p^m$, where $p \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ and $m \in \mathbb{N}$. For a comprehensive survey of equation (1.1) and other Lebesgue-Ramanunjan-Nagell type equations, see Le and Soydan [6] with over 350 references. In this paper, we study (1.1) when $D = 3^a \cdot 5^b \cdot 11^c \cdot 19^d$. It can be deduced from our work all solutions to (1.1) when the set of prime divisors of D is a *proper* subset of $\{3, 5, 11, 19\}$. The main result is the following.

Theorem 1.1. *All integer solutions* (n, a, b, c, d, x, y) *to the equation*

$$x^2 + 3^a \cdot 5^b \cdot 11^c \cdot 19^d = 4y^n$$

with

(i) $n \geq 3$, $a, b, c, d \geq 0$, $x, y > 0$, $\gcd(x, y) = 1$,

(ii) $(a, b, c, d) \not\equiv (1, 1, 1, 1) \pmod 2$ *if* $5 \mid n$,

are given in Tables 1, 4, 5, 7, and 8.

Our main tool is the so-called primitive divisor theorem of Lucas numbers by Bilu, Hanrot, and Voutier [2].

2. Preliminaries

Let α and β be two algebraic integers such that $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers, and $\frac{\alpha}{\beta}$ is not a root of unity. The Lucas sequence $(L_n)_{n\geq 1}$ is defined by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all} \quad n \geq 1.$$

A prime number p is called a primitive divisor of L_n if

$$p \mid L_n \quad \text{but} \quad p \nmid (\alpha - \beta)^2 L_1 \cdots L_{n-1}.$$

From the work of Bilu, Hanrot, and Voutier's [2] we know

(i) if q is a primitive divisor of L_n, then $n \mid q - \left(\frac{(\alpha-\beta)^2}{q}\right)$,

(ii) if $n > 30$, then L_n has a primitive divisor,

(iii) for all $4 < n \leq 30$, if L_n does not have a primitive divisor, then (n, α, β) can be derived from Table 1 in [2].

3. Proof of Theorem 1.1

From

$$x^2 + 3^a \cdot 5^b \cdot 11^c \cdot 19^d = 4y^n \tag{3.1}$$

we have $2 \nmid x$. Reducing (3.1) mod 4 gives $1 + (-1)^{a+c+d} \equiv 0 \pmod 4$. Hence, $2 \nmid a+c+d$. Note that $x, y > 0$, $\gcd(x, y) = 1$, and $n \geq 3$. Write $3^a \cdot 5^b \cdot 11^c \cdot 19^d = AB^2$, where $A, B \in \mathbb{Z}^+$ and A is square-free. Here $A \in \{3, 11, 15, 19, 55, 95, 627, 3135\}$. Let $K = \mathbb{Q}(\sqrt{-A})$. Let $h(K)$ and $\mathcal{O}_K$ be the class number and the ring of integers of K respectively. Then $h(K) \in \{1, 2, 4, 8, 40\}$ and $K = \mathbb{Z}\left[\frac{1+\sqrt{-A}}{2}\right]$.

Assume now that n is an odd prime not dividing $h(K)$. Then

$$\left(\frac{x + B\sqrt{-A}}{2}\right)\left(\frac{x - B\sqrt{-A}}{2}\right) = (y)^n. \tag{3.2}$$

Since x and AB^2 are odd, the two ideals $\left(\frac{x+B\sqrt{-A}}{2}\right)$ and $\left(\frac{x-B\sqrt{-A}}{2}\right)$ are coprime. We also have $n \nmid h(A)$, so (3.2) implies that

$$\frac{x + B\sqrt{-A}}{2} = u\alpha^n, \tag{3.3}$$

where u is a unit in $\mathcal{O}_K$ and $\alpha \in \mathcal{O}_K$. Since the order of the unit group of $\mathcal{O}_K$ is a power of 2, it is coprime to n. Therefore, in (3.3) u can be absorbed into α. So we can assume $u = 1$. Let $\alpha = \frac{r+s\sqrt{-A}}{2}$ and $\beta = \frac{r-s\sqrt{-A}}{2}$, where $r, s \in \mathbb{Z}$ and $r \equiv s \pmod 2$. We claim r and s are coprime odd integers. If r and s are even, let $r_1 = \frac{r}{2}$ and $s_1 = \frac{s}{2}$. Then

$$x = \frac{\alpha^n + \beta^n}{2} = 2\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} r_1^{n-2k}(-A)^k s_1^{2k},$$

impossible since $2 \nmid x$. Therefore r and s are odd. Then

$$x = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k} r^{n-2k}(-A)^k s^{2k}.$$

Let $l = \gcd(r, s)$. Then $l \mid x$ and $l \mid \frac{r^2+As^2}{4}$. Hence, $l \mid \gcd(x, y)$. Therefore $l = 1$. So $\gcd(r, s) = 1$. Let $q = \gcd(r, A)$. Since $|y| = \frac{r^2+As^2}{4}$, we have $q \mid y$. Since $x^2 + AB^2 = 4y^n$, we have $q \mid x^2$. Since $\gcd(x, y) = 1$, we have $q = 1$. Since $\alpha + \beta = r$ and $\alpha\beta = \frac{r^2+As^2}{4}$, we have $\alpha + \beta$ and $\alpha\beta$ are coprime integers.

The proof of Theorem 1.1 is now achieved by means of the following four lemmas. We only require the condition $(a, b, c, d) \not\equiv (1, 1, 1, 1) \pmod 2$ in the Lemma 3.5. So Lemmas 3.1, 3.2, 3.3, 3.4 give all solutions to (1.1) in each case of n with $\gcd(x, y) = 1$.

Lemma 3.1. *All solutions* (n, a, b, c, d, x, y) *to* (3.1) *with* $n = 3$ *are given in Table 1.*

Table 1. Solutions to (3.1) with $n = 3$ and $\gcd(x, y) = 1$.

(n, a, b, c, d, x, y)	(n, a, b, c, d, x, y)
$(3, 1, 0, 0, 0, 1, 1)$	$(3, 1, 0, 0, 0, 37, 7)$
$(3, 1, 0, 0, 2, 17, 7)$	$(3, 1, 0, 0, 4, 719, 61)$
$(3, 7, 1, 0, 4, 19307, 766)$	$(3, 7, 1, 2, 0, 15599, 394)$
$(3, 7, 1, 4, 2, 111946687, 146326)$	$(3, 7, 3, 1, 2, 2043331, 10144)$

(3, 7, 3, 4, 0, 2073287, 10246)	(3, 7, 3, 4, 2, 2495189, 12424)
(3, 3, 1, 0, 0, 11, 4)	(3, 3, 1, 6, 0, 96433, 1336)
(3, 9, 1, 0, 2, 443531, 3664)	(3, 3, 1, 2, 0, 7, 16)
(3, 3, 7, 14, 2, 380377270937, 47690296)	(3, 3, 1, 2, 2, 5771, 214)
(3, 3, 9, 1, 2, 2, 397447, 3436)	(3, 3, 1, 2, 2, 28267, 586)
(3, 3, 1, 2, 2, 154757, 1816)	(3, 3, 7, 2, 2, 43847521, 78334)
(3, 3, 3, 0, 0, 2761, 124)	(3, 3, 3, 0, 2, 1883, 106)
(3, 3, 3, 2, 0, 3107, 136)	(3, 3, 3, 2, 2, 1271, 334)
(3, 3, 5, 0, 4, 271051, 2764)	(3, 27, 5, 1, 1, 1291606603, 1184566)
(3, 3, 5, 10, 2, 10684962781, 3063094)	(3, 4, 1, 0, 1, 9673, 286)
(3, 5, 1, 0, 0, 623, 46)	(3, 5, 1, 0, 2, 781, 64)
(3, 11, 1, 1, 1, 74333, 1126)	(3, 5, 1, 1, 1, 1824473, 9406)
(3, 5, 1, 2, 0, 101, 34)	(3, 11, 1, 2, 0, 11877401, 32794)
(3, 5, 1, 2, 4, 873907, 5806)	(3, 5, 7, 2, 4, 1169073209, 699154)
(3, 5, 1, 4, 0, 713, 166)	(3, 11, 1, 4, 6, 1399486399, 862744)
(3, 5, 3, 0, 6, 778921, 7984)	(3, 5, 3, 2, 2, 41803, 916)
(3, 5, 5, 6, 0, 8694731, 26794)	

Proof. Write $a = 6a_1 + \epsilon_1$, $b = 6b_1 + \epsilon_2$, $c = 6c_1 + \epsilon_3$, and $d = 6d_1 + \epsilon_4$, where $a_1, b_1, c_1, d_1 \in \mathbb{N}$ and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{0, 1, \ldots, 5\}$. Let $D_1 = 3^{\epsilon_1} \cdot 5^{\epsilon_2} \cdot 11^{\epsilon_3} \cdot 19^{\epsilon_4}$. From (3.1) we have

$$Y^2 = X^3 - 16D_1, \tag{3.4}$$

where $X = \frac{4y}{3^{2a_1} \cdot 5^{2b_1} \cdot 11^{2c_1} \cdot 19^{2d_1}}$ and $Y = \frac{4x}{3^{3a_1} \cdot 5^{3b_1} \cdot 11^{3c_1} \cdot 19^{3d_1}}$. Since $2 \nmid a + c + d$, we have $2 \nmid \epsilon_1 + \epsilon_3 + \epsilon_4$. We use Magma [3] to search for S-integral points on (3.4), where $S = \{3, 5, 11, 19\}$. Solutions to (3.1) deduced from these S-integral points are listed in Table 1. We are able to find S- integral points on (3.4) for all but the cases of $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ listed in Table 2.

Table 2

$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$	$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$	$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$	$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$
(0, 1, 5, 4)	(0, 4, 5, 4)	(1, 1, 5, 5)	(1, 2, 1, 5)
(1, 2, 3, 5)	(1, 2, 5, 3)	(1, 2, 5, 5)	(1, 3, 3, 5)
(1, 3, 5, 3)	(1, 3, 5, 5)	(1, 4, 1, 5)	(1, 4, 3, 5)
(1, 4, 5, 1)	(1, 4, 5, 5)	(1, 5, 3, 3)	(1, 5, 5, 3)
(1, 5, 5, 5)	(3, 1, 5, 3)	(3, 1, 5, 5)	(3, 3, 1, 5)
(3, 3, 5, 3)	(3, 4, 5, 3)	(3, 5, 3, 5)	(4, 1, 3, 4)
(4, 1, 5, 4)	(4, 3, 5, 4)	(4, 4, 4, 5)	(4, 5, 1, 4)
(4, 5, 3, 4)	(4, 5, 4, 5)	(4, 5, 5, 2)	(4, 5, 5, 4)
(5, 0, 3, 5)	(5, 0, 5, 5)		

We will show that (3.1) has no solutions for these cases of $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. Since

$3 \nmid h(K)$, there exist coprime odd integers r, s such that

$$\frac{x + B\sqrt{-A}}{2} = \left(\frac{r + s\sqrt{-A}}{2}\right)^3.$$

Comparing the imaginary parts gives

$$4B = s(3r^2 - As^2). \tag{3.5}$$

Notice that $B = 3^{3a_1+u_1} \cdot 5^{3b_1+u_2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4}$, where $u_i = \lfloor \frac{\epsilon_i}{2} \rfloor$ for $i = 1, 2, 3, 4$. Hence,

$$4 \cdot 3^{3a_1+u_1} \cdot 5^{3b_1+u_2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4} = s(3r^2 - As^2). \tag{3.6}$$

Case 1: $A = 11$. Then $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (0, 4, 5, 4)$. Hence, (3.6) reduces to

$$4 \cdot 3^{3a_1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} = s(3r^2 - 11s^2). \tag{3.7}$$

If $11 \mid r$, then $11 \nmid s$. Hence, $11^2 \nmid s(3r^2 - 11s^2)$. Thus, (3.7) is impossible. So $11 \nmid r$. Hence, $11^{3c_1+2} \mid s$. Since $\left(\frac{3 \cdot 11}{5}\right) = -1$ and $\gcd(r, s) = 1$, we have $5 \nmid 3r^2 - 11s^2$. Hence, $5^{3b_1+2} \mid s$. Since $\left(\frac{3 \cdot 11}{19}\right) = -1$, we have $19 \nmid 3r^2 - 11s^2$. Therefore $19^{3d_1+2} \mid s$.

Case 1.1: $a_1 > 0$. Reducing (3.7) mod 3 gives $3 \mid s$. Hence, $3^{3a_1} \mid s$. Since $2 \nmid s$, we have $s = 3^{3a_1-1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 \in \{\pm 1\}$. Then (3.7) reduces to

$$4 = s_1(r^2 - 3^{6a_1-3} \cdot 5^{6b_1+4} \cdot 11^{6c_1+5} \cdot 19^{6d_1+4}). \tag{3.8}$$

Since $a_1 > 0$, we have $6a_1 - 3 > 0$. Reducing (3.8) mod 3 shows $s_1 = 1$. Then (3.8) reduces to

$$4 = r^2 - 3^{6a_1-3} \cdot 5^{6b_1+4} \cdot 11^{6c_1+5} \cdot 19^{6d_1+4}. \tag{3.9}$$

Reducing mod 7 shows

$$4 \equiv r^2 - 6 \pmod 7,$$

impossible mod 7 since $\left(\frac{10}{7}\right) = -1$.

Case 1.2: $a_1 = 0$. Since $2 \nmid s$, we have $s = \pm 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2}$. Then (3.7) reduces to

$$4 = \pm(3r^2 - 11s^2),$$

impossible mod 5 since $5 \mid s$, $5 \nmid r$, and $\left(\frac{\pm 3}{5}\right) = -1$.

Case 2: $A = 19$. Then $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (4, 4, 4, 5)$. Hence, (3.6) reduces to

$$4 \cdot 3^{3a_1+2} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} = s(3r^2 - 19s^2). \tag{3.10}$$

If $19 \mid r$, then $19 \nmid s$. Hence, $19^2 \nmid s(3r^2 - 19s^2)$, so (3.8) is impossible mod 19^2. Therefore $19 \nmid r$. Hence, $19^{3d_1+2} \mid s$. Since $\left(\frac{3 \cdot 19}{5}\right) = \left(\frac{3 \cdot 19}{11}\right) = -1$, we have $5 \nmid 3r^2 - 19s^2$ and $11 \nmid 3r^2 - 19s^2$. Hence, $5^{3b_1+2} \cdot 11^{3c_1+2} \mid s$. Reducing (3.8) mod 3 shows that $3 \mid s$. Hence, $3^{3a_1+1} \mid s$. Therefore $s = 3^{3a_1+1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 \in \{\pm 1\}$. Then (3.10) reduces to

$$4 = s_1(r^2 - 3^{6a_1+1} \cdot 5^{6b_1+2} \cdot 11^{6c_1+4} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.11}$$

Reducing (3.11) mod 3 shows $s_1 \equiv 1 \pmod 3$. Hence, $s_1 = 1$. Then

$$4 = r^2 - 3^{6a_1+1} \cdot 5^{6b_1+2} \cdot 11^{6c_1+4} \cdot 19^{6d_1+5}. \tag{3.12}$$

Write (3.12) as

$$4 = Y^2 - 3 \cdot 5^2 \cdot 11 \cdot 19^2 \cdot X^3, \tag{3.13}$$

where $Y = r$ and $X = 3^{2a_1} \cdot 5^{2b_1+1} \cdot 11^{2c_1+1} \cdot 19^{2d_1+1}$.

Magma [3] shows (3.13) only has integer solutions $(X, Y) = (0, \pm 2)$. Hence, (3.12) has no solutions.

Case 3: $A = 55$. Then $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (0, 1, 5, 4)$, $(4, 1, 3, 4)$, $(4, 5, 3, 4)$, $(4, 5, 5, 2)$, $(4, 5, 5, 4)$. Equation (3.6) reduces to

$$4 \cdot 3^{3a_1+u_1} \cdot 5^{3b_1+u_2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4} = s(3r^2 - 55s^2). \tag{3.14}$$

Since $\left(\frac{3 \cdot 55}{19}\right) = -1$, we have $19 \nmid 3r^2 - 55s^2$.

Case 3.1: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (0, 1, 5, 4)$. Equation (3.14) reduces to

$$4 \cdot \cdot 5^{3b_1} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} = s(3r^2 - 55s^2). \tag{3.15}$$

Since $3b_1 = 0$ or $3b_1 \geq 3$, from (3.15) have $5^{3b_1} \mid s$. From (3.15) we also have $11^{3c_1+2} \mid s$. Therefore $s = 5^{3b_1} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} s_1$, where $s_1 \in \{\pm 1\}$. Equation (3.15) reduces to

$$4 = \pm 3r^2 - 55s^2,$$

impossible mod 5 since $5 \nmid r$ and $\left(\frac{\pm 3}{5}\right) = -1$.

Case 3.2: $3a_1 + u_1 > 0$.

Case 3.2.1: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (4, 1, 3, 4)$. Then (3.14) reduces to

$$4 \cdot 3^{3a_1+2} \cdot 5^{3b_1} \cdot 11^{3c_1+1} \cdot 19^{3d_1+2} = s(3r^2 - 55s^2). \tag{3.16}$$

Reducing (3.16) mod 3 gives $3 \mid s$. Hence, $3^{3a_1+1} \mid s$. If $5 \mid r$, then $5 \nmid s$. Hence, $5^2 \nmid s(3r^2 - 55s^2)$. Therefore, (3.16) is impossible mod 5^{3b_1}. Hence, $5 \nmid r$. Thus $5^{3b_1} \mid s$.

• $11 \nmid s$. Then $11 \mid r$. Hence, $11^2 \nmid s(3r^2 - 55s^2)$. From (3.16) we have $3c_1 + 1 = 1$. Let $s = 3^{3a_1+1} \cdot 5^{3b_1} \cdot 19^{3d_1+u_4} s_1$, where $s_1 \in \mathbb{Z}$ and $r = 11r_1$, where $r_1 \in \mathbb{Z}$. Then (3.16) reduces to

$$4 = s_2(11r_1^2 - 3^{6a_1+2} \cdot 5^{6b_1+1} \cdot 19^{6d_1+4} \cdot s_1^2). \tag{3.17}$$

Reducing (3.17) mod 3 shows that $s_1 \equiv -1 \pmod 3$. Hence, $s_1 = -1$. Then (3.17) reduces to

$$4 = 3^{6a_1+2} \cdot 5^{6b_1+1} \cdot 19^{6d_1+4} - 11r_2^2,$$

impossible mod 19 since $\left(\frac{-11}{19}\right) = -1$.

• $11 \mid s$. Then $11^{3c_1+1} \mid s$. Let $s = 3^{3a_1+2} \cdot 5^{3b_1} \cdot 11^{3c_1+1} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 \in \{\pm 1\}$. Then (3.16) reduces to

$$4 = s_1(r^2 - 3^{6a_1+3} \cdot 5^{6b_1+1} \cdot 11^{6c_1+1} \cdot 19^{6d_1+4} \cdot s_1^2). \tag{3.18}$$

Reducing (3.18) mod 3 gives $s_1 \equiv 1 \pmod 3$. Hence, $s_1 = 1$. Then (3.18) reduces to

$$4 = r^2 - 3^{6a_1+3} \cdot 5^{6b_1+1} \cdot 11^{6c_1+1} \cdot 19^{6d_1+4}. \tag{3.19}$$

Reducing mod 13 shows

$$4 \equiv r^2 - 1 \pmod{13}$$

impossible since $\left(\frac{5}{13}\right) = -1$.

Case 3.3.2: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (4,5,3,4), (4,5,5,2), (4,5,5,4)$. Then (3.16) reduces to

$$4 \cdot 3^{3a_1+2} \cdot 5^{3b_1+2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4} = s(3r^2 - 55s^2). \tag{3.20}$$

Then $3^{3a_1+1} \cdot 5^{3b_1+2} \cdot 19^{3d_1+u_4} \mid s$.

• $11 \mid s$. Then $11^{3c_1+u_3} \mid s$. Hence, $s = 3^{3a_1+1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4} \cdot s_1$, where $s_1 \in \mathbb{Z}$. Then (3.20) reduces to

$$4 = s_1(r^2 - 3^{6a_1+1} \cdot 5^{6b_1+4} \cdot 11^{6c_1+2u_3+1} \cdot 19^{6d_1+2u_4} \cdot s_1^2). \tag{3.21}$$

Reducing (3.21) mod 3 gives $s_1 \equiv 1 \pmod 3$. Hence, $s_1 = 1$. Then

$$4 = r^2 - 3^{6a_1+1} \cdot 5^{6b_1+4} \cdot 11^{6c_1+\epsilon_3} \cdot 19^{6d_1+\epsilon_4}). \tag{3.22}$$

Write (3.22) as a cubic

$$Y^2 = 4 + 3 \cdot 5 \cdot 11^{v_1} \cdot 19^{v_2} \cdot X^3, \tag{3.23}$$

where $Y = r$, X only has prime divisors $5, 11, 19$, and $(v_1, v_2) = (0,1), (2,2), (2,1)$. Equation (3.23) only has integer solutions $(X, Y) = (0, \pm 2), (1, 17)$ as

$$2^2 = 4 + 3 \cdot 5 \cdot 11^{v_1} \cdot 19^{v_2} \cdot 0^3,$$
$$17^2 = 4 + 3 \cdot 5 \cdot 19 \cdot 1^2.$$

None of these solutions gives solutions to (3.22).

• $11 \nmid s$. Reducing (3.20) mod 11 shows $11 \mid r$. Since $11^2 \nmid s(3r^2 - 55s^2)$, in (3.20) we must have $3c_1 + u_3 = 1$. Hence, $(\epsilon_1, \epsilon_2, \epsilon_3) = (4,5,3,4)$. Then (3.16) reduces to

$$4 \cdot 3^{3a_1+2} \cdot 5^{3b_1+2} \cdot 11 \cdot 19^{3d_1+2} = s(3r^2 - 55s^2). \tag{3.24}$$

Let $s = 3^{3a_1+1} \cdot 5^{3b_1+2} \cdot 19^{3d_1+2} \cdot s_1$ and $r = 11r_1$, where $s_1, r_1 \in \mathbb{Z}$. Then (3.24) reduces to

$$4 = s_2(11r_1^2 - 3^{6a_1+1} \cdot 5^{6b_1+4} \cdot 19^{6d_1+4} \cdot s_2^2). \tag{3.25}$$

Reducing (3.25) mod 3 shows $s_2 \equiv -1 \pmod 3$. Hence, $s_2 = -1$. Therefore

$$4 = 3^{6a_1+1} \cdot 5^{6b_1+4} \cdot 19^{6d_1+4} \cdot s_2^2 - 11r_1^2,$$

impossible mod 19 since $\left(\frac{-11}{19}\right) = -1$.

Case 4: $A = 95$. Then $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (4, 5, 4, 5)$. Then (3.16) reduces to

$$4 \cdot 3^{3a_1+2} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} = s(3r^2 - 95s^2). \tag{3.26}$$

Then $3^{3a_1+1} \mid s$, $5^{3b_1+2} \mid s$, $19^{3d_1+2} \mid s$. Since $\left(\frac{3\cdot 93}{11}\right) = -1$, (3.26) implies $11^{3c_1+2} \mid s$. Let $s = 3^{3a_1+1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 = \pm 1$. Then (3.26) reduces to

$$4 = s_1(r^2 - 3^{6a_1+1} \cdot 5^{6b_1+5} \cdot 11^{6c_1+4} \cdot 19^{6d_1+5} \cdot s_1^2).$$

Reducing mid 3 shows $s_1 \equiv 1 \pmod 3$. Hence, $s_1 = 1$. Then

$$4 = r^2 - 3^{6a_1+1} \cdot 5^{6b_1+5} \cdot 11^{6c_1+4} \cdot 19^{6d_1+5}. \tag{3.27}$$

Write (3.27) as a cubic curve

$$Y^2 = 4 + 3 \cdot 5^2 \cdot 11 \cdot 19^2 \cdot X^3, \tag{3.28}$$

where $Y = r$ and $X = 3^{2a_1} \cdot 5^{2b_1+1} \cdot 11^{2c_1+1} \cdot 19^{2d_1+1}$. Magma shows that equation (3.28) only has integer solutions $(X, Y) = (0, \pm 2)$. Hence, (3.27) has no solutions.

Case 5: $A = 3 \cdot 11 \cdot 19$. Then $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 2, 1, 5)$, $(1, 2, 3, 5)$, $(1, 2, 5, 3)$, $(1, 2, 5, 5)$, $(1, 4, 1, 5)$, $(1, 4, 3, 5)$, $(1, 4, 5, 1)$, $(1, 4, 5, 5)$, $(3, 4, 5, 3)$, $(5, 0, 3, 5)$, $(5, 0, 5, 5)$. Then (3.16) reduces to

$$4 \cdot 3^{3a_1+u_1-1} \cdot 5^{3b_1+u_2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4} = s(r^2 - 209s^2). \tag{3.29}$$

Since r and s is odd, we have $8 \mid r^2 - 209s^2$. Therefore equation (3.29) is impossible mod 8.

Case 6: $A = 3 \cdot 5 \cdot 11 \cdot 19$. Then $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 1, 1, 5)$, $(1, 3, 3, 5)$, $(1, 3, 5, 3)$, $(1, 3, 5, 5)$, $(1, 5, 3, 3)$, $(1, 5, 5, 3)$, $(1, 5, 5, 5)$, $(3, 1, 5, 3)$, $(3, 1, 5, 5)$, $(3, 3, 1, 5)$, $(3, 3, 5, 3)$, $(3, 5, 3, 5)$. Hence, (3.16) reduces to

$$4 \cdot 3^{3a_1+u_1-1} \cdot 5^{3b_1+u_2} \cdot 11^{3c_1+u_3} \cdot 19^{3d_1+u_4} = s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2). \tag{3.30}$$

Notice that s can only have prime factors $3, 5, 11, 19$. Dividing both sides of (3.30) by s^3 gives a quartic equation of the form

$$Y^2 = 5 \cdot 11 \cdot 19 + 4 \cdot 3^{\gamma_1} \cdot 5^{u_2} \cdot 11^{u_3} \cdot 19^{u_4} \cdot X^3, \tag{3.31}$$

where $Y = \frac{r}{s}$, X can only have prime factors $3, 5, 11, 19$, and $\gamma_1 = u_1$ if $u_1 \geq 1$, $\gamma_1 = 2$ if $u_1 = 0$. We use Magma to search for S-integral points on (3.31), where $S = \{3, 5, 11, 19\}$. The result is given in Table 4, where UD means Magma is not able to find S-integral points.

Table 3. Solutions to (3.31).

$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$	$(\gamma_1, u_2, u_3, u_4)$	(X, Y)
$(1,1,1,5)$	$(2,0,0,2)$	$\emptyset$
$(1,3,3,5)$	$(2,1,1,2)$	$\emptyset$
$(1,3,5,3)$	$(2,1,2,1)$	$\emptyset$
$(1,3,5,5)$	$(2,1,2,2)$	UD
$(1,5,3,3)$	$(2,2,1,1)$	$\emptyset$
$(1,5,5,3)$	$(2,2,2,1)$	UD
$(1,5,5,5)$	$(2,2,2,2)$	UD
$(3,1,5,3)$	$(0,0,2,1)$	$\emptyset$
$(3,1,5,5)$	$(0,0,2,2)$	$\emptyset$
$(3,3,1,5)$	$(0,1,0,2)$	$\emptyset$
$(3,3,5,3)$	$(0,1,2,1)$	$\emptyset$
$(3,5,3,5)$	$(0,2,1,2)$	UD

Case 6.1: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (3, 5, 3, 5)$. Equation (3.16) reduces to

$$4 \cdot 3^{3a_1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+1} \cdot 19^{3d_1+2} = s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2). \tag{3.32}$$

Case 6.1.1: $11 \mid r$. Then $3c_1 + 1 = 1$. Let $r = 11r_1$ and $s = 5^{3b_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $r_1, s_1 \in \mathbb{Z}$. Then (3.32) reduces to

$$4 \cdot 3^{3a_1} = s_1(11r_1^2 - 5^{6b_1+5} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.33}$$

- $s_1 = 1$. Then (3.33) reduces to

$$4 \cdot 3^{3a_1} = 11r_1^2 - 5^{6b_1+5} \cdot 19^{6d_1+5}.$$

Since $\left(\frac{3}{19}\right) = -1$ and $\left(\frac{11}{19}\right) = 1$, we have $2 \mid 3a_1$. Let $a_1 = 2a_2$, where $a_2 \in \mathbb{N}$. Then

$$4 \cdot 3^{6a_1} = 11r_1^2 - 5^{6b_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$4 \equiv 4r_1^2 - 2 \pmod 7,$$

impossible mod 7 since $\left(\frac{6}{7}\right) = -1$.

- $s_1 = -1$. Then (3.33) reduces to

$$4 \cdot 3^{3a_1} = 11r_1^2 - 5^{6b_1+5} \cdot 19^{6d_1+5}.$$

Since $\left(\frac{3}{19}\right) = \left(\frac{-11}{19}\right) = -1$, we have $3 \nmid a_1$. Hence, $a_1 = 2a_2 + 1$, where $a_2 \in \mathbb{N}$. Then

$$4 \cdot 3^{6a_1+3} = 11r_1^2 - 5^{6b_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$3 \equiv 4r_1^2 - 2 \pmod 7,$$

impossible mod 7 since $\left(\frac{5}{7}\right) = -1$.

Case 6.1.2: $11 \mid s$. Then $11^{3c_1+1} \mid s$. Let $s = 5^{3b_1+2} \cdot 11^{3c_1+1} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 \in \mathbb{Z}$. Then (3.33) reduces to

$$4 \cdot 3^{3a_1} = s_1(r^2 - 5^{6b_1+5} \cdot 11^{6c_1+3} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.34}$$

• $3 \nmid s_1$. Since $11 \nmid r$ and $\left(\frac{3}{11}\right) = 1$, from (3.34) we have

$$\left(\frac{s_1}{11}\right) = \left(\frac{s_1 r^2}{11}\right) = \left(\frac{4 \cdot 3^{3a_1}}{11}\right) = 1.$$

Since $s_1 \in \{-1, 1\}$, we have $s_1 = 1$. Then (3.34) reduces to

$$4 \cdot 3^{3a_1} = r^2 - 5^{6b_1+5} \cdot 11^{6c_1+3} \cdot 19^{6d_1+5}.$$

Hence, $\left(\frac{4 \cdot 3^{3a_1}}{19}\right) = 1$. Since $\left(\frac{-3}{19}\right) = -1$, we have $2 \mid a_1$. Let $a_1 = 2a_2$, where $a_2 \in \mathbb{N}$. Then

$$4 \cdot 3^{6a_2} = r^2 - 5^{6b_1+5} \cdot 11^{6c_1+3} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$4 \equiv r^2 - 2 \pmod 7,$$

impossible since $\left(\frac{6}{7}\right) = -1$.

Case 6.2: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 3, 5, 5)$. Then (3.33) reduces to

$$4 \cdot 3^{3a_1-1} \cdot 5^{3b_1+1} \cdot 11^{3d_1+2} \cdot 19^{3d_1+2} = s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2). \tag{3.35}$$

Case 6.2.1: $5 \mid r$. Since $5^2 \nmid s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2)$, we have $3b_1 + 1 = 1$. Let $r = 5r_1$ and $s = 11^{3c_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $r_1, s_1 \in \mathbb{Z}$. Then (3.35) reduces to

$$4 \cdot 3^{3a_1-1} = s_1(5r_1^2 - 11^{6c_1+5} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.36}$$

Notice that $\left(\frac{3}{11}\right) = \left(\frac{5}{11}\right) = 1$. Hence, (3.36) gives $\left(\frac{s_1}{11}\right) = 1$.

• $3 \nmid s_1$. Since $\left(\frac{-1}{11}\right) = -1$, we have $s_1 = 1$. Then (3.36) reduces to

$$4 \cdot 3^{3a_1-1} = 5r_1^2 - 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Hence,

$$\left(\frac{4 \cdot 3^{3a_1-1}}{19}\right) = \left(\frac{5r_1^2}{19}\right) = 1.$$

Since $\left(\frac{3}{19}\right) = -1$, we have $2 \mid 3a_1 - 1$. Hence,, $2 \nmid a_1$. Let $a_1 = 2a_2 + 1$, where $a_2 \in \mathbb{N}$. Then

$$4 \cdot 3^{6a_2+2} = 5r_1^2 - 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 5 gives $4(-1)^{3a_2+1} \equiv 1 \pmod 5$. Hence, $2 \mid a_2$. Let $a_2 = 2a_3$, where $a_3 \in \mathbb{N}$. Then

$$4 \cdot 3^{12a_3+2} = 5r_1^2 - 11^{6c_1+5} \cdot 19^{6d_1+5}. \tag{3.37}$$

Let $c_1 = 2c_2 + i_1$ and $d_1 = 2d_2 + i_2$ where $i_1, i_2 \in \{0, 1\}$. From (3.37) we have

$$Y^2 = X(X^2 + 5^3 \cdot 6^4 \cdot 11^{5+6i_1} \cdot 19^{5+6i_2}), \tag{3.38}$$

where $X = \frac{20 \cdot 3^{6a_1+2}}{11^{6c_2} \cdot 19^{6d_2}}$, $Y = \frac{100 \cdot 3^{3a_1+2} \cdot r_1}{11^{12c_2} \cdot 19^{12d_2}}$. Magma [3] shows that the only $\{11, 19\}$-integral point on (3.38) is $(0, 0)$. Hence, (3.37) has no solutions.

• $3 \mid s_1$. Since $\left(\frac{s_1}{11}\right) = 1$, we have $s_1 = 3^{3a_1-1}$, then (3.36) reduces to

$$4 = 5r_1^2 - 3^{6a_1-2} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5},$$

impossible mod 3 since $\left(\frac{5}{3}\right) = -1$.

Case 6.2.2: $5 \mid s$. Then $s = 5^{3b_1+1} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 \in \mathbb{Z}$. Then (3.35) reduces to

$$4 \cdot 3^{3a_1-1} = s_1(r^2 - 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.39}$$

• $3 \nmid s_1$. If $s_1 = 1$, then (3.39) reduces to

$$4 \cdot 3^{3a_1-1} = r^2 - 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}. \tag{3.40}$$

Hence, $\left(\frac{4 \cdot 3^{3a_1-1}}{19}\right) = \left(\frac{r^2}{19}\right) = 1$. Since $\left(\frac{3}{19}\right) = -1$, we have $2 \mid 3a_1 - 1$. Let $a_1 = 2a_2 + 1$, where $a_2 \in \mathbb{N}$. Then (3.40) reduces to

$$4 \cdot 3^{6a_2+2} = r^2 - 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 13 gives

$$10 \equiv r^2 - 8 \pmod{13},$$

impossible since $\left(\frac{18}{13}\right) = -1$.

If $s_1 = -1$, then (3.39) reduces to

$$4 \cdot 3^{3a_1-1} = 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5} - r^2. \tag{3.41}$$

Hence, $\left(\frac{4 \cdot 3^{3a_1-1}}{19}\right) = \left(\frac{-1}{19}\right) = -1$. Since $\left(\frac{3}{19}\right) = -1$, we have $2 \nmid 3a_1 - 1$. Let $a_1 = 2a_2$, where $a_2 \in \mathbb{N}$. Then (3.41) reduces to

$$4 \cdot 3^{6a_1-1} = 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5} - r^2,$$

impossible mod 5 since $\left(\frac{-3}{5}\right) = -1$.

• $3 \mid s_1$. Then $s_1 = 3^{3a_1-1} \cdot s_2$, where $s_2 \in \mathbb{Z}$. Hence, (3.39) reduces to

$$4 = s_2(r^2 - 3^{6a_1-2} \cdot 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5} \cdot s_2^2).$$

Hence, $s_2 r^2 \equiv 4 \pmod{19}$. Therefore $\left(\frac{s_2}{19}\right) = 1$. Thus, $s_2 = 1$. Then

$$4 = r^2 - 3^{6a_1-2} \cdot 5^{6b_1+3} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 13 gives

$$4 \equiv r^2 - 11 \pmod{13},$$

impossible since $\left(\frac{15}{13}\right) = -1$.

Case 6.3: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 5, 5, 3)$. Then (3.33) reduces to

$$4 \cdot 3^{3a_1-1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+1} = s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2). \tag{3.42}$$

Case 6.3.1: $19 \mid r$. Then $19 \nmid s$. Thus, $19^2 \nmid s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2)$. Thus, in (3.42), we must have $d_1 = 0$. So (3.42) reduces to

$$4 \cdot 3^{3a_1-1} = s_1(19r_1^2 - 5^{6b_1+5} \cdot 11^{6c_1+5}). \tag{3.43}$$

Since $\left(\frac{3}{11}\right) = 1$ and $\left(\frac{19}{11}\right) = -1$, we have from (3.43) that $\left(\frac{s_1}{11}\right) = -1$.

- $3 \nmid s_1$. Then $s_1 \in \{\pm 1\}$. Since $\left(\frac{s_1}{11}\right) = -1$, we have $s_1 = -1$. Therefore (3.43) reduces to

$$4 \cdot 3^{3a_1-1} = 5^{6b_1+5} \cdot 11^{6c_1+5} - 19 \cdot r_1^2. \tag{3.44}$$

Thus, $\left(\frac{4 \cdot 3^{3a_1-1}}{19}\right) = \left(\frac{5 \cdot 11}{19}\right) = 1$. Since $\left(\frac{3}{19}\right) = -1$, we have $2 \mid 3a_1 - 1$. Thus, $a_1 = 2a_2 + 1$, where $a_2 \in \mathbb{N}$. Then (3.44) reduces to

$$4 \cdot 3^{6a_1+2} = 5^{6b_1+5} \cdot 11^{6c_1+5} - 19 \cdot r_1^2.$$

Reducing mod 13 gives

$$10 \equiv 9 - 6 \cdot r_1^2 \pmod{13},$$

impossible since $\left(\frac{-6}{13}\right) = -1$

- $3 \mid s_1$. Then $s_1 \in \{\pm 3^{3a_1+1}\}$. Since $\left(\frac{s_1}{11}\right) = -1$, we have $s_1 = -3^{3a_1+1}$. Therefore (3.42) reduces to

$$4 = 3^{6a_1-2} \cdot 5^{6b_1+5} \cdot 11^{6c_1+5} - 19 \cdot r_1^2,$$

impossible mod 3 since $\left(\frac{-19}{3}\right) = -1$.

Case 6.3.2: $19 \mid s$. Then $s = 5^{3b_1+2} \cdot 11^{3b_1+2} \cdot 19^{3d_1+1} \cdot s_1$, where $s_1 \in \mathbb{Z}$. Then (3.42) reduces to

$$4 \cdot 3^{3a_1-1} = s_1(r^2 - 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.45}$$

Since $\left(\frac{3}{11}\right) = 1$, we have $\left(\frac{s_1}{11}\right) = 1$.

- $3 \nmid s_1$. Then $s_1 = 1$. Hence, (3.45) reduces to

$$4 \cdot 3^{3a_1-1} = r^2 - 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Since $\left(\frac{3}{19}\right) = -1$, we have $2 \mid 3a_1 - 1$. Let $a_1 = 2a_2 + 1$, where $a_2 \in \mathbb{N}$. Then

$$4 \cdot 3^{6a_2+2} = r^2 - 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$1 \equiv r^2 - 4 \pmod{7},$$

impossible mod 7 since $\left(\frac{5}{7}\right) = -1$.

- $3 \mid s_1$. Then $s_1 = 3^{3a_1-1}$. Hence, (3.45) reduces to

$$4 = r^2 - 3^{6a_1-2} \cdot 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$4 \equiv r^2 - 2 \pmod 7,$$

impossible since $\left(\frac{6}{7}\right) = -1$

Case 6.4: $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 5, 5, 5)$. Then (3.33) reduces to

$$4 \cdot 3^{3a_1-1} \cdot 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} = s(r^2 - 5 \cdot 11 \cdot 19 \cdot s^2). \tag{3.46}$$

Thus, $s = 5^{3b_1+2} \cdot 11^{3c_1+2} \cdot 19^{3d_1+2} \cdot s_1$, where $s_1 \in \mathbb{Z}$. Therefore

$$4 \cdot 3^{3a_1-1} = s_1(r^2 - 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5} \cdot s_1^2). \tag{3.47}$$

Since $\left(\frac{3}{11}\right) = 1$, we have $\left(\frac{s_1}{11}\right) = 1$. Notice that $\left(\frac{-1}{11}\right) = -1$.

- $3 \nmid s_1$. Then $s_1 = 1$. Hence, (3.47) reduces to

$$4 \cdot 3^{3a_1-1} = r^2 - 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$(-1)^{1+a_1} \equiv r^2 - 4 \pmod 7,$$

impossible since $\left(\frac{4\pm1}{7}\right) = -1$.

- $3 \mid s_1$. Then $s_1 = 3^{3a_1-1}$. Hence, (3.47) reduces to

$$4 = r^2 - 3^{6a_1-2} \cdot 5^{6b_1+5} \cdot 11^{6c_1+5} \cdot 19^{6d_1+5}.$$

Reducing mod 7 gives

$$4 \equiv r^2 - 2 \pmod 7,$$

impossible since $\left(\frac{6}{7}\right) = -1$. □

Lemma 3.2. *All solutions (n, a, b, c, d, x, y) with $3 \mid n$ and $n > 3$ to (3.1) are given in Table 4.*

Table 4. Solutions to (3.1) with $3 \mid n$, $n > 3$, and $\gcd(x, y) = 1$.

(n, a, b, c, d, x, y)
$(n, 1, 0, 0, 0, 1, 1)$
$(6, 3, 1, 0, 0, 11, 2)$
$(6, 3, 1, 2, 0, 7, 4)$
$(12, 3, 1, 2, 0, 7, 2)$
$(6, 5, 1, 0, 2, 781, 8)$
$(9, 5, 1, 0, 2, 781, 4)$
$(18, 5, 1, 0, 2, 781, 2)$

Proof. Let $n = 3k$, where $k \in \mathbb{Z}^+$ and $k > 1$. Let $y_1 = y^k$. Then (3.1) reduces to

$$x^2 + 3^a \cdot 5^b \cdot 11^c \cdot 19^d = 4y_1^3. \tag{3.48}$$

We apply Lemma 3.1 to equation (3.48). Notice that solutions in Table 2 are deduced from solutions in Table 1. For example, solution

$$(n, a, b, c, d, x, y) = (3, 3, 1, 0, 0, 11, 4)$$

from Table 1 gives us a solution

$$(n, a, b, c, d, x, y) = (6, 3, 1, 0, 0, 11, 2)$$

in Table 2. □

Lemma 3.3. *All solutions* (n, a, b, c, d, x, y) *to* (3.1) *with* $n = 4$ *are list in Table 5.*

Table 5. Solutions to (3.1) with $n = 4$.

(n, a, b, c, d, x, y)	(n, a, b, c, d, x, y)	(n, a, b, c, d, x, y)	(n, a, b, c, d, x, y)
$(4, 4, 0, 0, 1, 31, 5)$	$(4, 4, 8, 3, 0, 141407, 353)$	$(4, 0, 1, 1, 0, 3, 2)$	$(4, 0, 5, 1, 0, 1557, 28)$
$(4, 4, 1, 2, 1, 947, 26)$	$(4, 0, 1, 2, 1, 1147, 24)$	$(4, 0, 1, 3, 2, 237, 28)$	$(4, 0, 1, 1, 0, 7, 3)$
$(4, 8, 3, 4, 1, 270973, 524)$	$(4, 0, 3, 0, 1, 53, 6)$	$(4, 0, 3, 1, 2, 1923, 32)$	$(4, 1, 0, 0, 0, 1, 1)$
$(4, 5, 0, 4, 0, 7199, 61)$	$(4, 1, 4, 1, 1, 195937, 313)$	$(4, 1, 0, 2, 2, 65521, 181)$	$(4, 1, 1, 0, 0, 7, 2)$
$(4, 1, 2, 2, 0, 23, 7)$	$(4, 2, 0, 1, 0, 49, 5)$	$(4, 2, 4, 2, 1, 10033, 73)$	$(4, 2, 1, 0, 1, 13, 4)$
$(4, 2, 1, 1, 0, 23, 4)$	$(4, 6, 1, 1, , 0, 337, 14)$	$(4, 2, 2, 0, 1, 73, 7)$	$(4, 2, 2, 0, 1, 233, 11)$
$(4, 2, 3, 1, 2, 937, 34)$	$(4, 3, 1, 2, 0, 7, 8)$	$(4, 3, 2, 0, 0, 337, 13)$	

Proof. Let $a = 4a_1 + i_1$, $b = 4b_1 + i_2$, $c = 4c_1 + i_3$, $d = 4d_1 + i_4$, where $a_1, b_1, c_1, d_1 \in \mathbb{N}$ and $0 \le i_1, i_1, i_3, i_4 \le 3$. From (3.1) we have

$$Y^2 = 4X^4 - 3^{i_1} \cdot 5^{i_2} \cdot 11^{i_3} \cdot 19^{i_4}, \tag{3.49}$$

where $X = \frac{y}{3^{a_1} \cdot 5^{b_1} \cdot 11^{c_1} \cdot 19^{d_1}}$, $Y = \frac{x}{3^{2a_1} \cdot 5^{2b_1} \cdot 11^{2c_1} \cdot 19^{2d_1}}$, $a_1, b_1, c_1, d_1 \in \mathbb{N}$, $0 \le i_1, i_2, i_3, i_4 \le 3$, and $2 \nmid i_1 + i_3 + i_4$. Magma [3] is able to find S-integral points on (3.49) for all but the case $(i_1, i_2, i_3, i_4) = (3, 3, 3, 3)$, where $S = \{3, 5, 11, 19\}$. We list all cases of (i_1, i_2, i_3, i_4) where (3.49) has solutions in Table 6, the case $(i_1, i_2, i_3, i_4) = (3, 3, 3, 3)$ is undetermined (or UD).

Table 6. Solutions to (3.49).

(i_1, i_2, i_3, i_4)	(X, Y)	(n, a, b, c, x, y)
$(0, 0, 0, 1)$	$(\pm 5/3, \pm 31/9)$	$(4, 4, 0, 0, 1, 31, 5)$
$(0, 0, 3, 0)$	$(\pm 353/75, \pm 141407/5625)$	$(4, 4, 8, 3, 0, 141407, 353)$
$(0, 1, 1, 0)$	$(\pm 2, \pm 3)$	$(4, 0, 1, 1, 0, 3, 2)$
$(0, 1, 1, 0)$	$(\pm 28/5, \pm 1557/25)$	$(4, 0, 5, 1, 0, 1557, 28)$
$(0, 1, 2, 1)$	$(\pm 22/3, \pm 77/9)$	$\emptyset$
$(0, 1, 2, 1,)$	$(\pm 26/3, \pm 947/9)$	$(4, 4, 1, 2, 1, 947, 26)$

$(0,1,2,1)$	$(\pm 24, \pm 1147)$	$(4,0,1,2,1,1147,24)$
$(0,1,3,2)$	$(\pm 28, \pm 237)$	$(4,0,1,3,2,237,28)$
$(0,2,0,1)$	$(\pm 5, \pm 45)$	$\emptyset$
$(0,2,1,0)$	$(\pm 3, \pm 7)$	$(4,0,2,1,0,7,3)$
$(0,2,2,1)$	$(\pm 11, \pm 33)$	$\emptyset$
$(0,3,0,1)$	$(\pm 524/99, \pm 270973/9801)$	$(4,8,3,4,1,270973,524)$
$(0,3,0,1)$	$(\pm 6, \pm 53)$	$(4,0,3,0,1,53,6)$
$(0,3,1,2)$	$(\pm 32, \pm 1923)$	$(4,0,3,1,2,1923,32)$
$(1,0,0,0)$	$(\pm 1, \pm 1)$	$(4,1,0,0,0,1,1)$
$(1,0,0,0)$	$(\pm 61/33, \pm 7199/1089)$	$(4,5,0,4,0,7199,61)$
$(1,0,1,1)$	$(\pm 313/5, \pm 195937/25)$	$(4,1,4,1,1,195937,313)$
$(1,0,2,2)$	$(\pm 181, \pm 65521)$	$(4,1,0,2,2,65521,181)$
$(1,1,0,0)$	$(\pm 2, \pm 7)$	$(4,1,1,0,0,7,2)$
$(1,1,0,2)$	$(\pm 76/3, \pm 11533/9)$	$\emptyset$
$(1,1,1,1)$	$(\pm 28, \pm 1567)$	$\emptyset$
$(1,2,0,2)$	$(\pm 19, \pm 703)$	$\emptyset$
$(1,2,2,0)$	$(\pm 7, \pm 23)$	$(4,1,2,2,0,23,7)$
$(2,0,1,0)$	$(\pm 3, \pm 15)$	$\emptyset$
$(2,0,1,0)$	$(\pm 5, \pm 49)$	$(4,2,0,1,0,49,5)$
$(2,0,2,1)$	$(\pm 73/5, \pm 10033/25)$	$(4,2,4,2,1,10033,73)$
$(2,1,0,1)$	$(\pm 4, \pm 13)$	$(4,2,1,0,1,13,4)$
$(2,1,1,0)$	$(\pm 4, \pm 23)$	$(4,2,1,1,0,23,4)$
$(2,1,0,1)$	$(\pm 14/3, \pm 337/9)$	$(4,6,1,1,0,337,14)$
$(2,1,2,1)$	$(\pm 22, \pm 913)$	$\emptyset$
$(2,2,0,1)$	$(\pm 7, \pm 73)$	$(4,2,2,0,1,73,7)$
$(2,2,0,1)$	$(\pm 11, \pm 233)$	$(4,2,2,0,1,233,11)$
$(2,2,1,0)$	$(\pm 5, \pm 5)$	$\emptyset$
$(2,2,3,2)$	$(\pm 575/3, \pm 654595/9)$	$\emptyset$
$(2,2,3,2)$	$(\pm 775, \pm 1201205)$	$\emptyset$
$(2,3,1,2)$	$(\pm 34, \pm 937)$	$(4,2,3,1,2,937,34)$
$(3,1,2,0)$	$(\pm 8, \pm 7)$	$(4,3,1,2,0,7,8)$
$(3,2,0,0)$	$(\pm 13, \pm 337)$	$(4,3,2,0,0,337,13)$
$(3,3,3,3)$	UD	UD

We consider the case $(i_1, i_2, i_3, i_4) = (3,3,3,3)$. Then (3.1) reduces to

$$(2y^2 - x)(2y^2 + x) = 3^{4a_1+3} \cdot 5^{4b_1+3} \cdot 11^{4c_1+3} \cdot 19^{4d_1+3}.$$

Hence,

$$4y^2 = A_1 + B_1, \tag{3.50}$$

where $A_1, B_1 \in \mathbb{Z}^+$ ad $A_1 B_1 = 3^{4a_1+3} \cdot 5^{4b_1+3} \cdot 11^{4c_1+3} \cdot 19^{4d_1+3}$. Without loss of generality, we can assume that $3 \mid A_1$.

Case 1: $3 \mid A_1$ and $19 \mid B_1$. Then $3^{4a_1+3} \mid A_1$. Since $\left(\frac{5}{19}\right) = \left(\frac{11}{19}\right) = 1$ and $\left(\frac{3}{19}\right) = -1$, we have $\left(\frac{A_1}{19}\right) = -1$. Hence, equation (3.50) is impossible mod 19.

Case 2: $3 \cdot 19 \mid A_1$ and $5 \mid B_1$. Since $3^{4a_1+3} \cdot 19^{4b_1+3} \mid A_1$, $\left(\frac{11}{5}\right) = \left(\frac{19}{5}\right) = 1$ and $\left(\frac{3}{5}\right) = -1$, we have $\left(\frac{A_1}{5}\right) = -1$, impossible since we deduce from (3.50) that

$$\left(\frac{A_1}{5}\right) = \left(\frac{4y^2}{5}\right) = 1.$$

Case 3: $3 \cdot 5 \cdot 19 \mid A_1$ and $11 \mid B_1$. Then $A_1 = 3^{3a_1+3} \cdot 5^{3b_1+3} \cdot 19^{3d_1+3}$ and $B_1 = 11^{3b_1+3}$. Equation (3.50) becomes

$$4y^2 = 3^{4a_1+3} \cdot 5^{4b_1+3} \cdot 19^{4d_1+3} + 11^{4c_1+3},$$

impossible mod 11 since $\left(\frac{19}{11}\right) = -1$ and $\left(\frac{3}{11}\right) = \left(\frac{5}{11}\right) = 1$.

Case 4: $3 \cdot 5 \cdot 11 \cdot 19 \mid A_1$. Then $A_1 = 3^{4a_1+3} \cdot 5^{4b_1+3} \cdot 11^{4c_1+3} \cdot 19^{4d_1+3}$ and $B_1 = 1$. Equation (3.50) reduces to

$$4y^2 = 1 + 3^{4a_1+3} \cdot 5^{4b_1+3} \cdot 11^{4c_1+3} \cdot 19^{4d_1+3}.$$

Then $(2y, 3^{2a_1+1} \cdot 5^{2b_1+1} \cdot 11^{2c_1+1} \cdot 19^{2d_1+1})$ is a solution to the Pell equation

$$X^2 - 3 \cdot 5 \cdot 11 \cdot 19 \cdot Y^2 = 1. \tag{3.51}$$

The fundamental solution to (3.51) is $(X, Y) = (56, 1)$. We look for $k \in \mathbb{Z}^+$ such that

$$3^{2a_1+1} \cdot 5^{2b_1+1} \cdot 11^{2c_1+1} \cdot 19^{2d_1+1} = Y_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}, \tag{3.52}$$

where $\lambda_1 = 56 + \sqrt{3135}$ and $\lambda_2 = 56 - \sqrt{3135}$.

If $k > 30$, then from the work of Bilu, Hanrot, and Voutier [2] we know that Y_k has a primitive divisor q such that $k \mid q - \left(\frac{(\lambda_1-\lambda_2)^2}{q}\right)$, impossible since $q \in \{3, 5, 11, 19\}$ and $k > 30$.

Therefore $k \leq 30$. Checking the values of k in the range $1 \leq k \leq 30$ shows that (3.52) is impossible for all $1 \leq k \leq 30$.

We conclude that all solutions to (3.1) with $n = 4$ is given in Table 5. □

Lemma 3.4. *Solutions* (n, a, b, c, d, x, y) *to* (3.1) *with* $4 \mid n$ *and* $n > 4$ *are listed in Table 7.*

Table 7. Solutions to (3.1) with $n4 \mid n$, $n > 4$, and $\gcd(x, y) = 1$.

(n, a, b, c, d, x, y)
$(n, 10, 0, 0, 1, 1)$
$(8, 2, 1, 0, 1, 13, 2)$
$(8, 2, 1, 1, 0, 23, 2)$
$(12, 3, 1, 2, 0, 7, 2)$

Proof. Let $n = 4k$, where $k \in \mathbb{Z}^+$ and $k > 1$. Let $y_1 = y^k$. Then

$$x^2 + 3^a \cdot 5^b \cdot 11^c \cdot 19^d = 4y_1^4. \tag{3.53}$$

We use Table 5 in Lemma 3.3 to find solutions to (3.53) and get Table 7. □

Lemma 3.5. *All solutions to* (3.1) *with* $n \geq 5$, $3 \nmid n$, $4 \nmid n$, $(a, b, c, d) \neq (1, 1, 1, 1) \pmod 2$, *and* $\gcd(x, y) = 1$ *are given in Table 8.*

Table 8. Solutions to (3.1) with $n \geq 5$, $3 \nmid n$, $4 \nmid n$, and $\gcd(x, y) = 1$.

(n, a, b, c, d, x, y)
$(n, 1, 0, 0, 0, 1, 1)$
$(5, 2, 0, 5, 2, 38599, 55)$
$(5, 0, 4, 5, 2, 41261, 99)$
$(5, 0, 3, 5, 0, 25289, 44)$

Proof. We can assume that n is an odd prime ≥ 5. Then

$$\frac{x + B\sqrt{-A}}{2} = \left(\frac{r + s\sqrt{-A}}{2}\right)^n,$$

where r, s are odd coprime integers. Therefore

$$\frac{B}{s} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = L_n,$$

where $\alpha = \frac{r+s\sqrt{-A}}{2}$ and $\beta = \frac{r-\sqrt{-A}}{2}$. If $\frac{\alpha}{\beta}$ is a root of unity, then $\frac{\alpha}{\beta} = \zeta_m$, a primitive m-root of unity. Since $|\mathbb{Q}(\zeta_m) : \mathbb{Q}| = \phi(m)$ and $\mathbb{Q}(\frac{\alpha}{\beta}) = 2$, we have $\phi(m) = 2$. Therefore $m \in \{3, 4\}$. Hence, $\zeta_m \in \left\{\pm\sqrt{-1}, \frac{\pm 1 \pm \sqrt{-3}}{2}\right\}$. Therefore $A = 3$ and $\alpha = \pm\frac{1 \pm \sqrt{-3}}{2}$. Hence, $y = 1$. We deduce that $(n, a, b, c, d, x, y) = (n, 1, 0, 0, 0, 1, 1)$.

We consider the case $\frac{\alpha}{\beta}$ is not a root of unity. If L_n has a primitive divisor q, then $n \mid q - \left(\frac{(\alpha-\beta)^2}{q}\right)$. Since $q \in \{3, 5, 11, 19\}$ and $n \geq 5$, we deduce that $n = 5$, $q = 19$, and $\left(\frac{(\alpha-\beta)^2}{q}\right) = -1$. Since $(\alpha - \beta)^2 = -As^2$, we have $\left(\frac{-A}{19}\right) = -1$. Since $19 \nmid A$ and $A \in \{3, 11, 15, 19, 55, 95, 627\}$, we have $A \in \{11, 55\}$. Let $B = 3^i \cdot 5^j \cdot 11^k \cdot 19^l$, where $i, j, k, l \in \mathbb{N}$. Since

$$B = \frac{\alpha^5 - \beta^5}{\sqrt{-A}},$$

we have

$$16B = s(5r^4 - 10Ar^2s^2 + A^2s^4). \tag{3.54}$$

Notice that $s \mid B$, so s only has prime divisors $3, 5, 11, 19$. Dividing both sides of (3.54) by s^5 gives a quartic curve

$$\gamma Y^2 = 5X^4 - 10AX^2 + A^2, \tag{3.55}$$

where $\gamma = \pm 3^{i_1} \cdot 5^{i_2} \cdot 11^{i_3} \cdot 19^{i_4}$, $i_1, i_2, i_3, i_4 \in \{0, 1\}$, $X = \frac{r}{s}$, $Y \in \mathbb{Q}$ and Y only has prime divisors $3, 5, 11, 19$. We use Magma to search for S-integral points on (3.55), where $S = \{3, 5, 11, 19\}$. We list the value of γ where (3.55) has S-integral points and the corresponding tuples (n, a, b, c, d, x, y) in Table 9. □

Table 9. Solutions to (3.55).

A	(3.55)	(X, Y)	(r, s)	(n, a, b, c, d, x, y)
11	$-19Y^2 = 5X^4 - 110X^2 + 121$	$(\pm 11/3, \pm 44/9)$	$(\pm 11, \pm 3)$	$(5, 2, 0, 5, 2, 38599, 55)$
11	$-95Y^2 = 5X^4 - 110X^2 + 121$	$(\pm 78/25, \pm 1399/625)$	$(\pm 78, \pm 25)$	$\emptyset$
		$(\pm 11/5, \pm 44/25)$	$(\pm 11, \pm 5)$	$(5, 0, 4, 5, 2, 41261, 99)$
		$(\pm 8/5, \pm 29/25)$	$(\pm 8, \pm 5)$	$\emptyset$
55	$Y^2 = X^4 - 110X^2 + 605$	$(\pm 11, \pm 40)$	$(\pm 11, \pm 1)$	$(5, 0, 3, 5, 0, 25289, 44)$

Remark 3.6. We need the condition $5 \nmid h(\mathbb{Q}(\sqrt{-A}))$ in the proof of Lemma 3.5. Since the class number of $\mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 11 \cdot 19})$ is 40, the condition $(a, b, c, d) \not\equiv (1, 1, 1, 1) \pmod 2$ in Lemma 3.5 and Theorem 1.1 is indispensable.

When $(a, b, c, d) = (1, 1, 1, 1) \pmod 2$, then $a = 10a_1 + i_1$, $b = 10b_1 + i_2$, $c = 10c_1 + i_3$, and $d = 10d_1 + i_4$, where $a_1, b_1, c_1, d_1 \in \mathbb{N}$ and $i_1, i_2, i_3, i_4 \in \{1, 3, 5, 7, 9\}$. Then (3.1) reduces to

$$Y^2 = 4X^5 + 3^{i_1} \cdot 5^{i_2} \cdot 11^{i_3} \cdot 19^{i_4}, \tag{3.56}$$

where

$$Y = \frac{x}{3^{5i_1} \cdot 5^{5i_2} \cdot 11^{5i_3} \cdot 19^{5i_4}} \quad \text{and} \quad X = \frac{y}{3^{2i_1} \cdot 5^{2i_2} \cdot 11^{2i_3} \cdot 19^{2i_4}}.$$

Equation (3.56) represents a curve of genus 2, and we need to find $\{3, 5, 11, 19\}$-integral points on this curve. It might be possible to attack (3.56) using the method in [5] but the author of this paper has not been able to proceed in this way.

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References

[1] S. Bhatter, A. Hoque, R. Sharma: *On the solutions of a Lebesgue–Nagell type equation*, Acta Mathematica Hungarica 158.1 (2019), pp. 17–26, doi: https://doi.org/10.1007/s10474-018-00901-6.

[2] Y. Bilu, G. Hanrot, P. M. Voutier: *Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte*, Journal für die reine und angewandte Mathematik 539.3 (2001), pp. 75–122, doi: https://doi.org/10.1515/crll.2001.080.

[3] W. Bosma, J. Cannon, C. Playoust: *The Magma algebra system. I. The user language*, Journal of Symbolic Computation 24.3-4 (1997), pp. 235–265, doi: https://doi.org/10.1006/jsco.1996.0125.

[4] K. Chakraborty, A. Hoque, R. Sharma: *Complete solutions of certain Lebesgue–Ramanujan–Nagell type equations*, Publicationes Mathematicae Debrecen 97.3-4 (2020), url: http://publi.math.unideb.hu/load_jpg.php?p=2405.

[5] H. R. Gallegos-Ruiz: *S-integral points on hyperelliptic curves*, International Journal of Number Theory 7.3 (2011), pp. 803–824, doi: https://doi.org/10.1142/S1793042111004435.

[6] M. Le, G. Soydan: *A brief survey on the generalized Lebesgue–Ramanujan–Nagell equation*, Surveys in Mathematics and its Applications 15 (2020), pp. 473–523, URL: https://www.utgjiu.ro/math/sma/v15/p15_20.pdf.

[7] F. Luca, S. Tengely, A. Togbé: *On the Diophantine equation* $x^2 + C = 4y^n$, Annales mathématiques du Québec 33.2 (2009), pp. 171–184, URL: http://www.labmath.uqam.ca/~annales/volumes/33-2/PDF/171-184.pdf.