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INTEGRAL INEQUALITIES FOR EXPONENTIALLY HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL OPERATORS

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Abstract. The main aim of this paper is to derive some new integral inequalities related to Hermite-Hadamard type by using Riemann-Liouville fractional integral operator for the class of exponentially harmonically convex functions. The formal technique of this paper may enhance further research in this field.

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1. INTRODUCTION AND PRELIMINARIES

Theory of convex functions had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general frame work to study a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions, see [2, 4–10, 13, 15–18, 23–30].

The class of harmonic convex function was introduced by Anderson *et al.* in [2] and in [8], İşcan has proved some new integral inequalities for this class of functions. It is natural to unify these different concepts.

An important class of convex functions, which is called an exponentially convex functions, was introduced and studied by Antczak in [3], Dragomir *et al.* in [6] and Noor *et al.* in [17]. In [1], Alirezai and Mathar have investigated their mathematical properties along with their potential applications in statistics and information theory, see [1, 19]. Due to its significance, in [4], Awan et al and also in [20], Pecaric and Jaksetic defined another kind of exponential convex functions, have shown that the class of exponential convex functions unifies various concepts in different manners.

The advantages of fractional calculus have been described and pointed out in the last few decades by many authors. Fractional calculus is based on derivatives and integrals of fractional order, fractional differential equations and methods of their solution. The most celebrated inequality has been studied extensively since it was established by Hermite, is the Hermite-Hadamard inequality not only established for classical integrals but also for fractional integrals, see [5, 10–14, 22, 25, 26, 28].

We now recall some known basic results and concepts, which are necessary to obtain the main results.

Definition 1 ([8]). A function $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y),\ \forall x,y\in I,\ t\in[0,1].$$

We now define the concept of exponentially convex function, which is mainly due to Antczak [3], Dragomir [6] and Noor et al [17].

Definition 2 ([3, 6, 17]). Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is an exponentially convex function, if f is positive, $\forall a,b\in I$ and $t\in[0,1]$, we have

$$e^{f((1-t)a+tb)} \leq [(1-t)e^{f(a)}+te^{f(b)}], \quad a,b\in I,\ t\in[0,1]. \tag{1.1}$$

We recall the following special functions and inequality.

(1) The Gamma function:

$$\Gamma(x,y) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x,y>0$$

(2) The Beta function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,y>0$$

(3) The hypergeometric function (see [15]):

$${}_2F(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad c>b>0, \quad |z|<1.$$

Lemma 1 ([22, 30]). *For $0<\alpha\leq 1$ and $0\leq a<b$, we have*

$$|a^\alpha - b^\alpha| \leq (b-a)^\alpha.$$

We now give the definition of the fractional integral, which is mainly due to [21].

Definition 3 ([21]). Let $\alpha>0$ with $n-1<\alpha\leq n,\ n\in\mathbb{N}$, and $1<x<v$. The left- and right-hand side Riemann-Liouville fractional integrals of order α of function f are given by

$$J^\alpha_{u_+}f(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-t)^{\alpha-1}f(t)dt, \tag{1.2}$$

and

$$J^{\alpha}_{v_-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t)dt, \tag{1.3}$$

where $\Gamma(\alpha)$ is the Gamma function.

In this article, we aim to study a new class of harmonically convex functions, which is called exponentially harmonically convex functions. We also derive some new integral inequalities via Riemann-Liouville fractional integrals for exponentially harmonically convex functions by using a new integral identity. Innovative ideas and techniques of this paper may stimulate further research in this dynamic field.

2. MAIN RESULTS

In this section, we derive our main results. Firstly, we will define the concept of exponentially harmonic convex functions, which is the main motivation of this paper.

Definition 4. A function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be an exponentially harmonically convex function, if

$$e^{f\left(\frac{xy}{tx+(1-t)y}\right)} \leq (1-t)e^{f(x)} + te^{f(y)}, \ \forall x,y \in I, \ t \in [0,1]. \tag{2.1}$$

Also note that for $t = \frac{1}{2}$ in Definition 4, we have Jensen type exponentially harmonic convex functions with

$$e^{f\left(\frac{2xy}{x+y}\right)} \leq \frac{1}{2}[e^{f(x)} + e^{f(y)}], \ \forall x,y \in I.$$

Theorem 1. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an exponentially harmonic convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then one has the following inequalities:*

$$e^{f(\frac{2ab}{a+b})} \leq \frac{\Gamma(\alpha+1)}{2} \big(\frac{ba}{b-a}\big)^{\alpha} \big[J^{\alpha}_{\frac{1}{a}-} e^{f\circ g(\frac{1}{b})} + J^{\alpha}_{\frac{1}{b}+} e^{f\circ g(\frac{1}{a})}\big] \leq \frac{e^{f(a)} + e^{f(b)}}{2}. \tag{2.2}$$

Proof. Since f is an exponentially harmonic convex function, for $t = \frac{1}{2}$ in inequality (2.1), we have

$$e^{f(\frac{2xy}{x+y})} \leq \frac{1}{2}[e^{f(x)+e^{f(y)}}], \quad \forall x,y \in I.$$

Substituting $x = \frac{ab}{tb+(1-t)a}$, $y = \frac{ab}{ta+(1-t)b}$, we get

$$e^{f(\frac{2ab}{a+b})} \leq \frac{1}{2}[e^{f(\frac{ab}{tb+(1-t)a})} + e^{f(\frac{ab}{ta+(1-t)b})}]. \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{e^{f(\frac{2ab}{a+b})}}{\alpha} \leq \frac{1}{2}\left[\int_0^1 t^{\alpha-1} e^{f(\frac{ab}{tb+(1-t)a})} dt + \int_0^1 t^{\alpha-1} e^{f(\frac{ab}{ta+(1-t)b})} dt\right]$$

$$= \frac{\alpha}{2}\Big(\frac{ba}{b-a}\Big)^{\alpha}\Bigg[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\alpha-1} e^{f(\frac{1}{x})}dx + \int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{a}-x\right)^{\alpha-1} e^{f(\frac{1}{x})}dx\Bigg]$$

$$= \frac{\alpha\Gamma(\alpha)}{2}\Big(\frac{ba}{b-a}\Big)^{\alpha}\big[J^{\alpha}_{\frac{1}{a}-} e^{f\circ g(\frac{1}{b})} + J^{\alpha}_{\frac{1}{b}+} e^{f\circ g(\frac{1}{a})}\big]$$

$$= \frac{\Gamma(\alpha+1)}{2}\Big(\frac{ba}{b-a}\Big)^{\alpha}\big[J^{\alpha}_{\frac{1}{a}-} e^{f\circ g(\frac{1}{b})} + J^{\alpha}_{\frac{1}{b}+} e^{f\circ g(\frac{1}{a})}\big].$$

This completes the proof of the left hand side of (2.2).

Since f is an exponentially harmonic convex function, then we can write

$$e^{f\left(\frac{ab}{tb+(1-t)a}\right)} \leq te^{f(a)} + (1-t)e^{f(b)}$$

and

$$e^{f\left(\frac{ab}{ta+(1-t)b}\right)} \leq te^{f(b)} + (1-t)e^{f(a)}.$$

By adding these inequalities, we have

$$e^{f\left(\frac{ab}{tb+(1-t)a}\right)} + e^{f\left(\frac{ab}{ta+(1-t)b}\right)} \leq e^{f(a)+e^{f(b)}}.$$

Multiplying both sides of the above inequality by $t^{\alpha-1}$, and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}e^{f\left(\frac{ab}{tb+(1-t)a}\right)}dt + \int_0^1 t^{\alpha-1}e^{f\left(\frac{ab}{ta+(1-t)b}\right)}dt \leq \int_0^1 t^{\alpha-1}[e^{f(a)+e^{f(b)}}]dt.$$

Thus

$$\Gamma(\alpha+1)\Big(\frac{ba}{b-a}\Big)^{\alpha}\big[J^{\alpha}_{\frac{1}{a}-} e^{f\circ g(\frac{1}{b})} + J^{\alpha}_{\frac{1}{b}+} e^{f\circ g(\frac{1}{a})}\big] \leq [e^{f(a)+e^{f(b)}}],$$

which completes the proof of the right hand side of (2.2). □

Now, we are in a position that we can discuss a special new case of Theorem 1.

Corollary 1. *If we take $\alpha = 1$, then we have a new Hadamard type result for harmonically convex functions:*

$$e^{f(\frac{2ab}{a+b})} \leq \frac{ab}{b-a}\int_a^b \frac{e^{f(x)}}{x^2}dx \leq \frac{e^{f(a)}+e^{f(b)}}{2}.$$

Lemma 2. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $f \in L[a,b]$, then the following equality holds:*

$$\Phi_f(g;\alpha,a,b) = \frac{ab(b-a)}{2}\int_0^1 \frac{[t^{\alpha}-(1-t)^{\alpha}]}{M_t^2}e^{f\left(\frac{ab}{M_t}\right)}f'\Big(\frac{ab}{M_t}\Big)dt, \qquad (2.4)$$

where

$$\Phi_f(g;\alpha,a,b) = \frac{e^{f(a)}+e^{f(b)}}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}-}\, e^{f\circ g(\frac{1}{b})} + J^{\alpha}_{\frac{1}{b}+}\, e^{f\circ g(\frac{1}{a})}\right],$$

$M_t = ta+(1-t)b$ *and* $g(x)=\frac{1}{x}$.

Proof. Consider

$$\begin{aligned} I &= \frac{ab(b-a)}{2}\int_0^1 \frac{[t^{\alpha}-(1-t)^{\alpha}]}{M_t^2} e^{f\left(\frac{ab}{M_t^2}\right)} f'\Big(\frac{ab}{M_t^2}\Big)dt \qquad (2.5)\\ &= \frac{ab(b-a)}{2}\int_0^1 \frac{t^{\alpha}}{M_t^2} e^{f\left(\frac{ab}{M_t^2}\right)} f'\Big(\frac{ab}{M_t^2}\Big)dt - \frac{ab(b-a)}{2}\int_0^1 \frac{(1-t)^{\alpha}}{M_t^2} e^{f\left(\frac{ab}{M_t^2}\right)} f'\Big(\frac{ab}{M_t^2}\Big)dt\\ &= I_1+I_2. \end{aligned}$$

By applying integration by parts, we have

$$\begin{aligned} I_1 &= \frac{1}{2}\left[t^{\alpha} e^{f\left(\frac{ab}{M_t}\right)}\Big|_0^1 - \alpha\int_0^1 t^{\alpha-1} e^{f\left(\frac{ab}{M_t}\right)}dt\right] \qquad (2.6)\\ &= \frac{1}{2}\left[e^{f(b)} - \alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha} e^{f(\frac{1}{x})}dx\right]\\ &= \frac{1}{2}\left[e^{f(b)} - \alpha\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}+}\, e^{f\circ g(\frac{1}{a})}\right], \end{aligned}$$

similarly, we obtain

$$\begin{aligned} I_2 &= \frac{1}{2}\left[t^{\alpha} e^{f\left(\frac{ab}{M_t}\right)}\Big|_0^1 - \alpha\int_0^1 t^{\alpha-1} e^{f\left(\frac{ab}{M_t}\right)}dt\right] \qquad (2.7)\\ &= \frac{1}{2}\left[e^{f(b)} - \alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\alpha} e^{f(\frac{1}{x})}dx\right]\\ &= \frac{1}{2}\left[e^{f(b)} - \alpha\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}+}\, e^{f\circ g(\frac{1}{a})}\right]. \end{aligned}$$

Using (2.6) and (2.7) in (2.5), we get (2.4). □

Theorem 2. *Let* $f: I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be an exponentially differentiable function on the interior* I° *of* I *such that* $f'\in L[a,b]$*, where* $a,b\in I^{\circ}$ *with* $a<b$*. If* $|f'|^q$ *is*

harmonically convex, for some fixed $q \geq 1$, then the following inequality holds for Riemann-Liouville integral operators:

$$\Phi_f(g;\alpha,a,b) \leq \frac{ab(b-a)}{2}\Big(\eta_1(\alpha;a,b)\Big)^{1-\frac{1}{q}} \tag{2.8}$$
$$\Big[\eta_2(\alpha;a,b)|e^{f(a)}f'(a)|^q+\eta_3(\alpha;a,b)|e^{f(b)}f'(b)|^q+\eta_3(\alpha;a,b)\Delta_1(a,b)\Big]^{\frac{1}{q}},$$

where

$$\Delta_1(a,b) = |e^{f(b)}f'(a)|^q+|e^{f(a)}f'(b)|^q$$
$$\eta_1(\alpha;a,b) = \frac{1}{b^2(\alpha+1)}\Big[{}_2F_1\big(2,1;\alpha+2;1-\frac{a}{b}\big)+{}_2F_1\big(2,\alpha+1;\alpha+2;1-\frac{a}{b}\big)\Big],$$
$$\eta_2(\alpha;a,b) = \frac{1}{b^2(\alpha+3)}$$
$$\Big[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\big(2,3;\alpha+4;1-\frac{a}{b}\big)+2{}_2F_1\big(2,\alpha+3;\alpha+3;1-\frac{a}{b}\big)\Big],$$
$$\eta_3(\alpha;a,b) = \frac{1}{b^2(\alpha+3)}$$
$$\Big[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\big(2,\alpha+1;\alpha+4;1-\frac{a}{b}\big)+{}_2F_1\big(2,1;\alpha+4;1-\frac{a}{b}\big)\Big],$$
$$\eta_4(\alpha;a,b) = \frac{1}{b^2(\alpha+3)(\alpha+4)}$$
$$\Big[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\big(2,2;\alpha+4;1-\frac{a}{b}\big)+{}_2F_1\big(2,\alpha+2;\alpha+4;1-\frac{a}{b}\big)\Big].$$

Proof. Using Lemma 2, the property of modulus, the power mean inequality and by using harmonically convexity of $|e^f f'|$, we have

$$\Phi_f(g;\alpha,a,b) \leq \frac{ab(b-a)}{2}\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\Big|e^{f\left(\frac{ab}{M_t}\right)}f'\big(\frac{ab}{M_t}\big)\Big|dt$$
$$\leq \frac{ab(b-a)}{2}\Big(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\Big|e^{f\left(\frac{ab}{M_t}\right)}f'\Big(\frac{ab}{M_t}\Big)\Big|dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{ab(b-a)}{2}\Big(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\Big[t^2|e^{f(b)}f'(b)|^q$$
$$+(1-t)^2|e^{f(a)}f'(a)|^q+t(1-t)\big[|e^{f(b)}f'(a)|^q+|e^{f(a)}f'(b)|^q\big]\Big]dt\Big)^{\frac{1}{q}}.$$

Namely,

$$\Phi_f(g;\alpha,a,b) \leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt\right)^{1-\frac{1}{q}} \tag{2.9}$$

$$\left(\int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}\left[t^2|e^{f(b)}f'(b)|^q+(1-t)^2|e^{f(a)}f'(a)|^q+t(1-t)\Delta_1(a,b)\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{2}\Big(\eta_1(\alpha;a,b)\Big)^{1-\frac{1}{q}}$$

$$\left[\eta_2(\alpha;a,b)|e^{f(a)}f'(a)|^q+\eta_3(\alpha;a,b)|e^{f(b)}f'(b)|^q+\eta_3(\alpha;a,b)\Delta_1(a,b)\right]^{\frac{1}{q}}.$$

By calculating $\eta_1(\alpha;a,b)$, $\eta_2(\alpha;a,b)$, $\eta_3(\alpha;a,b)$ and $\eta_4(\alpha;a,b)$, we obtain

$$\eta_1(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}dt \tag{2.10}$$

$$= \frac{1}{b^2(\alpha+1)}\left[{}_2F_1\big(2,1;\alpha+2;1-\frac{a}{b}\big)+{}_2F_1\big(2,\alpha+1;\alpha+2;1-\frac{a}{b}\big)\right]$$

$$\eta_2(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}t^2dt \tag{2.11}$$

$$= \frac{1}{b^2(\alpha+3)}\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\big(2,3;\alpha+4;1-\frac{a}{b}\big)+2{}_2F_1\big(2,\alpha+3;\alpha+3;1-\frac{a}{b}\big)\right]$$

$$\eta_3(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}(1-t)^2dt \tag{2.12}$$

$$= \frac{1}{b^2(\alpha+3)}\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\big(2,\alpha+1;\alpha+4;1-\frac{a}{b}\big)+{}_2F_1\big(2,1;\alpha+4;1-\frac{a}{b}\big)\right]$$

$$\eta_4(\alpha;a,b) = \int_0^1 \frac{|t^\alpha-(1-t)^\alpha|}{M_t^2}(1-t)^2dt = \frac{1}{b^2(\alpha+3)(\alpha+4)} \tag{2.13}$$

$$\left[\frac{2}{(\alpha+1)(\alpha+2)}{}_2F_1\big(2,2;\alpha+4;1-\frac{a}{b}\big)+{}_2F_1\big(2,\alpha+2;\alpha+4;1-\frac{a}{b}\big)\right].$$

Thus, if we use (2.10), (2.11), (2.12) and (2.13) in (2.9), we obtain the required inequality (2.8). □

Theorem 3. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex, for some fixed $q > 1$ and $p^{-1}+q^{-1} = 1$, then the following inequality holds for Riemann-Liouville integral operators:*

$$\Phi_f(g;\alpha,a,b) \le \frac{a(b-a)}{2b}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\frac{2\left(|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\right)+\Delta_1(a,b)}{6}\right]^{\frac{1}{q}} \times \left[{}_2F_1^{\frac{1}{p}}\left(2p,1;p\alpha+2;1-\frac{a}{b}\right)+{}_2F_1^{\frac{1}{p}}\left(2p,p\alpha+1;p\alpha+2;1-\frac{a}{b}\right)\right]. \tag{2.14}$$

Proof. Using Lemma 2, the Hölder inequality and the exponentially harmonically convexity of $|f'|$, we find

$$\begin{aligned}
&\Phi_f(g;\alpha,a,b) \qquad (2.15)\\
&\le \frac{ab(b-a)}{2}\left[\int_0^1 \frac{(1-t)^\alpha}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt+\int_0^1 \frac{t^\alpha}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right]\\
&\le \frac{ab(b-a)}{2}\left[\left(\int_0^1 \frac{(1-t)^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1 \frac{t^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}\right.\\
&\le \frac{ab(b-a)}{2}\left\{\left(\int_0^1 \frac{(1-t)^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}+\left(\int_0^1 \frac{t^{p\alpha}}{M_t^2}dt\right)^{\frac{1}{p}}\right\}\left(\int_0^1 \left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right)^{\frac{1}{q}}\\
&\le \frac{ab(b-a)}{2}\left(\zeta_1^{\frac{1}{p}}+\zeta_2^{\frac{1}{p}}\right)\\
&\quad\left(\int_0^1 \left[(1-t)^2|e^{f(a)}f'(a)|^q+t^2|e^{f(b)}f'(b)|^q+t(1-t)\Delta_1(a,b)\right]dt\right)^{\frac{1}{q}}\\
&\le \frac{ab(b-a)}{2}\left(\zeta_1^{\frac{1}{p}}+\zeta_2^{\frac{1}{p}}\right)\left[\frac{2\left(|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\right)+\Delta_1(a,b)}{6}\right]^{\frac{1}{q}}.
\end{aligned}$$

Calculating ζ_1 and ζ_2, we have

$$\zeta_1 = \int_0^1 \frac{(1-t)^{p\alpha}}{M_t^2}dt = \frac{b^{-2p}}{1+\alpha p}\,{}_2F_1\left(2p,1;p\alpha+2;1-\frac{a}{b}\right), \tag{2.16}$$

$$\zeta_2 = \int_0^1 \frac{t^{p\alpha}}{M_t^2}dt = \frac{b^{-2p}}{1+\alpha p}{}_2F_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{a}{b}\right). \tag{2.17}$$

Thus, if we use (2.16) and (2.17) in (2.15), we obtain inequality (2.14).

This completes the proof. □

Theorem 4. *Let $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex, for some fixed $q > 1$ and $p^{-1}+q^{-1} = 1$, then the following inequality holds for Riemann-Liouville integral operators:*

$$\left|\Phi_f(g;\alpha,a,b)\right| \le \frac{a(b-a)}{2b}\frac{{}_2F_1^{\frac{1}{p}}(2p,1;2;1-\frac{a}{b})}{2^{\frac{1}{q}}\left((q\alpha+1)(q\alpha+2)(q\alpha+3)\right)} \tag{2.18}$$
$$\times\left[(q\alpha+2)^2|e^{f(a)}f'(a)|^q + |e^{f(b)}f'(b)|^q + (q\alpha+2)\Delta_1(a,b)\right]^{\frac{1}{q}}.$$

Proof. Using Lemma 2, the Hölder inequality and the exponentially harmonically convexity of $|f'|$, we can write

$$\left|\Phi_f(g;\alpha,a,b)\right| \le \frac{ab(b-a)}{2}\left[\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right. \tag{2.19}$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|^q\right)^{\frac{1}{q}}$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{M_t^2}dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|^{q\alpha}\left[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q\right.\right.$$
$$\left.\left.+t(1-t)\{|e^{f(a)}f'(b)|^q + |e^{f(b)}f'(a)|^q\}\right]\right)^{\frac{1}{q}}$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{M_t^2}dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1 |1-2t|^{q\alpha}\left[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q + t(1-t)\Delta_1(a,b)\right]\right)^{\frac{1}{q}}$$
$$\le \frac{ab(b-a)}{2}\zeta_3^{\frac{1}{p}}\left(\zeta_4|e^{f(a)}f'(a)|^q + \zeta_5|e^{f(b)}f'(b)|^q + \zeta_6\Delta_1(a,b)\right)^{\frac{1}{q}},$$

where

$$\zeta_3=\int_0^1 \frac{1}{M_t^2}dt=b^{-2p}\int_0^1\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2p}dt=b_2^{-2p}F_1\left(2p,1;2;1-\frac{a}{b}\right),$$

$$\zeta_4=\int_0^1 |1-2t|^{q\alpha}t^2dt=\frac{q\alpha+2}{2(q\alpha+1)(q\alpha+3)},$$

$$\zeta_5=\int_0^1 |1-2t|^{q\alpha}(1-t)^2dt=\frac{1}{2(q\alpha+1)(q\alpha+2)(q\alpha+3)},$$

and

$$\zeta_6=\int_0^1 |1-2t|^{q\alpha}(1-t)^2dt=\frac{1}{2(q\alpha+1)(q\alpha+3)}.$$

By computing ζ_3, ζ_4, ζ_5 and ζ_6, we obtain inequality (2.18). □

Theorem 5. *Let $f: I\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be an exponentially differentiable function on the interior I° of I such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex, for some fixed $q>1$, then the following inequality holds for Riemann-Liouville integral operators:*

$$\Phi_f(g;\alpha,a,b)\le\frac{a(b-a)}{2.6^{\frac{1}{q}}b}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(2\left[{}_2F_1\left(2q,3;4;1-\frac{a}{b}\right)|e^{f(a)}f'(a)|^q\right.\right. \tag{2.20}$$
$$\left.\left.+{}_2F_1\left(2q,1;4;1-\frac{a}{b}\right)|e^{f(b)}f'(b)|^q\right]+{}_2F_1\left(2q,2;4;1-\frac{a}{b}\right)\Delta_1(a,b)\right)^{\frac{1}{q}}.$$

Proof. By using Lemma 2, the Hölder inequality and the exponentially harmonically convexity of $|f'|$, we have

$$\Phi_f(g;\alpha,a,b)\le\frac{ab(b-a)}{2}\left[\int_0^1\frac{|(1-t)^\alpha-t^\alpha|}{M_t^2}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|dt\right. \tag{2.21}$$
$$\le\frac{ab(b-a)}{2}\left(\int_0^1|(1-t)^\alpha-t^\alpha|^pdt\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{M_t^{2q}}\left|e^{f\left(\frac{ab}{M_t}\right)}f'\left(\frac{ab}{M_t}\right)\right|^q\right)^{\frac{1}{q}}$$
$$\le\frac{ab(b-a)}{2}\left(\int_0^1|1-2t|^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{M_t^{2q}}\left[(1-t)^2|e^{f(a)}f'(a)|^q+t^2|e^{f(b)}f'(b)|^q\right.\right.$$
$$\left.\left.+t(1-t)\{|e^{f(a)}f'(b)|^q+|e^{f(b)}f'(a)|^q\}\right]\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^{p\alpha}dt\right)^{\frac{1}{p}}$$

$$\left(\int_0^1 \frac{1}{M_t^{2q}}\left[(1-t)^2|e^{f(a)}f'(a)|^q + t^2|e^{f(b)}f'(b)|^q + t(1-t)\Delta_1(a,b)\right]\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{2}\zeta_7^{\frac{1}{p}}\left(\zeta_8|e^{f(a)}f'(a)|^q + \zeta_9|e^{f(b)}f'(b)|^q + \zeta_{10}\Delta_1(a,b)\right)^{\frac{1}{q}},$$

where

$$\zeta_7 = \int_0^1 |1-2t|^{p\alpha}dt = \frac{1}{p\alpha+1},$$

$$\zeta_8 = \int_0^1 t^2 M_t^{-2q}dt = b^{-2q}\int_0^1 t^2\big(1-t(1-\frac{a}{b})\big)^{-2q}dt = \frac{1}{3b^{2q}}\,{}_2F_1\big(2q,3;4;1-\frac{a}{b}\big),$$

$$\zeta_9 = \int_0^1 (1-t)^2 M_t^{-2q}dt = b^{-2q}\int_0^1 (1-t)^2\big(1-t(1-\frac{a}{b})\big)^{-2q}dt$$

$$= \frac{1}{3b^{2q}}\,{}_2F_1\big(2q,1;4;1-\frac{a}{b}\big)$$

and

$$\zeta_{10} = \int_0^1 t(1-t)M_t^{-2q}dt = b^{-2q}\int_0^1 t(1-t)\big(1-t(1-\frac{a}{b})\big)^{-2q}dt \tag{2.22}$$

$$= \frac{1}{6b^{2q}}\,{}_2F_1\big(2q,2;4;1-\frac{a}{b}\big). \tag{2.23}$$

Thus, if we use ζ_7, ζ_8, ζ_9 and ζ_{10} in (2.21), we obtain inequality (2.20). The proof is completed. □

3. CONCLUSION

In this paper, the definition of exponentially harmonically convex functions is given and a new integral identity that includes Riemann-Liouville fractional integral operators is established. Depending on this new definition and identity, some new Hermite-Hadamard type inequalities for exponentially harmonically convex functions are built via Riemann-Liouville fractional forms. By choosing $\alpha = 1$, one can reduce our main results to provide integral inequalities for classical integrals, we omit the details.

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