Miskolc Mathematical Notes
Vol. 22 (2021), No. 2, pp. 755–762
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2021.3393

SOME PROPERTIES OF (AMPLY) G-RADICAL SUPPLEMENTED MODULES

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Received 23 June, 2020

Abstract. In this work, some new properties of (amply) g-radical supplemented modules are investigated. It is proved that every factor module and every homomorphic image of an amply g-radical supplemented module are amply g-radical supplemented. Let M be a π-projective and g-radical supplemented module. Then M is amply g-radical supplemented. Let M be a projective and g-radical supplemented module. Then every finitely M-generated module is amply g-radical supplemented. Let R be any ring. Then $_RR$ is g-radical supplemented if and only if every finitely generated R-module is amply g-radical supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: g-small submodules, radical, g-supplemented modules, g-radical supplemented modules

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. A submodule N of an R-module M is called an *essential submodule* of M, denoted by $N \trianglelefteq M$, if $K = 0$ for every $K \leq M$ with $K \cap N = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [14], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). Let M be an R-module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *local module* if M has the largest submodule, i. e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $M = U+T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U+V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U+V$, U has a supplement (g-supplement) V' in M with $V' \leq V$, then we say U has *ample supplements* (*g-supplements*) in M. If every submodule of M has ample supplements (g-supplements) in M, then M is called an *amply supplemented* (*g-supplemented*) *module*. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. M is said to be *semilocal* if $M/RadM$ is semisimple, i. e. every submodule of $M/RadM$ is a direct summand of $M/RadM$. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized (radical) supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M = U+V$, U has a Rad-supplement V' in M with $V' \leq V$, then we say U has *ample generalized* (*radical*) *supplements* (briefly, *ample Rad-supplements*) in M. If every submodule of M has ample Rad-supplements in M, then M is called an *amply generalized* (*radical*) *supplemented* (briefly, *amply Rad-supplemented*) *module*. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM (in [14], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. M is said to be *g-semilocal* if M/Rad_gM is semisimple, i. e. every submodule of M/Rad_gM is a direct summand of M/Rad_gM. Let M be an R-module. We say submodules X and Y of M are β^* equivalent, $X\beta^*Y$, if and only if $Y+K = M$ for every $K \leq M$ such that $X+K = M$ and $X+T = M$ for every $T \leq M$ such that $Y+T = M$. We say submodules X and Y of M are β_g^* equivalent, $X\beta_g^*Y$, if and only if $Y+K = M$ for every $K \trianglelefteq M$ such that $X+K = M$ and $X+T = M$ for every $T \trianglelefteq M$ such that $Y+T = M$. Let M be R-module and $X \leq Y \leq M$. If $Y/X \ll M/X$, then we say Y *lies above* X in M.

More information about (amply) supplemented modules are in [2, 6, 13]. More informations about g-small submodules and (amply) g-supplemented modules are in [4, 8, 11]. The definition of (amply) generalized supplemented modules and some properties of them are in [12]. The definition of g-semilocal modules and some properties of them are in [5]. The definition of β^* relation and some results of this relation are in [1]. The definition of β_g^* relation and some results of this relation are in [10].

Lemma 1. *Let M be an R-module. The following assertions hold.*

(1) For every $m \in Rad_gM$, $Rm \ll_g M$.
(2) If $N \leq M$, then $Rad_gN \leq Rad_gM$.
(3) $Rad_gM = \sum_{L \ll_g M} L$.

Proof. See [3, Lemma 2 and Lemma 3]. □

2. G-RADICAL SUPPLEMENTED MODULE

Definition 1. Let M be an R-module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq Rad_g V$, then V is called a g-radical supplement of U in M. If every submodule of M has a g-radical supplement in M, then M is called a g-radical supplemented module. (See [3, 7].)

Clearly we can see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. Every Rad-supplemented module is g-radical supplemented.

Proposition 1. *Let M be an R-module and $Rad_g V = V \cap Rad_g M$ for every $V \leq M$. Then M is g-radical supplemented if and only if M is g-semilocal.*

Proof. $(\Longrightarrow)$ Clear from [3, Theorem 1].
$(\Longleftarrow)$ Let $U \leq M$. Since M is g-semilocal, $(U+Rad_g M)/Rad_g M$ is a direct summand of $M/Rad_g M$. By this, there exists $V/Rad_g M \leq M/Rad_g M$ such that $\frac{U+Rad_g M}{Rad_g M} \oplus \frac{V}{Rad_g M} = \frac{M}{Rad_g M}$. Then $\frac{U+Rad_g M}{Rad_g M} + \frac{V}{Rad_g M} = \frac{M}{Rad_g M}$ and $\frac{U+Rad_g M}{Rad_g M} \cap \frac{V}{Rad_g M} = 0$. Here $\frac{M}{Rad_g M} = \frac{U+Rad_g M}{Rad_g M} + \frac{V}{Rad_g M} = \frac{U+Rad_g M+V}{Rad_g M} = \frac{U+V}{Rad_g M}$ and $0 = \frac{U+Rad_g M}{Rad_g M} \cap \frac{V}{Rad_g M} = \frac{(U+Rad_g M)\cap V}{Rad_g M} = \frac{U\cap V+Rad_g M}{Rad_g M}$. Since $\frac{U+V}{Rad_g M} = \frac{M}{Rad_g M}$, $U+V = M$. Since $\frac{U\cap V+Rad_g M}{Rad_g M} = 0$, $U \cap V + Rad_g M = Rad_g M$ and $U \cap V \leq Rad_g M$. By $U \cap V \leq V$ and $Rad_g V = V \cap Rad_g M$, $U \cap V \leq V \cap Rad_g M = Rad_g V$. Hence V is a g-radical supplement of U in M and M is g-radical supplemented. □

Proposition 2. *Let M be an R-module and $RadV = V \cap RadM$ for every $V \leq M$. Then M is Rad-supplemented if and only if M is semilocal.*

Proof. $(\Longrightarrow)$ Clear from [12, Proposition 2.6 (2)].
$(\Longleftarrow)$ Let $U \leq M$. Since M is semilocal, $(U+RadM)/RadM$ is a direct summand of $M/RadM$. By this, there exists $V/RadM \leq M/RadM$ such that $\frac{U+RadM}{RadM} \oplus \frac{V}{RadM} = \frac{M}{RadM}$. Then $\frac{U+RadM}{RadM} + \frac{V}{RadM} = \frac{M}{RadM}$ and $\frac{U+RadM}{RadM} \cap \frac{V}{RadM} = 0$. Here $\frac{M}{RadM} = \frac{U+RadM}{RadM} + \frac{V}{RadM} = \frac{U+RadM+V}{RadM} = \frac{U+V}{RadM}$ and $0 = \frac{U+RadM}{RadM} \cap \frac{V}{RadM} = \frac{(U+RadM)\cap V}{RadM} = \frac{U\cap V+RadM}{RadM}$. Since $\frac{U+V}{RadM} = \frac{M}{RadM}$, $U+V = M$. Since $\frac{U\cap V+RadM}{RadM} = 0$, $U \cap V + RadM = RadM$ and $U \cap V \leq RadM$. By $U \cap V \leq V$ and $RadV = V \cap RadM$, $U \cap V \leq V \cap RadM = RadV$. Hence V is a Rad-supplement of U in M and M is Rad-supplemented. □

Lemma 2. *Let $X\beta_g^* Y$ in M, Y be a g-radical supplement of U in M and $U \trianglelefteq M$. Then $U \cap X \leq Rad_g M$.*

Proof. Since Y is a g-radical supplement of U in M, $M = U+Y$ and $U \cap Y \leq Rad_g Y \leq Rad_g M$. Since $M = U+Y$ and $U \trianglelefteq M$ and $X\beta_g^* Y$, $M = U+X$. Let T be any essential maximal submodule of M. Here $U \cap Y \leq Rad_g M \leq T$. Assume that $U\cap X\nleq T$. Then $U\cap X+T=M$ and since $M=U+X$, by [2, Lemma 1.24], $X+U\cap T=M$. Since $U\trianglelefteq M$ and $T\trianglelefteq M$, $U\cap T\trianglelefteq M$. Since $X\beta_g^*Y$, $Y+U\cap T=M$ and since $U+T=M$, by [2, Lemma 1.24] again, $U\cap Y+T=M$. Then by $U\cap Y\leq T$, $M=U\cap Y+T=T$. This is a contradiction. Hence $U\cap X\leq T$ for every essential maximal submodule T of M and $U\cap X\leq Rad_gM$. □

Corollary 1. *Let $X\beta^*Y$ in M, Y be a g-radical supplement of U in M and $U\trianglelefteq M$. Then $U\cap X\leq Rad_gM$.*

Proof. Clear from Lemma 2. □

Corollary 2. *Let X lies above Y, Y be a g-radical supplement of U in M and $U\trianglelefteq M$. Then $U\cap X\leq Rad_gM$.*

Proof. Clear from Lemma 2. □

Definition 2. Let M be an R-module and $V\leq M$. If V is a g-radical supplement of an essential submodule in M, then V is called an eg-radical supplement submodule (briefly, eg-radical supplement) in M.

Lemma 3. *Let M be an R-module. If every submodule of M is β_g^* equivalent to an eg-radical supplement in M, then M is g-semilocal.*

Proof. Let $X/Rad_gM\leq M/Rad_gM$. Since $X\leq M$, by hypothesis, there exists an eg-radical supplement Y in M such that $X\beta^*Y$. Since Y is an eg-radical supplement in M, there exists $U\trianglelefteq M$ such that Y is a g-radical supplement of U in M. By Lemma 2, $U\cap X\leq Rad_gM$. Since $X\beta_g^*Y$ and $U\trianglelefteq M$ and $Y+U=M$, $X+U=M$. Then $\frac{M}{Rad_gM}=\frac{X+U}{Rad_gM}=\frac{X}{Rad_gM}+\frac{U+Rad_gM}{Rad_gM}$ and $\frac{X}{Rad_gM}\cap\frac{U+Rad_gM}{Rad_gM}=\frac{X\cap(U+Rad_gM)}{Rad_gM}=\frac{U\cap X+Rad_gM}{Rad_gM}=\frac{Rad_gM}{Rad_gM}=0$. Hence $\frac{M}{Rad_gM}=\frac{X}{Rad_gM}\oplus\frac{U+Rad_gM}{Rad_gM}$ and M/Rad_gM is semisimple. Thus M is g-semilocal. □

Corollary 3. *Let M be an R-module. If every submodule of M lies above an eg-radical supplement in M, then M is g-semilocal.*

Proof. Clear from Lemma 3. □

3. Amply g-radical supplemented modules

Definition 3. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a g-radical supplement V' in M with $V'\leq V$, then we say U has ample g-radical supplements in M. If every submodule of M has ample g-radical supplements in M, then M is called an amply g-radical supplemented module. (See also [9].)

Clearly we can see that every amply g-radical supplemented module is g-radical supplemented. Since $RadM\leq Rad_gM$ for every R-module M, every amply Rad-supplemented module is amply g-radical supplemented. Since every amply supplemented module is amply Rad-supplemented, every amply supplemented module is amply g-radical supplemented. Hollow and local modules are amply g-radical supplemented.

Proposition 3. *Let $M = M_1 + M_2 + \cdots + M_n$. If M_i is amply g-radical supplemented for every $i = 1, 2, \ldots, n$, then M is g-radical supplemented.*

Proof. Since M_i is amply g-radical supplemented for every $i = 1, 2, \ldots, n$, M_i is g-radical supplemented. Then by [3, Corollary 4], M is g-radical supplemented. □

Proposition 4. *Let M be an amply g-supplemented R-module. Then M is amply g-radical supplemented.*

Proof. Let $M = U + V$. Since M is amply g-supplemented, U has a g-supplement V' with $V' \leq V$. Here $M = U + V'$ and $U \cap V' \ll_g V'$. Since $U \cap V' \ll_g V'$, by Lemma 1, $U \cap V' \leq Rad_g V'$. Hence V' is a g-radical supplement of U in M. Moreover, $V' \leq V$. Hence M is amply g-radical supplemented. □

Proposition 5. *Let M be an R-module, $U_1, U_2 \leq M$ and $M = U_1 + U_2$. If U_1 and U_2 have ample g-radical supplements in M, then $U_1 \cap U_2$ has also ample g-radical supplements in M.*

Proof. Let $U_1 \cap U_2 + T = M$. Then by [2, Lemma 1.24], $M = U_1 + U_2 \cap T = U_2 + U_1 \cap T$. Since U_1 and U_2 have ample g-radical supplements in M, then U_1 has a g-radical supplement V_1 with $V_1 \leq U_2 \cap T$ and U_2 has a g-radical supplement V_2 with $V_2 \leq U_1 \cap T$. Since $M = U_1 + V_1$ and $V_1 \leq U_2$, by Modular Law, $U_2 = U_2 \cap (U_1 + V_1) = U_1 \cap U_2 + V_1$. Similarly we have $U_1 = U_1 \cap U_2 + V_2$. Then $M = U_1 + U_2 = U_1 \cap U_2 + V_2 + U_1 \cap U_2 + V_1 = U_1 \cap U_2 + V_1 + V_2$ and by Lemma 1, $U_1 \cap U_2 \cap (V_1 + V_2) = U_1 \cap (V_1 + U_2 \cap V_2) = U_1 \cap V_1 + U_2 \cap V_2 \leq Rad_g V_1 + Rad_g V_2 \leq Rad_g (V_1 + V_2)$. Hence $V_1 + V_2$ is a g-radical supplement of $U_1 \cap U_2$ and since $V_1 + V_2 \leq T$, $U_1 \cap U_2$ has ample g-radical supplements in M. □

Lemma 4. *Every factor module of an amply g-radical supplemented module is amply g-radical supplemented.*

Proof. Let M be any amply g-radical supplemented module and $K \leq M$. Let $U/K \leq M/K$ and $M/K = U/K + V/K$ with $V/K \leq M/K$. Then $M = U + V$ and since M is amply g-radical supplemented, there exists a g-radical supplement T of U with $T \leq V$. Then by [3, Lemma 8], $(T + K)/K$ is a g-radical supplement of U/K in M/K. Moreover, $(T + K)/K \leq V/K$. Hence U/K has ample g-radical supplements in M/K and M/K is amply g-radical supplemented. □

Corollary 4. *The homomorphic image of an amply g-radical supplemented module is amply g-radical supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let M be an R-module. If every submodule of M is g-radical supplemented, then M is amply g-radical supplemented.*

Proof. Let $U \leq M$ and $M = U + V$ with $V \leq M$. By hypothesis, V is g-radical supplemented and $U \cap V$ has a g-radical supplement T in V. Here $V = U \cap V + T$ and $U \cap V \cap T \leq Rad_g T$. Then $M = U + V = U + U \cap V + T = U + T$ and $U \cap T = U \cap V \cap T \leq Rad_g T$. Hence U has ample g-radical supplements in M and M is amply g-radical supplemented. □

Proposition 6. *Let R be any ring. Then every R-module is g-radical supplemented if and only if every R-module is amply g-radical supplemented.*

Proof. ($\Longrightarrow$) Let M be an R-module. Since every R-module is g-radical supplemented, every submodule of M is g-radical supplemented. Then by Lemma 5, M is amply g-radical supplemented, as desired.

($\Longleftarrow$) Clear. □

Lemma 6. *Let M be a π-projective and g-radical supplemented module. Then M is amply g-radical supplemented.*

Proof. Let $M = U + V$ and X be a g-radical supplement of U in M. Since M is π-projective and $M = U + V$, there exists an R-module homomorphism $f\colon M \to M$ such that $Imf \subset V$ and $Im(1-f) \subset U$. So, we have $M = f(M) + (1-f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a + (1-f)(x) \in U$. Thus $x \in U \cap X$ and so $a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \leq f(Rad_g X) \leq Rad_g f(X)$. This means that $f(X)$ is a g-radical supplement of U in M. Moreover, $f(X) \subset V$. Therefore M is amply g-radical supplemented. □

Corollary 5. *If M is a projective and g-radical supplemented module, then M is an amply g-radical supplemented module.*

Proof. Clear from Lemma 6. □

Lemma 7. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. If M_i is projective and g-radical supplemented for every $i = 1, 2, \ldots, n$, then M is amply g-radical supplemented.*

Proof. Since M_i projective for every $i = 1, 2, \ldots, n$, by [2, 4.3], M is projective. Since M_i is g-radical supplemented for every $i = 1, 2, \ldots, n$, by [3, Corollary 4], M is g-radical supplemented. Then by Corollary 5, M is amply g-radical supplemented, as desired. □

Lemma 8. *Let M be a projective and g-radical supplemented module. Then every finitely M-generated module is amply g-radical supplemented.*

Proof. Let N be a finitely M-generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f\colon M^{(\Lambda)} \longrightarrow N$. Since M is projective and g-radical supplemented, by Lemma 7, $M^{(\Lambda)}$ is amply g-radical supplemented. Then by Corollary 4, N is amply g-radical supplemented. □

Proposition 7. *Let R be any ring. The following assertions are equivalent.*

(i) *${}_RR$ is g-radical supplemented.*
(ii) *${}_RR$ is amply g-radical supplemented.*
(iii) *${}_RR^{(\Lambda)}$ is g-radical supplemented for every finite index set Λ.*
(iv) *Every finitely generated $R-$module is g-radical supplemented.*
(v) *Every finitely generated $R-$module is amply g-radical supplemented.*

Proof. (i) $\Longleftrightarrow$ (ii) Clear from Corollary 5, since ${}_RR$ is projective.
(i) $\Longleftrightarrow$ (iii) Clear from [3, Corollary 4].
(i) $\Longleftrightarrow$ (iv) Clear from [3, Corollary 6].
(i) $\Longleftrightarrow$ (v) Clear from Lemma 8. □

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