A GENERALIZATION OF BANG'S LEMMA

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Abstract. We prove a common extension of Bang's and Kadets' lemmas for contact pairs, in the spirit of the Colourful Carathéodory Theorem. We also formulate a generalized version of the affine plank problem and prove it under special assumptions. In particular, we obtain a generalization of Kadets' theorem. Finally, we give applications to problems regarding translative coverings.

1. Plank problems

In 1950, Bang [Ba50, Ba51] proved the plank problem of Tarski [Ta32]: he showed that if a convex body $K \subset \mathbb{R}^d$ is covered by a finite number of planks, then the sum of their widths is not less than the minimal width of K. Here a *plank* P is the closed region of $\mathbb{R}^d$ between two parallel hyperplanes, whose distance apart is the *width* of P, denoted by $w(P)$. Let $\mathcal{K}^d$ stand for the family of convex bodies in $\mathbb{R}^d$. Given a convex body $K \in \mathcal{K}^d$ and a direction $u \in \mathbb{R}^d \setminus \{0\}$, the *width of K in direction u*, denoted by $w_u(K)$, is the width of the smallest plank containing K whose bounding hyperplanes are orthogonal to u. The *minimal width of K* is $w(K) = \min_u w_u(K)$.

In the same article, Bang suggested an affine invariant generalization of the problem. Given a convex body $K \subset \mathbb{R}^d$ and a plank $P \subset \mathbb{R}^d$, he defined the *width of P relative to K* as

$$w_K(P) = \frac{w(P)}{w_u(K)} \tag{1}$$

where $u \in \mathbb{R}^d \setminus \{0\}$ is normal to a boundary hyperplane of P.

Conjecture 1 (The affine plank problem, Bang [Ba51])**.** *Assume that the planks $P_1, \ldots, P_n$ cover the convex body $K \in \mathcal{K}^d$. Then $\sum_{i=1}^n w_K(P_i) \geq 1$.*

The statement was proved for symmetric K's by Ball [Ba91], but is still open for general convex bodies apart from the following special cases: only two planks in the plane [Ba54, Mo58, Al68], at most three planks in the plane [Hu93], or when the planks can be partitioned to two parallel subfamilies [AkKP19].

One of the main ingredients of Bang's proof of the plank problem is the following statement, which has been polished to its present form by Fenchel [Fe51] and Ball [Ba01]:

Lemma 1 (Bang's Lemma)**.** *Let $(u_i)_1^n$ be a sequence of unit vectors in $\mathbb{R}^d$ and $(w_i)_1^n$ a sequence of positive numbers. Then for any sequence $(m_i)_1^n$ of reals, there exists a point u*

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of the form

$$u = \sum \varepsilon_i u_i w_i$$

with $\varepsilon_i = \pm 1$ *for* $i \in [n]$*, so that*

$$|\langle u, u_k \rangle - m_k| \geq w_k$$

holds for each k*.*

Above and later on, $[n] = \{1, \dots, n\}$.

Bang's lemma has found numerous applications in the past decades. In particular, it is a crucial ingredient of Ball's proof for the symmetric case of the affine plank problem [Ba91], his lower bound on the density of sphere packings [Ba92] as well as Nazarov's solution of the coefficient problem [Na97].

In 2005, Kadets [Ka05] generalized the original plank problem to coverings with arbitrary convex bodies in $\mathbb{R}^d$. He proved that if a family of convex bodies $K_1, \dots, K_n \in \mathcal{K}^d$ covers $K \in \mathcal{K}^d$, then $\sum_{i=1}^n r(K_i) \geq r(K)$, where $r(K)$ denotes the inradius of K. The crux of his argument boils down to the following generalization of Theorem 1. Below, S^{d-1} denotes the d-dimensional unit sphere.

Lemma 2 (Kadets' Lemma)**.** *Assume that* $U_1, \dots, U_n \subset S^{d-1}$ *are finite sets of unit vectors in* $\mathbb{R}^d$ *so that* $0 \in \operatorname{conv} U_i$ *for each* i*. Let* $r_1, \dots, r_n > 0$ *be positive numbers. Then for every set of points* $o_1, \dots, o_n \in \mathbb{R}^d$ *there exist* $u_i \in U_i$*,* $i = 1, \dots, n$ *so that setting* $u = \sum_{i=1}^n r_i u_i$*,*

$$\langle u - o_k, u_k \rangle \geq r_k$$

holds for every k*.*

We note that the planar case of Kadets' theorem was also proved much earlier by Ohmann [Oh53], and later independently by Bezdek [Be07]. Prior to that, Bezdek and Bezdek [BeB95] solved Conway's potato problem and showed that if K is successively sliced by $n-1$ hyperplane cuts, dividing just one piece by each cut, then one of the remaining pieces must have inradius at least $\frac{1}{n} r(K)$. In a follow-up article [BeB96], they extended their result to K*-inradii* instead of inradii: given a convex body $K \in \mathcal{K}^d$ and a convex set $L \subset \mathbb{R}^d$, the K-inradius of L is defined as

$$r_K(L) = \sup\{\lambda \geq 0 : \ \lambda K + x \subset L \text{ for some } x \in \mathbb{R}^d.\} \tag{2}$$

Note that for a plank $P \subset \mathbb{R}^d$,

$$w_K(P) = r_K(P). \tag{3}$$

The connection to plank problems is provided by Alexander [Al68], who proved that for $K \in \mathcal{K}^d$, the sum of the K-inradii of n planks covering K is guaranteed to be at least 1 if and only if for an arbitrary set of $n-1$ hyperplanes, there exists a convex body $L \subset K$ with $r_K(L) \geq \frac{1}{n}$ not cut by any of these hyperplanes.

Along this direction, Akopyan and Karasev [AkK12] proved analogues of Kadets' result for K-inradii: among other results, they showed that if $K_1, \dots, K_n$ form an *inductive partition* of $K \in \mathcal{K}^d$, then $\sum r_K(K_i) \geq 1$ holds, moreover, the same statement is true in the plane for arbitrary convex partitions.

The goal of the present paper is to generalize Lemmas 1 and 2 in the spirit of Bárány's Colourful Carathéodory Theorem [Bá82]. The resulting statement may be applied to general covering problems involving K-inradii, and in particular, to translative covering problems.

Let $K, L \subset \mathbb{R}^d$ be convex bodies. It is a well-known fact that if K' is a maximal homothetic copy of K inscribed in L, then there exists a set of points $u_1, \dots, u_n \in \mathbb{R}^d$

with corresponding normal directions $v_1, \dots, v_n \in \mathbb{R}^d \setminus \{0\}$ such that u_i is a common boundary point of K' and L with corresponding (common) outer normal vector v_i for every i, moreover, $0 \in \operatorname{conv}\{v_1, \dots, v_n\}$. The pairs (u_i, v_i) are called *contact pairs* of K' and L. A set of contact pairs is called *complete* if $0 \in \operatorname{conv}\{v_1, \dots, v_n\}$. Carathéodory's theorem implies that in the above setting, there always exists a complete set of contact pairs of cardinality at most $d+1$.

We are going to generalize Bang's lemma to sets of contact pairs. The forthcoming arguments will use the following setup. For vectors $u, v \in \mathbb{R}^d$, we define $w \in \mathbb{R}^d \times \mathbb{R}^d$ as $w = (u, v)$. For any such vector $w = (u, v)$, let $\widehat{w} = (v, u)$. Here comes the main result of the paper.

Theorem 1. *Assume that $W_1, \dots, W_n \subset \mathbb{R}^d \times \mathbb{R}^d$ are finite sets such that $(0,0) \in \operatorname{conv} W_i$ for each $i \in [n]$. For any set of vectors $z_1, \dots, z_n \in \mathbb{R}^d \times \mathbb{R}^d$, there exist $w_i \in W_i$, $i \in [n]$ so that by setting $w = \sum_{i=1}^n w_i$,*

$$\langle w - z_k, \widehat{w}_k \rangle \geq \langle w_k, \widehat{w}_k \rangle \tag{4}$$

holds for each k.

Theorem 1 is formulated in the context of contact pairs (u_i, v_i). Setting $v_i = u_i$ and $y_i = x_i$ for every i, it takes the following simpler form.

Corollary 1. *Assume that all the finite vector sets $U_1, \dots, U_n \subset \mathbb{R}^d$ contain the origin in their convex hull. Then for any set of vectors $x_i, \dots, x_n \in \mathbb{R}^d$ we may select $u_i \in U_i$ for each $i \in [n]$ so that setting $u = \sum_i u_i$,*

$$\langle u - x_k, u_k \rangle \geq |u_k|^2$$

holds for every k.

When all the sets U_i consist of unit vectors, we recover Kadets' lemma, while the case $U_i = \{-u_i, u_i\}$ with $u_i \in \mathbb{R}^d$ corresponds to Bang's lemma.

Theorem 1 and Corollary 1 lead towards the following generalization of the affine plank problem. We say that the convex sets $C_1, \dots, C_n \subset \mathbb{R}^d$ *permit a translative covering of* $K \in \mathcal{K}^d$ if

$$K \subset \bigcup_{i=1}^n (C_i + x_i)$$

for some $x_1, \dots, x_n \in \mathbb{R}^d$.

Conjecture 2. *Assume that the convex sets $C_1, \dots, C_n \subset \mathbb{R}^d$ permit a translative covering of the convex body $B \in \mathcal{K}^d$. Then*

$$\sum_{i=1}^n r_B(C_i) \geq 1$$

holds.

Equation (3) shows that this is indeed an extension of Conjecture 1, the affine plank problem.

In addition to the special cases of the affine plank problem discussed earlier, Conjecture 2 has been proved if B is an ellipsoid [Oh53, Be07, Ka05] or if the sets C_i form a partition of B in the plane, or an inductive partition in higher dimensions [AkK12]. Corollary 1 implies that it also holds in a wide range of cases.

Theorem 2. *Conjecture 2 holds if for every $i \in [n]$ there exists some $o_i \in \mathbb{R}^d$ such that $r_B(C_i)B - o_i$ and $C_i - o_i$ have a complete set of contact pairs $W_i \subset \mathbb{R}^d \times \mathbb{R}^d$ with $(0,0) \in \operatorname{conv} W_i$, so that for any two such contact pairs $(u_i, v_i) \in W_i$, $(u_j, v_j) \in W_j$ with $i \neq j$,*

$$\langle u_i, v_j \rangle = \langle u_j, v_i \rangle \tag{5}$$

holds.

We immediately obtain the following generalization of Kadets' theorem [Ka05].

Corollary 2. *Conjecture 2 holds if for every $i \in [n]$ there exists some $o_i \in \mathbb{R}^d$ such that $r_B(C_i)B - o_i$ and $C_i - o_i$ have a complete set of contact pairs of the form (u, u).*

A particular case is when $o_i \in r_B(C_i)B$, and the contact points between $r_B(C_i)B$ and C_i are the local extrema of the radial function $|x - o_i|$ for $x \in \partial(r_B(C_i)B)$, provided that 0 is contained in the convex hull of these. Such a situation is illustrated on Figure 1.

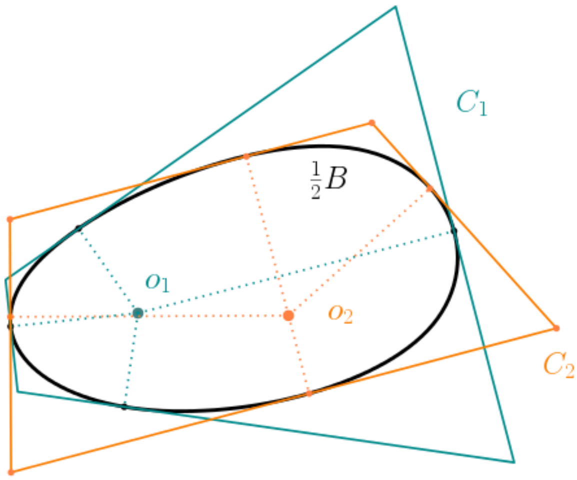

FIGURE 1. Convex discs with complete sets of contact pairs of the form (u, u)

The direct application of Theorem 1 yields another sufficient condition.

Proposition 1. *Conjecture 2 holds in $\mathbb{R}^{2d}$ if for every i there exists some $o_i \in \mathbb{R}^{2d}$ such that $r_B(C_i)B - x_i$ and $C_i - o_i$ have a complete set of contact pairs of the form $(w, \widehat{w})$.*

Applications of Theorem 1 to translative coverings are listed in Section 3.

Although the above results are formulated for finite vector sets/families of convex sets in $\mathbb{R}^d$, they may be extended to an arbitrary number of vectors/convex sets in finite dimensional real or complex Hilbert spaces using the standard techniques.

Further developments, historical and mathematical details related to the plank problem may be found in [Am10, Be14, FT22].

2. Proof of the main results

Proof of Theorem 1. For each i, let $w_i = (u_i, v_i)$ and $z_i = (x_i, y_i)$ with $u_i, v_i, x_i, y_i \in \mathbb{R}^d$. Select $w_i \in W_i$, $i \in [n]$ so as to maximize

$$\sum_{i \neq j} \langle u_i, v_j \rangle - \sum_i \langle x_i, v_i \rangle - \sum_j \langle u_j, y_j \rangle \tag{6}$$

and set $w = (u, v) = \sum_i w_i$, that is, $u = \sum_i u_i$ and $v = \sum v_i$. We will show that (4) holds for every k, that is,

$$\langle u - x_k, v_k \rangle + \langle u_k, v - y_k \rangle \geq 2\langle u_k, v_k \rangle. \tag{7}$$

Let $k \in [n]$ be arbitrary. By the condition of the theorem, there exist non-negative numbers $\alpha(w'_k)$, $w'_k \in W_k$ so that $\sum_{w'_k \in W_k} \alpha(w'_k) = 1$ and

$$\sum_{w'_k \in W_k} \alpha(w'_k) w'_k = (0, 0).$$

Moreover, since (6) is maximal, for each $w'_k = (u'_k, v'_k) \in W_k$,

$$0 \geq \sum_{i \neq k} \langle u_i, v'_k - v_k \rangle + \sum_{j \neq k} \langle u'_k - u_k, v_j \rangle - \langle x_k, v'_k - v_k \rangle - \langle u'_k - u_k, y_k \rangle.$$

Multiplying the above equation by $\alpha(w'_k)$ and summing up for all $w'_k \in W_k$ leads to

$$0 \geq \sum_{i \neq k} \langle u_i, -v_k \rangle + \sum_{j \neq k} \langle -u_k, v_j \rangle - \langle x_k, -v_k \rangle - \langle -u_k, y_k \rangle,$$

which directly implies (7). □

Proof of Theorem 2. We may assume that $0 \in B$. Let $\lambda_i = r_B(C_i)$ for every i, and $\lambda := \sum \lambda_i$. Assume on the contrary that $\lambda < 1$ and $B \subset \bigcup (C_i + x'_i)$ with some $x'_1, \ldots, x'_n \in \mathbb{R}^d$. Choose $\varepsilon > 0$ so that $(1 + \varepsilon)\lambda < 1$. For each i, let W_i be the complete set of contact pairs between $\lambda_i B - o_i$ and $C_i - o_i$ which contains $(0, 0)$ in its convex hull. Also, set $o = (1 + \varepsilon) \sum o_i$ and $x_i = x'_i + o_i - o$ for each i.

Apply Theorem 1 to the sets $(1 + \varepsilon)W_i$ and the corresponding points $z_k = (2x_i, 0)$. It implies the existence of $w_i = (u_i, v_i) \in W_i$, $i \in [n]$ so that setting $w = (u, v) = \sum (1 + \varepsilon) w_i$,

$$\langle u - 2x_k, (1 + \varepsilon)v_k \rangle + \langle v, (1 + \varepsilon)u_k \rangle \geq 2(1 + \varepsilon)^2 \langle u_k, v_k \rangle$$

holds for each k. Since (5) implies that $\langle u, v_k \rangle = \langle v, u_k \rangle$, the above equation simplifies to

$$\langle u - x_k, v_k \rangle \geq (1 + \varepsilon)\langle u_k, v_k \rangle > \langle u_k, v_k \rangle.$$

Since u_k is a boundary point of $C_k - o_k$ with outer normal v_k, the convexity of C_k implies that $u - x_k \notin C_k - o_k$, equivalently, $u + o \notin C_k + x'_k$ for any k. On the other hand, $u_i \in \lambda_i B - o_i$ for every i. Therefore,

$$u \in \sum_i (1 + \varepsilon)\lambda_i B - \sum_i (1 + \varepsilon) o_i = (1 + \varepsilon)\lambda B - o.$$

Since B is convex and $0 \in B$, $(1 + \varepsilon)\lambda B \subset B$. Hence, $u + o \in B$, but it is not covered by any of the sets $C_k + x'_k$, which is a contradiction. □

The proof of Proposition 1 is nearly identical, thus we leave it to the dedicated reader.

3. Applications to translative coverings

Corollary 1 readily implies the next statement regarding translative coverings.

Proposition 2. *Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a family of convex bodies in $\mathbb{R}^d$ containing the origin in their interior. For each i, let $V_i \subset S^{d-1}$ be a set of direction vectors for which $0 \in \operatorname{conv} V_i$. Denote by U_i the set of projection vectors of 0 onto the supporting hyperplanes of K_i corresponding to members of V_i. Then $\mathcal{K}$ does not permit a translative covering of $U_1 + \ldots + U_n$.*

A particular case is when all the K_i's are homothets of a fixed convex body $K \in \mathcal{K}^d$. Such homothetic coverings have been studied extensively, see e.g. [Na18], Section 3.2 of [BrMP05] and Section 15.4 of [FT22]. A related conjecture is due to Soltan [So90]:

Conjecture 3 (V. Soltan)**.** *Assume that $K \in \mathcal{K}^d$ and that $\lambda_1 K, \ldots, \lambda_n K$ permit a translative covering of K with $\lambda_i \in (0,1)$ for every i. Then*

$$\sum_{i=1}^{n} \lambda_i \geq d.$$

Let T^d denote the d-dimensional regular simplex. Setting $K = T^d$, $n = d+1$ and $\lambda_i = \frac{d}{d+1}$ shows that the above bound may not be improved.

Conjecture 3 was proved for $d = 2$ or $n = d+1$ by Soltan and Vásárhelyi [SoV93] and for $K = B^d$ by Glazyrin [Gl19], while Naszódi [Na10] showed that $\sum \lambda_i > \alpha\, d$ for any fixed $\alpha < 1$ if d is sufficiently large.

Böröczky asked whether the same bound holds for covering a triangle with its *negative* homothets. Vásárhelyi [Vá84] gave an affirmative answer. We conclude the article with the extension of this result to arbitrary dimensions.

Theorem 3. *Assume that $T \subset \mathbb{R}^d$ is a non-degenerate simplex, and $\lambda_1, \ldots, \lambda_n \geq 0$ are so that the family $-\lambda_1 T, \ldots, -\lambda_n T$ permits a translative covering of T. Then*

$$\sum_{i=1}^{n} \lambda_i \geq d. \tag{8}$$

Proof. We may assume that $T = T^d$ with its centre at 0. Let V be the set of normal directions of the facets of T^d and U be the set of projection vectors of 0 onto the facets of T^d. It is well-known that $\operatorname{conv} U = -\frac{1}{d} T^d$. Applying Proposition 2 with $K_i = -\lambda_i T^d$, $V_i = V$ and $U_i = \lambda_i U$ yields an uncovered point in

$$U_1 + \ldots + U_n \subset (\lambda_1 + \ldots + \lambda_n) T^d,$$

which implies (8). □

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