A GENERALIZATION OF BANG’S LEMMA

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Abstract. We prove a common extension of Bang’s and Kadets’ lemmas for contact pairs, in the spirit of the Colourful Carathéodory Theorem. We also formulate a generalized version of the affine plank problem and prove it under special assumptions. In particular, we obtain a generalization of Kadets’ theorem. Finally, we give applications to problems regarding translative coverings.

1. Plank problems

In 1950, Bang [Ba50, Ba51] proved the plank problem of Tarski [Ta32]: he showed that if a convex body $K \subset \mathbb{R}^d$ is covered by a finite number of planks, then the sum of their widths is not less than the minimal width of $K$. Here a plank $P$ is the closed region of $\mathbb{R}^d$ between two parallel hyperplanes, whose distance apart is the width of $P$, denoted by $w(P)$. Let $K^d$ stand for the family of convex bodies in $\mathbb{R}^d$. Given a convex body $K \in K^d$ and a direction $u \in \mathbb{R}^d \setminus \{0\}$, the width of $K$ in direction $u$, denoted by $w_u(K)$, is the width of the smallest plank containing $K$ whose bounding hyperplanes are orthogonal to $u$. The minimal width of $K$ is $w(K) = \min_u w_u(K)$.

In the same article, Bang suggested an affine invariant generalization of the problem. Given a convex body $K \subset \mathbb{R}^d$ and a plank $P \subset \mathbb{R}^d$, he defined the width of $P$ relative to $K$ as

$$w_K(P) = \frac{w(P)}{w_u(K)}$$

where $u \in \mathbb{R}^d \setminus \{0\}$ is normal to a boundary hyperplane of $P$.

Conjecture 1 (The affine plank problem, Bang [Ba51]). Assume that the planks $P_1, \ldots, P_n$ cover the convex body $K \in K^d$. Then $\sum_{i=1}^n w_K(P_i) \geq 1$.

The statement was proved for symmetric $K$’s by Ball [Ba91], but is still open for general convex bodies apart from the following special cases: only two planks in the plane [Ba54, Mo58, Al68], at most three planks in the plane [Hu93], or when the planks can be partitioned to two parallel subfamilies [AkKP19].

One of the main ingredients of Bang’s proof of the plank problem is the following statement, which has been polished to its present form by Fenchel [Fe51] and Ball [Ba01]:

Lemma 1 (Bang’s Lemma). Let $(u_i)_i^n$ be a sequence of unit vectors in $\mathbb{R}^d$ and $(w_i)_i^n$ a sequence of positive numbers. Then for any sequence $(m_i)_i^n$ of reals, there exists a point $u$
of the form
\[ u = \sum \varepsilon_i u_i w_i \]
with \( \varepsilon_i = \pm 1 \) for \( i \in [n] \), so that
\[ |\langle u, u_k \rangle - m_k| \geq w_k \]
holds for each \( k \).

Above and later on, \([n] = \{1, \ldots, n\}\).

Bang’s lemma has found numerous applications in the past decades. In particular, it is a crucial ingredient of Ball’s proof for the symmetric case of the affine plank problem [Ba91], his lower bound on the density of sphere packings [Ba92] as well as Nazarov’s solution of the coefficient problem [Na97].

In 2005, Kadets [Ka05] generalized the original plank problem to coverings with arbitrary convex bodies in \( \mathbb{R}^d \). He proved that if a family of convex bodies \( K_1, \ldots, K_n \in \mathcal{K}^d \) covers \( K \in \mathcal{K}^d \), then \( \sum_{i=1}^n r(K_i) \geq r(K) \), where \( r(K) \) denotes the inradius of \( K \). The crux of his argument boils down to the following generalization of Theorem 1. Below, \( S^{d-1} \) denotes the \( d \)-dimensional unit sphere.

**Lemma 2 (Kadets’ Lemma).** Assume that \( U_1, \ldots, U_n \subset S^{d-1} \) are finite sets of unit vectors in \( \mathbb{R}^d \) so that \( 0 \in \text{conv} U_i \) for each \( i \). Let \( r_1, \ldots, r_n > 0 \) be positive numbers. Then for every set of points \( o_1, \ldots, o_n \in \mathbb{R}^d \) there exist \( u_i \in U_i, i = 1, \ldots, n \) so that setting \( u = \sum_{i=1}^n r_i u_i \),
\[ \langle u - o_k, u_k \rangle \geq r_k \]
holds for every \( k \).

We note that the planar case of Kadets’ theorem was also proved much earlier by Ohmann [Oh53], and later independently by Bezdek [Be07]. Prior to that, Bezdek and Bezdek [BeB95] solved Conway’s potato problem and showed that if \( K \) is successively sliced by \( n-1 \) hyperplane cuts, dividing just one piece by each cut, then one of the remaining pieces must have inradius at least \( \frac{1}{n} r(K) \). In a follow-up article [BeB96], they extended their result to \( K \)-inradius instead of inradii: given a convex body \( K \in \mathcal{K}^d \) and a convex set \( L \subset \mathbb{R}^d \), the \( K \)-inradius of \( L \) is defined as
\[ r_K(L) = \sup \{ \lambda \geq 0 : \lambda K + x \subset L \text{ for some } x \in \mathbb{R}^d \} \]
Note that for a plank \( P \subset \mathbb{R}^d \),
\[ w_K(P) = r_K(P). \]

The connection to plank problems is provided by Alexander [Al68], who proved that for \( K \in \mathcal{K}^d \), the sum of the \( K \)-inradii of \( n \) planks covering \( K \) is guaranteed to be at least 1 if and only if for an arbitrary set of \( n - 1 \) hyperplanes, there exists a convex body \( L \subset \mathbb{R}^d \) with \( r_K(L) \geq \frac{1}{n} \) not cut by any of these hyperplanes.

Along this direction, Aopyan and Karasev [AkK12] proved analogues of Kadets’ result for \( K \)-inradii: among other results, they showed that if \( K_1, \ldots, K_n \) form an inductive partition of \( K \in \mathcal{K}^d \), then \( \sum r_K(K_i) \geq 1 \) holds, moreover, the same statement is true in the plane for arbitrary convex partitions.

The goal of the present paper is to generalize Lemmas 1 and 2 in the spirit of Bárány’s Colourful Carathéodory Theorem [Bá82]. The resulting statement may be applied to general covering problems involving \( K \)-inradii, and in particular, to translative covering problems.

Let \( K, L \subset \mathbb{R}^d \) be convex bodies. It is a well-known fact that if \( K' \) is a maximal homothetic copy of \( K \) inscribed in \( L \), then there exists a set of points \( u_1, \ldots, u_n \in \mathbb{R}^d \)
with corresponding normal directions $v_1, \ldots, v_n \in \mathbb{R}^d \setminus \{0\}$ such that $u_i$ is a common boundary point of $K'$ and $L$ with corresponding (common) outer normal vector $v_i$ for every $i$, moreover, $0 \in \text{conv}\{v_1, \ldots, v_n\}$. The pairs $(u_i, v_i)$ are called contact pairs of $K'$ and $L$. A set of contact pairs is called complete if $0 \in \text{conv}\{v_1, \ldots, v_n\}$. Carathéodory’s theorem implies that in the above setting, there always exists a complete set of contact pairs of cardinality at most $d + 1$.

We are going to generalize Bang’s lemma to sets of contact pairs. The forthcoming arguments will use the following setup. For vectors $u, v \in \mathbb{R}^d$, we define $w \in \mathbb{R}^d \times \mathbb{R}^d$ as $w = (u, v)$. For any such vector $w = (u, v)$, let $\hat{w} = (v, u)$. Here comes the main result of the paper.

**Theorem 1.** Assume that $W_1, \ldots, W_n \subset \mathbb{R}^d \times \mathbb{R}^d$ are finite sets such that $(0, 0) \in \text{conv} W_i$ for each $i \in [n]$. For any set of vectors $z_1, \ldots, z_n \in \mathbb{R}^d \times \mathbb{R}^d$, there exist $w_i \in W_i$, $i \in [n]$ so that by setting $w = \sum_{i=1}^n w_i$,

\[(4) \quad \langle w - z_k, \hat{w}_k \rangle \geq \langle w_k, \hat{w}_k \rangle\]

holds for each $k$.

Theorem 1 is formulated in the context of contact pairs $(u_i, v_i)$. Setting $v_i = u_i$ and $y_i = x_i$ for every $i$, it takes the following simpler form.

**Corollary 1.** Assume that all the finite vector sets $U_1, \ldots, U_n \subset \mathbb{R}^d$ contain the origin in their convex hull. Then for any set of vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ we may select $u_i \in U_i$ for each $i \in [n]$ so that setting $u = \sum_i u_i$,

\[\langle u - x_k, u_k \rangle \geq |u_k|^2\]

holds for every $k$.

When all the sets $U_i$ consist of unit vectors, we recover Kadets’ lemma, while the case $U_i = \{-u_i, u_i\}$ with $u_i \in \mathbb{R}^d$ corresponds to Bang’s lemma.

Theorem 1 and Corollary 1 lead towards the following generalization of the affine plank problem. We say that the convex sets $C_1, \ldots, C_n \subset \mathbb{R}^d$ permit a translative covering of $K \in \mathcal{K}^d$ if

\[K \subset \bigcup_{i=1}^n (C_i + x_i)\]

for some $x_1, \ldots, x_n \in \mathbb{R}^d$.

**Conjecture 2.** Assume that the convex sets $C_1, \ldots, C_n \subset \mathbb{R}^d$ permit a translative covering of the convex body $B \in \mathcal{K}^d$. Then

\[\sum_{i=1}^n r_B(C_i) \geq 1\]

holds.

Equation (3) shows that this is indeed an extension of Conjecture 1, the affine plank problem.

In addition to the special cases of the affine plank problem discussed earlier, Conjecture 2 has been proved if $B$ is an ellipsoid [Oh53, Be07, Ka05] or if the sets $C_i$ form a partition of $B$ in the plane, or an inductive partition in higher dimensions [AkK12]. Corollary 1 implies that it also holds in a wide range of cases.
Theorem 2. Conjecture 2 holds if for every $i \in [n]$ there exists some $o_i \in \mathbb{R}^d$ such that $r_B(C_i)B - o_i$ and $C_i - o_i$ have a complete set of contact pairs $W_i \subset \mathbb{R}^d \times \mathbb{R}^d$ with $(0, 0) \in \text{conv} W_i$, so that for any two such contact pairs $(u_i, v_i) \in W_i$, $(u_j, v_j) \in W_j$ with $i \neq j$,

\[
\langle u_i, v_j \rangle = \langle u_j, v_i \rangle
\]

holds.

We immediately obtain the following generalization of Kadets’ theorem [Ka05].

Corollary 2. Conjecture 2 holds if for every $i \in [n]$ there exists some $o_i \in \mathbb{R}^d$ such that $r_B(C_i)B - o_i$ and $C_i - o_i$ have a complete set of contact pairs of the form $(u, u)$.

A particular case is when $o_i \in r_B(C_i)B$, and the contact points between $r_B(C_i)B$ and $C_i$ are the local extrema of the radial function $|x - o_i|$ for $x \in \partial(r_B(C_i)B)$, provided that $0$ is contained in the convex hull of these. Such a situation is illustrated on Figure 1.

![Figure 1. Convex discs with complete sets of contact pairs of the form $(u, u)$](image)

The direct application of Theorem 1 yields another sufficient condition.

Proposition 1. Conjecture 2 holds in $\mathbb{R}^{2d}$ if for every $i$ there exists some $o_i \in \mathbb{R}^{2d}$ such that $r_B(C_i)B - o_i$ and $C_i - o_i$ have a complete set of contact pairs of the form $(w, \hat{w})$.

Applications of Theorem 1 to translative coverings are listed in Section 3.

Although the above results are formulated for finite vector sets/families of convex sets in $\mathbb{R}^d$, they may be extended to an arbitrary number of vectors/convex sets in finite dimensional real or complex Hilbert spaces using the standard techniques.

Further developments, historical and mathematical details related to the plank problem may be found in [Am10, Be14, FT22].

2. Proof of the main results

Proof of Theorem 1. For each $i$, let $w_i = (u_i, v_i)$ and $z_i = (x_i, y_i)$ with $u_i, v_i, x_i, y_i \in \mathbb{R}^d$. Select $w_i \in W_i$, $i \in [n]$ so as to maximize

\[
\sum_{i \neq j} \langle u_i, v_j \rangle - \sum_i \langle x_i, v_i \rangle - \sum_j \langle u_j, y_j \rangle
\]
and set \( w = (u, v) = \sum_{i} w_i \), that is, \( u = \sum_{i} u_i \) and \( v = \sum_{i} v_i \). We will show that (4) holds for every \( k \), that is,

\[
(u - x_k, v_k) + (u_k, v - y_k) \geq 2(u_k, v_k).
\]

Let \( k \in [n] \) be arbitrary. By the condition of the theorem, there exist non-negative numbers \( \alpha(w'_k) \), \( w'_k \in W_k \) so that \( \sum_{w'_k \in W_k} \alpha(w'_k) = 1 \) and

\[
\sum_{w'_k \in W_k} \alpha(w'_k)w'_k = (0, 0).
\]

Moreover, since (6) is maximal, for each \( w'_k = (u'_k, v'_k) \in W_k \),

\[
0 \geq \sum_{i \neq k} (u_i, v'_k - v_k) + \sum_{j \neq k} (u'_k - u_k, v_j) - (x_k, v'_k - v_k) - (u'_k - u_k, y_k).
\]

Multiplying the above equation by \( \alpha(w'_k) \) and summing up for all \( w'_k \in W_k \) leads to

\[
0 \geq \sum_{i \neq k} (u_i, -v_k) + \sum_{j \neq k} (-u_k, v_j) - (x_k, -v_k) - (-u_k, y_k),
\]

which directly implies (7).

**Proof of Theorem 2.** We may assume that \( 0 \in B \). Let \( \lambda_i = r_B(C_i) \) for every \( i \), and \( \lambda := \sum \lambda_i \). Assume on the contrary that \( \lambda < 1 \) and \( B \subseteq \bigcup (C_i + x'_i) \) with some \( x'_1, \ldots, x'_n \in \mathbb{R}^d \). Choose \( \varepsilon > 0 \) so that \((1 + \varepsilon)\lambda < 1 \). For each \( i \), let \( W_i \) be the complete set of contact pairs between \( \lambda_i B - o_i \) and \( C_i - o_i \) which contains \((0, 0)\) in its convex hull. Also, set \( o = (1 + \varepsilon)\sum a_i \) and \( x_i = x'_i + o_i - o \) for each \( i \).

Apply Theorem 1 to the sets \((1 + \varepsilon)W_i \) and the corresponding points \( z_k = (2x_i, 0) \). It implies the existence of \( w_i = (u_i, v_i) \in W_i \), \( i \in [n] \) so that setting \( w = (u, v) = \sum (1 + \varepsilon)w_i \),

\[
(u - 2x_k, (1 + \varepsilon)u_k) + (v, (1 + \varepsilon)u_k) \geq 2(1 + \varepsilon)^2(u_k, v_k)
\]

holds for each \( k \). Since (5) implies that \( (u, v_k) = (v, u_k) \), the above equation simplifies to

\[
(u - x_k, v_k) \geq (1 + \varepsilon)(u_k, v_k) > (u_k, v_k).
\]

Since \( u_k \) is a boundary point of \( C_k - o_k \) with outer normal \( v_k \), the convexity of \( C_k \) implies that \( u - x_k \not\in C_k - o_k \), equivalently, \( u + o \not\in C_k + x'_k \) for any \( k \). On the other hand, \( u_i \in \lambda_i B - o_i \) for every \( i \). Therefore,

\[
u \in \sum_i (1 + \varepsilon)\lambda_i B - \sum_i (1 + \varepsilon)\lambda_i o_i = (1 + \varepsilon)\lambda B - o.
\]

Since \( B \) is convex and \( 0 \in B \), \((1 + \varepsilon)\lambda B \subseteq B \). Hence, \( u + o \in B \), but it is not covered by any of the sets \( C_k + x'_k \), which is a contradiction.

\[ \square \]

The proof of Proposition 1 is nearly identical, thus we leave it to the dedicated reader.

### 3. Applications to Translative Coverings

Corollary 1 readily implies the next statement regarding translative coverings.

**Proposition 2.** Let \( \mathcal{K} = \{ K_1, \ldots, K_n \} \) be a family of convex bodies in \( \mathbb{R}^d \) containing the origin in their interior. For each \( i \), let \( V_i \subseteq S^{d-1} \) be a set of direction vectors for which \( 0 \in \text{conv} V_i \). Denote by \( U_i \) the set of projection vectors of \( 0 \) onto the supporting hyperplanes of \( K_i \) corresponding to members of \( V_i \). Then \( \mathcal{K} \) does not permit a translative covering of \( U_1 + \ldots + U_n \).
A particular case is when all the $K_i$’s are homothets of a fixed convex body $K \in \mathcal{K}^d$. Such homothetic coverings have been studied extensively, see e.g. [Na18], Section 3.2 of [BrMP05] and Section 15.4 of [FT22]. A related conjecture is due to Soltan [So90]:

**Conjecture 3** (V. Soltan). Assume that $K \in \mathcal{K}^d$ and that $\lambda_1 K, \ldots, \lambda_n K$ permit a translatative covering of $K$ with $\lambda_i \in (0, 1)$ for every $i$. Then

$$\sum_{i=1}^{n} \lambda_i \geq d.$$  

Let $T^d$ denote the $d$-dimensional regular simplex. Setting $K = T^d$, $n = d + 1$ and $\lambda_i = \frac{d}{d+1}$ shows that the above bound may not be improved.

Conjecture 3 was proved for $d = 2$ or $n = d + 1$ by Soltan and Vásárhelyi [SoV93] and for $K = B^d$ by Glazyrin [Gl19], while Naszódi [Na10] showed that $\sum \lambda_i > \alpha d$ for any fixed $\alpha < 1$ if $d$ is sufficiently large.

Böröczky asked whether the same bound holds for covering a triangle with its negative homothets. Vásárhelyi [Vá84] gave an affirmative answer. We conclude the article with the extension of this result to arbitrary dimensions.

**Theorem 3.** Assume that $T \subset \mathbb{R}^d$ is a non-degenerate simplex, and $\lambda_1, \ldots, \lambda_n \geq 0$ are so that the family $-\lambda_1 T, \ldots, -\lambda_n T$ permits a translatative covering of $T$. Then

$$\sum_{i=1}^{n} \lambda_i \geq d.$$  

**Proof.** We may assume that $T = T^d$ with its centre at 0. Let $V$ be the set of normal directions of the facets of $T^d$ and $U$ be the set of projection vectors of 0 onto the facets of $T^d$. It is well-known that $\text{conv } U = -\frac{1}{d} T^d$. Applying Proposition 2 with $K_i = -\lambda_i T^d$, $V_i = V$ and $U_i = \lambda_i U$ yields an uncovered point in

$$U_1 + \ldots + U_n \subseteq (\lambda_1 + \ldots + \lambda_n) T^d,$$

which implies (8).  

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I would like to dedicate this piece of work to the loving memory of my father, Imre Ambrus (1953-2021.)

**References**


