GROUPS OF WORLDVIEW TRANSFORMATIONS IMPLIED BY ISOTROPY OF SPACE

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ABSTRACT. Given any Euclidean ordered field, Q, and any 'reasonable' group, G, of (1+3)-dimensional spacetime symmetries, we show how to construct a model $\mathcal{M}_{\mathsf{G}}$ of kinematics for which the set $\mathbb{W}$ of worldview transformations between inertial observers satisfies $\mathbb{W} = \mathsf{G}$. This holds in particular for all relevant subgroups of Gal $c\mathsf{Poi}$, and $c\mathsf{Eucl}$ (the groups of Galilean, Poincaré and Euclidean transformations, respectively, where $c \in Q$ is a model-specific parameter corresponding to the speed of light in the case of Poincaré transformations).

In doing so, by an elementary geometrical proof, we demonstrate our main contribution: spatial isotropy is enough to entail that the set $\mathbb{W}$ of worldview transformations satisfies either $\mathbb{W} \subseteq \mathsf{Gal}$, $\mathbb{W} \subseteq c\mathsf{Poi}$, or $\mathbb{W} \subseteq c\mathsf{Eucl}$ for some $c > 0$. So assuming spatial isotropy is enough to prove that there are only 3 possible cases: either the world is classical (the worldview transformations between inertial observers are Galilean transformations); the world is relativistic (the worldview transformations are Poincaré transformations); or the world is Euclidean (which gives a nonstandard kinematical interpretation to Euclidean geometry). This result considerably extends previous results in this field, which assume a priori the (strictly stronger) special principle of relativity, while also restricting the choice of Q to the field $\mathbb{R}$ of reals.

As part of this work, we also prove the rather surprising result that, for any G containing translations and rotations fixing the time-axis $\mathbf{t}$, the requirement that G be a subgroup of one of the groups Gal, $c\mathsf{Poi}$ or $c\mathsf{Eucl}$ is logically equivalent to the somewhat simpler requirement that, for all $g \in \mathsf{G}$: $g[\mathbf{t}]$ is a line, and if $g[\mathbf{t}] = \mathbf{t}$ then g is a trivial transformation (i.e. g is a linear transformation that preserves Euclidean length and fixes the time-axis setwise).

1. INTRODUCTION

Physical theories conventionally define coordinate systems and transformations using values and functions defined over the field of reals, $\mathbb{R}$. However, this assumption is not well-founded in physical observation because all physical measurements yield only finite-accuracy values — even quantum electrodynamics (QED), one of the most precisely tested physical theories, is only accurate to around 12 decimal digits [OHDG06]. Since we have no empirical reason to make this assumption, it is worth investigating what happens to our expectations of physical theories if we generalize by assuming less about the physical quantities used in measurements. In this paper, we assume only that every positive element in the ordered field of quantities has a square root, but it is worth noting that special relativity can also be modelled over the field of rational numbers [MS13], in which even this assumption fails. It remains an open question whether the new results presented here generalize over arbitrary ordered fields.

Starting in 1910, Ignatovsky's [Ign10a, Ign10b, Ign11] attempt to derive special relativity assuming only Einstein's principle of relativity initiated a new research direction investigating the consequences of assuming the principle of relativity without Einstein's light postulate. However, Frank and Rothe [FR11] quickly identified (1911) that hidden assumptions were implicitly used by both Einstein and Ignatovsky, and it is still not uncommon over a century later to find hidden assumptions in related works.

One notable investigation was that of Borisov [Bor78] (see also [Gut82, §10, pp. 60-61]). Borisov explicitly introduced all the assumptions used in his framework investigating the consequences of the principle of relativity. Then he showed that there are basically two possible cases: either the world is classical and the worldview transformations between inertial observers are Galilean; or the world is relativistic and the worldview transformations are Poincaré transformations.[1]

In [MSS19], we made Borisov's framework even more explicit using first-order logic, and investigated the role of his assumption that the structure of physical quantities is the field of real numbers. We showed that over non-Archimedean fields there is a third possibility: the worldview transformations can also be Euclidean isometries.[2]

In this paper, we present a general axiom system for kinematics using a simple language talking only about quantities, inertial observers (coordinate systems), and the worldview transformations between them. Our axiom system is based on just a few natural assumptions, e.g., instead of assuming that the structure of physical quantities is the field of real numbers we assume only that it is an ordered field Q in which all non-negative values have square roots. Using this framework, we investigate what happens if instead of the principle of relativity we make the weaker assumption that space is isotropic. We show that isotropy is already enough to ensure that the worldview transformations are either Euclidean isometries, or Galilean or Poincaré transformations; see Theorem 5.5 (Classification).

The investigation presented in this paper is part of the Andréka–Németi school's general project of logic-based axiomatic foundations of relativity theories, see e.g., [AMN06, AMN07, AMNS12, AN14]. Friend and Molinini [Fri15, FM15] discuss the significance of this project and the underlying methodology from the viewpoints of epistemology and explanation in science. One important feature of using a first-order logic-based axiomatic framework is that it helps avoid hidden assumptions, which is fundamental in foundational analyses of this nature. Another feature is that it opens up the possibility of machine verification of the results, see e.g., [SN14, GBT15].

2. Framework

We are concerned in this paper with two sorts of objects, *(inertial)* **observers** and **quantities**, which we represent as elements of non-empty sets IOb and Q, respectively.

Observers are interpreted to be labels for inertial coordinate systems. Quantities are used to specify coordinates, lengths and related quantities, and we assume that

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[1] Metric geometries corresponding to these two structures also appear among Cayley-Klein geometries; see, e.g., [Str16] and [PSS17, §6].

[2] That the principle of relativity is consistent with world view transformations being Euclidean isometries has previously been shown by Gyula Dávid [Dáv90].

Q is equipped with the usual binary operations, $\cdot$ (multiplication) and $+$ (addition); constants, 0 and 1 (additive and multiplicative identities); and a binary relation, $\leq$ (ordering).

Although the results presented here can also be generalized to higher-dimensional spaces (though not necessarily lower-dimensional ones — see Sect. 8), we assume for definiteness that observers inhabit 4-dimensional spacetime, Q^4, and locations in spacetime are accordingly represented as 4-tuples over Q. We often write $\vec{p}$, $\vec{q}$ and $\vec{r}$ to denote generic spacetime locations.

For each pair of observers $k, h \in IOb$, we assume the existence of a function $\mathsf{w}_{kh} \colon Q^4 \to Q^4$, called the ***worldview transformation*** *from the worldview of h to the worldview of k*, which we interpret as representing the idea that observers may see (i.e. coordinatize) the same events, but at different spacetime locations: whatever is seen by h at $\vec{p}$ is seen by k at $\mathsf{w}_{kh}\left(\vec{p}\right)$.[3]

Formally, this framework corresponds to using a two sorted first-order language where the models are of the following form

$$\mathcal{M} = (IOb, Q, +, \cdot, 0, 1, \leq, \mathsf{w}),$$

where: IOb and Q are two sorts; $+$ and $\cdot$ are binary operations on Q; 0 and 1 are constants on Q; $\leq$ is a binary relation on Q; and w is a function from $IOb \times IOb \times Q^4$ to Q^4. In this language, the worldview transformation between fixed observers k and h can be introduced as:

$$\mathsf{w}_{kh}\left(t, x, y, z\right) \overset{\text{def}}{=} \mathsf{w}(k, h, t, x, y, z).$$

3. Axioms

In this section, we describe the general axiom system, KIN, used to represent kinematics in this paper. Additional axioms representing spatial isotropy and the special principle of relativity will be introduced in Section 4.

3.1. **Quantities.** We assume that $(Q, +, \cdot, 0, 1, \leq)$ exhibits the most fundamental algebraic properties expected of the real numbers ($\mathbb{R}$), so that calculations can be performed and results compared with one another. We also assume that square-roots are defined for non-negative values (i.e. that Q is a ***Euclidean field*** [EOM20]).

> $\underline{\mathsf{AxEField}}$**:** $(Q, +, \cdot, 0, 1, \leq)$ is a Euclidean field, i.e. a linearly ordered field in which every non-negative element has a square root.

Assuming $\mathsf{AxEField}$ also means that the derived operations of subtraction ($-$), division ($/$), square root ($\sqrt{\ }$), dot product of vectors ($\cdot$), Euclidean length of vectors, etc., are well-defined on their domains, and allows us to assume the usual vector space structure of Q^4 over Q. We will generally omit the multiplication symbol.

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[3] In more general theories, for example in general relativity, this relation need not be a function or even defined on the whole Q^4, because an event seen by k may be invisible to h or may appear at one or more different spacetime locations from h's point of view, but in this paper we assume that all observers completely and unambiguously coordinatize the same universe — they all see the same events, albeit in different locations relative to one another.

3.2. Worldview transformations. The following axiom states informally that: (i) the worldview transformation from an observer's worldview to itself is just the identity transformation, $\mathsf{Id}\colon Q^4 \to Q^4$; and (ii) switching from k's worldview to h's and then to m's has the same effect as switching directly from k's worldview to m's.

> AxWvt: For all $k, h, m \in IOb$:
>   (i) $\mathsf{w}_{kk} = \mathsf{Id}$;
>   (ii) $\mathsf{w}_{mh} \circ \mathsf{w}_{hk} = \mathsf{w}_{mk}$.

3.3. Lines, worldlines and motion. By assumption, all of the locations under discussion in this paper are points in Q^4. We often write (t, x, y, z) to indicate the coordinates of a generic point in Q^4. Given any $n > 0$ and $\vec{p} = (p_1, p_2, \ldots, p_n) \in Q^n$, its **squared length**, $|\vec{p}|^2$, is defined by

$$|\vec{p}|^2 \stackrel{\text{def}}{=} p_1^2 + \ldots + p_n^2.$$

(This is just the standard Euclidean squared length of $\vec{p}$.)

To simplify our notation, we write $\vec{\mathsf{o}} \stackrel{\text{def}}{=} (0, 0, 0, 0)$ for the zero-vector (origin) in Q^4. More generally, we sometimes write $\vec{0}$ for any tuple of zeroes (the length will always be clear from context). We define the **time-axis**, $\mathbf{t}$, and the **present simultaneity**, $\mathbf{S}$, to be the set

$$\mathbf{t} \stackrel{\text{def}}{=} \{(t, 0, 0, 0) : t \in Q\}.$$

and the spatial hyperplane

$$\mathbf{S} \stackrel{\text{def}}{=} \{(0, x, y, z) : x, y, z \in Q\},$$

respectively. We write $\vec{\mathsf{t}}$ for the unit time vector $(1, 0, 0, 0)$, and likewise $\vec{\mathsf{x}} \stackrel{\text{def}}{=} (0, 1, 0, 0)$, $\vec{\mathsf{y}} \stackrel{\text{def}}{=} (0, 0, 1, 0)$ and $\vec{\mathsf{z}} \stackrel{\text{def}}{=} (0, 0, 0, 1)$. If $\vec{p} = (t, x, y, z) \in Q^4$, we call $\vec{p}_t \stackrel{\text{def}}{=} t$ the **time component**, and $\vec{p}_s \stackrel{\text{def}}{=} (x, y, z)$ the **space component**, of $\vec{p}$. Finally, if $t \in Q$ and $\vec{s} \in Q^3$, we write $(t, \vec{s})$ for the point with time component t and space component $\vec{s}$.

The **worldline** of observer h according to observer k is defined as

$$\mathsf{wl}_k(h) \stackrel{\text{def}}{=} \mathsf{w}_{kh}[\mathbf{t}].$$

In particular, if we assume AxWvt and take $k = h$, we have $\mathsf{wl}_h(h) = \mathsf{w}_{hh}[\mathbf{t}] = \mathbf{t}$. This corresponds to the convention that observers consider themselves to be at the spatial origin relative to which measurements are made: from their own viewpoint their worldline is the time-axis; and $\mathsf{wl}_k(h) = \mathsf{w}_{kh}[\mathbf{t}] = \mathsf{w}_{kh}[\mathsf{wl}_h(h)]$ describes the same worldline but from k's point of view.

When we say that one observer moves *inertially* with respect to another, we mean that neither of them accelerates relative to the other, so that linear motions seen by one remain linear when seen by the other. Since each observer considers its own worldline to be the line $\mathbf{t}$, we would expect all inertial observers to agree that each others' world lines are lines.

Formally, a subset $\ell \subseteq Q^4$ is a **line** iff there are $\vec{p}, \vec{v} \in Q^4$, where $\vec{v} \neq \vec{\mathsf{o}}$ and $\ell = \{\vec{p} + \lambda\vec{v} : \lambda \in Q\}$. The next axiom states that worldlines of observers according to observers are lines.

> AxLine: For every $k, h \in IOb, \mathsf{wl}_k(h)$ is a line.

According to AxLine, the worldlines of observers are lines, and by AxWvt each observer considers its own worldline to be the time-axis; we can therefore express the idea that *observer k is **moving according to** observer m* by saying that $\mathsf{wl}_m(k)$ is not parallel to $\mathbf{t}$,[4] or more simply, that w_{mk} takes the time-unit vector $\vec{\mathsf{t}}$ and the zero-vector $\vec{\mathsf{o}}$ to coordinate points having different spatial components, i.e. $\mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s \neq \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s$. In the same spirit, we say that k ***is at rest according to*** m iff $\mathsf{w}_{mk}(\vec{\mathsf{t}})_s = \mathsf{w}_{mk}(\vec{\mathsf{o}})_s$.

We will sometimes need to assume explicitly the existence of observers moving relative to one another, which we express using the following formula:

$\underline{\exists\mathsf{MovingIOb}}$**:** There are observers $m, k \in IOb$ such that $\mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s \neq \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s$.

3.4. **Trivial transformations.** We say that a linear transformation $T : Q^4 \to Q^4$ is a ***linear trivial transformation*** provided it fixes (setwise) both the time-axis and the present simultaneity, and preserves squared lengths in both, i.e.

- if $\vec{p} \in \mathbf{t}$, then $T(\vec{p}) \in \mathbf{t}$ and $T(\vec{p})_t^2 = \vec{p}_t^2$; and
- if $\vec{p} \in \mathbf{S}$, then $T(\vec{p}) \in \mathbf{S}$ and $|T(\vec{p})_s|^2 = |\vec{p}_s|^2$.

Remark 3.1. Assuming AxEField, the statement that T is a linear trivial transformation is equivalent to the statement that T is a linear transformation that preserves Euclidean length and fixes the time-axis setwise.[5]   □

A map $f : Q^4 \to Q^4$ is a ***translation*** iff there is $\vec{q} \in Q^4$ such that $f(\vec{p}) = \vec{p} + \vec{q}$ for every $\vec{p} \in Q^4$. We write Trans for the *set of all translations.*

A transformation is called a ***trivial transformation*** if it is a linear trivial transformation composed with a translation. We write Triv for the *set of all trivial transformations.*

We say that two observers k and k' are ***co-located*** if they consider themselves to share the same worldline: $\mathsf{wl}_k(k) = \mathsf{wl}_k(k')$ (assuming AxWvt, this relationship is symmetric; see Lemma 6.3.5 (Equal Worldlines)). The following axiom says that, if observers k and k' are co-located, then their worldviews are related to one another by a trivial transformation. In other words, even though inertial observers following the same worldline may use different coordinate systems, these coordinate systems can only differ by using a different orthonormal basis for coordinatizing space and/or a different direction and origin of time.[6]

$\underline{\mathsf{AxColocate}}$**:** For all $k, k' \in IOb$, if $\mathsf{wl}_k(k) = \mathsf{wl}_k(k')$, then $\mathsf{w}_{kk'} \in$ Triv.

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[4]As one would expect, being in motion relative to another observer — and likewise being at rest — are symmetric relations; see Lemma 6.6.2 (Rest).

[5] This claim follows by Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$), but can also be proven directly. Suppose T is linear, preserves Euclidean length and fixes $\mathbf{t}$ setwise. It follows immediately that $T(\vec{\mathsf{t}}) = \pm\vec{\mathsf{t}}$. Now choose any $(0, \vec{s}) \in \mathbf{S}$, and suppose $T(0, \vec{s}) = (t', \vec{s}')$. Then $|T(\pm 1, \vec{s})|^2 = |T(0, \vec{s}) \pm T(\vec{\mathsf{t}})|^2 = (t' \pm 1)^2 + |\vec{s}'|^2$. Since $|(1, \vec{s})|^2 = |(-1, \vec{s})|^2$ and T preserves Euclidean length, we therefore require $(t' + 1)^2 + |\vec{s}'|^2 = (t' - 1)^2 + |\vec{s}'|^2$, whence $t' = 0$. Thus, T also fixes $\mathbf{S}$, so it is a linear trivial transformation. The converse is trivial.

[6] By AxWvt, if k and k' are co-located, i.e. $\mathsf{wl}_k(k) = \mathsf{wl}_k(k')$, then $\mathsf{w}_{kk'}[\mathbf{t}] = \mathbf{t}$. This is why we do not need to assume explicitly in the statement of AxColocate that co-located observers share the same time-axis.

3.5. **Spatial rotations.** A linear trivial transformation $R : Q^4 \rightarrow Q^4$ is called a **spatial rotation** iff it preserves the direction of time and the orientation of space, i.e. $R(\vec{t}) = \vec{t}$ and the determinant of 3×3 matrix $[R(\vec{x})_s, R(\vec{y})_s, R(\vec{z})_s]$ is positive.[7] We denote the *set of all spatial rotations* by SRot.

The following axiom says that translated and spatially rotated versions of any inertial coordinate system are also inertial coordinate systems.[8]

> AxRelocate: For all $k \in IOb$ and for all $T \in$ Trans $\cup$ SRot, there is $h \in IOb$ such that $\mathsf{w}_{kh} = T$.

The underlying axiom system with which we are concerned in this paper is

$$\mathsf{KIN} \stackrel{\mathrm{def}}{=} \{\mathsf{AxEField}, \mathsf{AxWvt}, \mathsf{AxLine}, \mathsf{AxRelocate}, \mathsf{AxColocate}\},$$

which defines our basic theory of the **kinematics** of inertial observers.

4. The special principle of relativity, isotropy and set of worldview transformations

There are many different formal interpretations of the principle of relativity [Göm15, GS15, MSS17]. In this paper, we interpret the **special principle of relativity** (SPR) to mean that all inertial observers agree as to how they are related to other observers, so that no observer can be distinguished from any other in terms of the things they can and cannot (potentially) observe. We express this via the following axiom:

> AxSPR: For every $k, k^*, h \in IOb$, there exists $h^* \in IOb$ such that $\mathsf{w}_{kh} = \mathsf{w}_{k^*h^*}$,

that is, given observers k, k^*, h, there must (potentially) be some h^* which is related to k^* in exactly the same way that h is related to k, i.e. the geometrical structure of spacetime cannot forbid such an observer.

In contrast, **isotropy** refers to the weaker constraint that there is no distinguished direction in space, i.e. no matter which direction we face, we should be able to perform the same experiments and observe the same outcomes. Isotropy can be expressed in much the same way as SPR, except that we only require equivalence as to what can be observed (h) when the relevant observers (k and k^*) are related via a spatial rotation (see Figure 1):

> AxIsotropy: For every $k, k^*, h \in IOb$, if $\mathsf{w}_{kk^*} \in$ SRot, there exists $h^* \in IOb$ such that $\mathsf{w}_{kh} = \mathsf{w}_{k^*h^*}$.

In order to investigate these ideas, we will need to consider various sets of worldview transformations, and attempt to establish both their algebraic properties and

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[7] This can be expressed in our formal language without any assumption about the structure of quantities as: $R(\vec{x})_2 R(\vec{y})_3 R(\vec{z})_4 + R(\vec{x})_4 R(\vec{y})_2 R(\vec{z})_3 + R(\vec{x})_3 R(\vec{y})_4 R(\vec{z})_2 > R(\vec{x})_4 R(\vec{y})_3 R(\vec{z})_2 + R(\vec{x})_2 R(\vec{y})_4 R(\vec{z})_3 + R(\vec{x})_3 R(\vec{y})_2 R(\vec{z})_4$, here $R(\vec{p})_2$, $R(\vec{p})_3$, and $R(\vec{p})_4$ denotes the second, third and fourth component of $R(\vec{p}) \in Q^4$, i.e. if $R(\vec{p}) = (t, x, y, z)$, then $R(\vec{p})_2 = x$, $R(\vec{p})_3 = y$, and $R(\vec{p})_4 = z$.

[8] The quantification over T in AxRelocate appears at first sight to be second-order. However, because translations are determined by the image of the origin, while spatial rotations are determined by the images of the three spatial unit vectors, this axiom can be formalized in our first-order logic language by quantifying over the 4 parameters representing the image of the origin and the 12 parameters representing the images of the three spatial unit vectors.
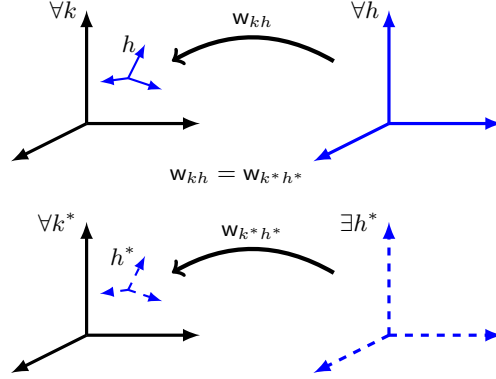
FIGURE 1. Isotropy and the special principle of relativity. The special principle, AxSPR, says that given any k, h and k^*, there exists an h^* that is related to k^* the same way that h is related to k (i.e. there are no distinguished inertial coordinate systems). Spatial isotropy, AxIsotropy, is similar, except that we only require h^* to exist when w_{kk^*} is a spatial rotation (i.e. rotating ones spatial coordinate system has no effect on what can and cannot potentially be seen).

the relationships between them. The set $\mathbb{W}_k$ of worldview transformations associated with a specific observer $k \in IOb$ will be defined by

$$\mathbb{W}_k \overset{\text{def}}{=} \{\mathsf{w}_{kh} : h \in IOb\}$$

and the set of *all* worldview transformations is then given by

$$\mathbb{W} \overset{\text{def}}{=} \{\mathsf{w}_{kh} : k, h \in IOb\} = \bigcup \{\mathbb{W}_k : k \in IOb\}.$$
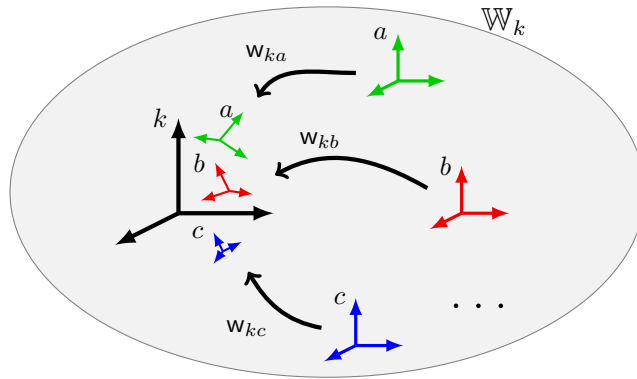
FIGURE 2. The set $\mathbb{W}_k$ of all worldview transformations into k's coordinate system. For each observer $a, b, c, \ldots$, the set $\mathbb{W}_k$ contains the associated transformation $\mathsf{w}_{ka}, \mathsf{w}_{kb}, \mathsf{w}_{kc}, \ldots$.

Using these notations AxSPR can be reformulated as saying that all inertial observers have essentially the same worldview, i.e. $\mathbb{W}_k = \mathbb{W}_{k^*}$ for all $k, k^* \in IOb$.

Although it is not immediately obvious that any $\mathbb{W}_k$ can form a group, if we assume AxWvt it can be proven that AxSPR is equivalent to saying that there is at least one k for which $\mathbb{W}_k$ forms a group under composition, which is itself equivalent to saying that $\mathbb{W}_k = \mathbb{W}$. For the proof of this and other equivalent formulations of AxSPR, see [MSS19, Prop. 2.1]. Similarly, AxIsotropy is equivalent to saying, for all $k, k^* \in IOb$, if $\mathsf{w}_{kk^*} \in \mathsf{SRot}$, then $\mathbb{W}_k = \mathbb{W}_{k^*}$.

Remark 4.1. We have already noted that AxSPR entails AxIsotropy, so that the special principle of relativity is at least as strong assumption as spatial isotropy. In fact, it is strictly stronger, because $\mathbb{W}$ is a group in all models of KIN + AxIsotropy, but $\mathbb{W}_k$ need not be. In particular, therefore, KIN + AxIsotropy does not imply AxSPR. This remains true even if we add the restriction that $(Q, +, \cdot, 0, 1, \leq)$ is the ordered field of real numbers. However, if we add the assumption that co-located observers agree on the direction of time, then it can be shown that KIN + AxIsotropy implies AxSPR.

For easy reference, Table 1 summarizes the axioms used in this paper and discussed above.

TABLE 1. Our axioms and their intuitive meanings.

KIN	Axiom	Description
✓	AxEField	the set Q of quantities is an ordered field in which all non-negative values have square roots
✓	AxWvt	w_{kk} transforms k's worldview to itself identically; and going from k's worldview to h's and then to m's is same as going directly from k's worldview to m's
✓	AxLine	inertial observers see each other's worldlines as lines
✓	AxColocate	if two observers are co-located, their worldviews are trivially related to one another
✓	AxRelocate	translated and spatially rotated versions of inertial coordinate systems are also inertial
	AxSPR	the special principle of relativity
	AxIsotropy	isotropy of space

5. Main theorems

First let us introduce the transformations that will be used in this paper to characterize the worldviews of observers. In this section, we assume that $(Q, +, \cdot, 0, 1)$ is a field. Table 2 summarizes the various transformation groups referred to in the theorems.

5.1. κ-isometries. Given $\vec{p} = (t, x, y, z)$, the *(squared) κ-length* of $\vec{p}$ is defined by
$$\|(t, x, y, z)\|_\kappa^2 \stackrel{\text{def}}{=} t^2 - \kappa(x^2 + y^2 + z^2),$$
or in other words,
$$\|\vec{p}\|_\kappa^2 \stackrel{\text{def}}{=} \vec{p}_t^2 - \kappa|\vec{p}_s|^2.$$

TABLE 2. Transformation groups considered in this paper.

Trans	translations		
SRot	spatial rotations		
Triv	trivial transformations		
$_\kappa$Iso	κ-isometries		
cPoi	c-Poincaré transformations	$=$	$_{1/c^2}$Iso
cEucl	c-Euclidean transformations	$=$	$_{-1/c^2}$Iso
Gal	Galilean transformations	$=$	$_0$Iso

Taking $\kappa = 1$ gives the squared *Minkowski length* $\|\vec{p}\,\|_1^2 = t^2 - (x^2 + y^2 + z^2)$ of $\vec{p}$, while $\kappa = -1$ gives its squared *Euclidean length*, $\|\vec{p}\,\|_{-1}^2 = |\vec{p}\,|^2 = t^2 + (x^2 + y^2 + z^2)$.

Definition 5.1.1 (κ-isometry, $\kappa \neq 0$)**.** If $\kappa \neq 0$, we call a linear transformation $f : Q^4 \to Q^4$ a ***linear κ-isometry*** provided it preserves κ-length, i.e. for every $\vec{p} \in Q^4$,

$$\|f(\vec{p})\|_\kappa^2 = \|\vec{p}\,\|_\kappa^2 \,.$$

In the case of $\kappa = 0$, we require more than simply preserving 0-length, for while 0-length takes account of temporal extent, it ignores spatial structure. We therefore need to add an extra condition to the definition of 0-isometry to ensure that spatial structure is also respected when considering points with equal time coordinates.[9]

Definition 5.1.2 (κ-isometry, $\kappa = 0$)**.** Let $f : Q^4 \to Q^4$ be a linear transformation. We call f a ***linear 0-isometry*** provided, for every $\vec{p} \in Q^4$,

$$(5.1) \qquad f(\vec{p})_t^2 = \vec{p}\,_t^2 \text{ and } \left(\vec{p}\,_t = 0 \;\Rightarrow\; |f(\vec{p})_s|^2 = |\vec{p}\,_s|^2 \right).$$

We call the composition of a linear κ-isometry and a translation a κ-***isometry***, and write $_\kappa$Iso for the *set of all κ-isometries*.

Definition 5.1.3 (cPoi, cEucl and Gal)**.** For $c > 0$, $1/c^2$-isometries will be called c-***Poincaré transformations*** and $-1/c^2$-isometries will be called c-***Euclidean isometries***. Parameter c in c-Poincaré transformations corresponds to the "speed of light". A 0-isometry is also called a ***Galilean symmetry***. We denote these sets of transformations by cPoi, cEucl and Gal, respectively.

It is easily verified that each of these sets forms a group under function composition. In general, when we speak about a set G of transformations as a group, we mean G under function composition, i.e. $(\mathsf{G}, \circ)$. As usual, we write $\mathsf{H} \leq \mathsf{G}$ to mean that H is a subgroup of G, and $\mathsf{H} < \mathsf{G}$ to mean that the inclusion is proper.

We note that 1-Poincaré transformations form the usual group Poi of Poincaré transformations and 1-Euclidean isometries form the usual group Eucl of Euclidean isometries. Notice also that trivial transformations, translations and spatial rotations are κ-isometries for all values of κ. Moreover, by Lemma 6.3.2 ($\mathsf{Triv} = \bigcap {}_\kappa\mathsf{Iso}$),

$$(5.2) \qquad \mathsf{Trans} \cup \mathsf{SRot} \subset \mathsf{Triv} = \bigcap_{\kappa \in Q} {}_\kappa\mathsf{Iso} = {}_x\mathsf{Iso} \cap {}_y\mathsf{Iso}$$

for any two distinct $x, y \in Q$. It follows immediately that $\mathsf{Trans} \cup \mathsf{SRot} \subset c\mathsf{Poi} \cap c\mathsf{Eucl} \cap \mathsf{Gal}$.

[9]Although every 0-isometry preserves 0-length, the converse is not true.

5.2. **The theorems.** Our first result, Theorem 5.1 (Characterisation), tells us that if space is isotropic then all worldview transformations are κ-isometries for some κ, and shows how to calculate the value of κ in the case that two observers can be found which move relative to one another.

Theorem 5.1 (Characterisation)**.** Assume KIN+AxIsotropy. Then there is a $\kappa \in Q$ such that the set of worldview transformations is a set of κ-isometries, i.e.

$$\mathbb{W} \subseteq {}_{\kappa}\mathsf{Iso}.$$

In other terms,

$$\text{either } \mathbb{W} \subseteq c\mathsf{Poi}, \ \mathbb{W} \subseteq \mathsf{Gal}, \text{ or } \mathbb{W} \subseteq c\mathsf{Eucl} \text{ for some } c > 0.$$

Moreover,

- if ¬∃MovingIOb is assumed, then $\mathbb{W} \subseteq \mathsf{Triv}$;
- if ∃MovingIOb is assumed, this κ is uniquely determined by the w_{mk}-images of $\vec{\mathsf{o}}$ and $\vec{\mathsf{t}}$ where m and k are observers moving relative to one another, and can be calculated as

$$\kappa = \frac{\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right|^2 - 1}{\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s \right|^2}.$$

$\square$

For all positive $c \in Q$, the group $c\mathsf{Poi}$ is isomorphic to group Poi (via natural inner automorphisms of the affine group, representing the effects of changing the spatial or temporal units of measurements) and similarly group $c\mathsf{Eucl}$ is isomorphic to the Euclidean transformation group Eucl (via the same inner automorphisms); see [MSS19, Prop. 6.9]. So essentially there are only three nontrivial cases: either all the worldview transformations are relativistic; all of them are classical; or all of them are Euclidean isometries. Subject to this constraint, however, Theorem 5.3 (Model Construction) says that all 'reasonable' transformation groups (groups containing the translations and spatial rotations, which we know must be present) can occur as the group of worldview transformations in a model of KIN + AxSPR.

To present a general model construction, let us write $\mathsf{Sym}(Q^4)$ for the set of all permutations of Q^4. Given any transformation group $\mathsf{G} \leq \mathsf{Sym}(Q^4)$, we define a model $\mathcal{M}_{\mathsf{G}}$ of our language by taking $IOb := \mathsf{G}$ and $\mathsf{w}_{mk} := m \circ k^{-1}$ for $k, m \in \mathsf{G}$. Theorem 5.2 (Satisfaction) connects the axioms of KIN to properties of G.

Theorem 5.2 (Satisfaction)**.** Let $\mathsf{G} \leq \mathsf{Sym}(Q^4)$. Then

- (a) $\mathcal{M}_{\mathsf{G}}$ satisfies AxWvt, AxSPR and $\mathbb{W} = \mathsf{G}$.
- (b) $\mathcal{M}_{\mathsf{G}}$ satisfies AxRelocate iff $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathsf{G}$.
- (c) $\mathcal{M}_{\mathsf{G}}$ satisfies AxLine iff $g[\mathbf{t}]$ is a line for all $g \in \mathsf{G}$.
- (d) $\mathcal{M}_{\mathsf{G}}$ satisfies AxColocate iff $g \in \mathsf{Triv}$ whenever $g \in \mathsf{G}$ and $g[\mathbf{t}] = \mathbf{t}$.

Theorem 5.3 (Model Construction)**.** Assume AxEField. Let G be a group such that

- $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathsf{G} \leq c\mathsf{Poi}$ for some $c \in Q$; or
- $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathsf{G} \leq c\mathsf{Eucl}$ for some $c \in Q$; or
- $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathsf{G} \leq \mathsf{Gal}$.

Then $\mathcal{M}_{\mathsf{G}}$ is a model of KIN + AxSPR for which $\mathbb{W} = \mathsf{G}$.

By Theorem 5.1 (Characterisation), Theorem 5.3 (Model Construction) and Theorem 5.2 (Satisfaction), in order to determine whether a group of symmetries has to be a subgroup of one of the groups $c\mathsf{Poi}$, $c\mathsf{Eucl}$ and Gal, it is sufficient to consider its members' actions on $\mathbf{t}$:

Theorem 5.4 (Determination)**.** Let $(Q, +, \cdot, 0, 1, \leq)$ be a Euclidean field, and let G be a group satisfying $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathsf{G} \leq \mathsf{Sym}(Q^4)$. Then

(i) For all $g \in \mathsf{G}$, $g[\mathbf{t}]$ is a line, and $\qquad$ (ii) $\mathsf{G} \leq c\mathsf{Poi}$, $\mathsf{G} \leq c\mathsf{Eucl}$ or $\mathsf{G} \leq \mathsf{Gal}$
$\quad$ if $g[\mathbf{t}] = \mathbf{t}$, then $g \in \mathsf{Triv}$. $\qquad \Longleftrightarrow \qquad$ for some positive $c \in Q$.

Our next result, Theorem 5.5 (Classification), tells us that we can classify all possible models by looking at how observers' clocks run relative to one another. Based on the difference between the time components of the w_{mk}-image of $\vec{\mathsf{t}}$ and $\vec{\mathsf{o}}$, we can decide whether observer k's clock is fast, slow or accurate relative to observer m's clock; see Figure 3. Using these notions, we can capture the following situations:

$\exists\underline{\mathsf{SlowClock}}$**:** There are observers $m, k \in IOb$ such that
$$\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right| > 1.$$

$\exists\underline{\mathsf{FastClock}}$**:** There are observers $m, k \in IOb$ such that
$$\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right| < 1.$$

$\exists\underline{\mathsf{MovingAccurateClock}}$**:** There are observers $m, k \in IOb$ such that
$$\mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s \neq \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s \ \text{ and } \ \left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right| = 1.$$

$\forall\underline{\mathsf{MovingClockSlow}}$**:** For all observers $m, k \in IOb$,
$$\text{if } \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s \neq \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s, \ \text{ then } \ \left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right| > 1.$$

$\forall\underline{\mathsf{MovingClockFast}}$**:** For all observers $m, k \in IOb$,
$$\text{if } \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s \neq \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s, \ \text{ then } \ \left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right| < 1.$$

PSfrag replacements

$\forall\underline{\mathsf{ClockAccurate}}$**:** For all observers $m, k \in IOb$, $\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right| = 1.$

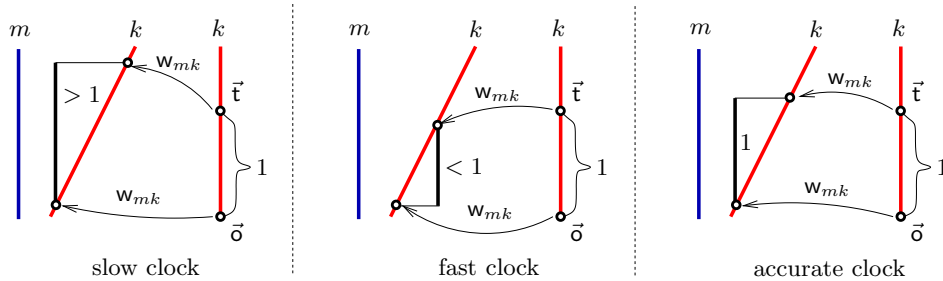FIGURE 3. k's clock can be fast, slow or accurate according to m

Theorem 5.5 (Classification). Assume KIN + AxIsotropy. Then precisely one of the following four cases holds:

(1) There exists a slow clock (∃SlowClock). In this case, there exists a moving observer (∃MovingIOb), all moving clocks are slow (∀MovingClockSlow), and

$$\mathbb{W} \subseteq c\mathsf{Poi} \text{ for some positive } c \in Q.$$

(2) There exists a fast clock (∃FastClock). In this case, there exists a moving observer (∃MovingIOb), all moving clocks are fast (∀MovingClockFast), and

$$\mathbb{W} \subseteq c\mathsf{Eucl} \text{ for some positive } c \in Q.$$

(3) There exists a moving accurate clock (∃MovingAccurateClock). In this case, all clocks are accurate (∀ClockAccurate) and

$$\mathbb{W} \subseteq \mathsf{Gal}.$$

(4) There are no moving observers (¬∃MovingIOb). In this case,

$$\mathbb{W} \subseteq \mathsf{Triv}.$$

By Theorem 5.6 (Consistency), all of these situations can indeed arise.

Theorem 5.6 (Consistency). The following axiom systems are all consistent (they all have models):

(1) KIN + AxSPR + ∃SlowClock,
(2) KIN + AxSPR + ∃FastClock,
(3) KIN + AxSPR + ∃MovingAccurateClock,
(4) KIN + AxSPR + ¬∃MovingIOb.

6. Subsidiary theorems and lemmas

Because we use only a small number of basic axioms, we have a large number of intermediate lemmas to prove before we can prove our main theorems. This section is accordingly split into six subsections, each focussing on a key stage in the overall proof of our main findings. Each stage builds on its predecessor(s) and together they establish the following subsidiary theorems. Informally stated, they assert (subject to various conditions) that:

Theorem 6.1 (Observer Lines Lemma):
If ℓ is a possible worldline, then all lines of the same slope as ℓ are also possible worldlines.

Theorem 6.2 (Line-to-Line Lemma):
Each worldview transformation is a bijection taking lines to lines, planes to planes and hyperplanes to hyperplanes.

Theorem 6.3 (tx-Plane Lemma):
If w_{km} maps the tx-plane to itself, then it also maps the yz-plane to itself; moreover, if w_{km} is linear, there is some positive λ such that $|\mathsf{w}_{km}(\vec{p})| = \lambda|\vec{p}|$ for all $\vec{p}$ in the yz-plane.

Theorem 6.4 (Same-Speed Lemma):
Suppose at least one observer considers h and k to be travelling with the same speed. Then w_{hk} is a κ-isometry for some κ.

Theorem 6.5 (Fundamental Lemma):
Suppose no observers move with infinite speed, and that $\mathsf{speed}_k(m) = u > 0$. Then there exists $\varepsilon > 0$ for which, given any positive $v \leq u + \varepsilon$, there is some h with $\mathsf{speed}_k(h) = v$ and $\mathsf{speed}_m(h) = \mathsf{speed}_m(k)$.

Theorem 6.6 (Main Lemma):

There exists at least one observer k and one κ for which all worldview transformations w_{mk} involving observers m who agree with k about the origin are κ-isometries.

The order of implications in the proofs that follow is:
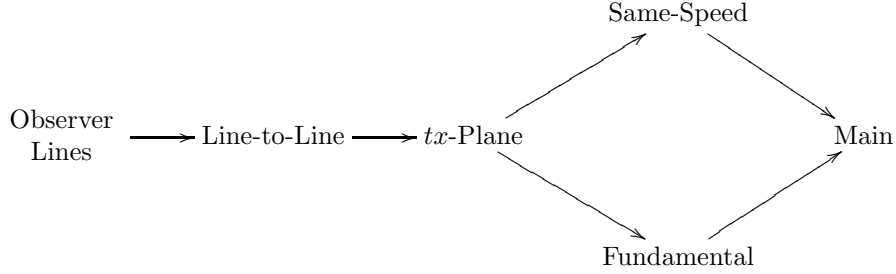
6.1. **Observer Lines Lemma.** We say that a subset $\ell \subseteq Q^4$ is an ***observer line for*** k if there is some observer h for which $\ell = \mathsf{wl}_k(h)$, and write $\mathsf{ObLines}(k)$ for the set of k-observer lines. We say that ℓ is an *observer line* if there is some k for which it is an observer line. By AxLine, all observer lines are lines (because they are worldlines). In this section, we prove that if k can see an observer travelling along a worldline, then every other line with the same slope is also a worldline as far as k is concerned; there are none of these lines from which observers are banned.

Now suppose $\mathsf{AxEField}$ holds. If ℓ is a line and $\vec{p}$, $\vec{q}$ are distinct points in ℓ, we define its ***slope*** by

$$\mathsf{slope}(\ell) \stackrel{\text{def}}{=} \begin{cases} |\vec{p}_s - \vec{q}_s|/|\vec{p}_t - \vec{q}_t| & \text{if } \vec{p}_t \neq \vec{q}_t, \\ \infty & \text{otherwise .} \end{cases}$$

Theorem 6.1 (Observer Lines Lemma)**.** Assume $\mathsf{AxEField}$, AxWvt, $\mathsf{AxRelocate}$, AxLine and $\mathsf{AxIsotropy}$. Suppose either

(a) $\mathsf{slope}(\ell) = \mathsf{slope}(\ell') \neq \infty$; or else
(b) $\mathsf{slope}(\ell) = \mathsf{slope}(\ell') = \infty$ and there exist $\vec{p} \in \ell$ and $\vec{q} \in \ell'$ whose time coordinates are equal.

Then for any observer k, we have $\ell \in \mathsf{ObLines}(k)$ iff $\ell' \in \mathsf{ObLines}(k)$. $\qquad\square$

In order to prove this result, we require various supporting lemmas (the more elementary ones are re-used in subsequent proofs). These lemmas refer to a concept we call F-*transformation* that relates the worldviews of any two observers via that of a third (see Figure 4). To illustrate the concept, suppose that I am observing two planets, k and k^*, in the night sky. From my point of view, people living on those planets would see the world quite differently, but they nonetheless see the same world I do, so I ought to be able to find some function (F) that transforms "what I think k sees" into "what I think k^* sees". From my point of view, I can say that "k^* is an 'F-transformed' version of k."

Definition 6.1.1 (F-transforms)**.** Given any bijection $F : Q^4 \rightarrow Q^4$, we say that k^* *is an F-**transformed** version of k according to h*, and write $k \stackrel{F}{\leadsto}_h k^*$ if

(6.1) $$\mathsf{w}_{hk^*} = F \circ \mathsf{w}_{hk}.$$

$\qquad\square$

Remark 6.1. Assuming AxWvt, $k \overset{\mathsf{Id}}{\leadsto}_h k^*$ is equivalent to $\mathsf{w}_{k^*k} = \mathsf{Id}$, in particular $k \overset{\mathsf{Id}}{\leadsto}_h k$; relations $k \overset{F}{\leadsto}_h k^*$ and $k^* \overset{G}{\leadsto}_h k'$ imply $k \overset{G \circ F}{\leadsto}_h k'$; and $k \overset{F}{\leadsto}_h k^*$ implies $k^* \overset{F^{-1}}{\leadsto}_h k$.

PSfrag replacements

w_{hk^*}

w_{hk}

worldview of h

k $\qquad$ k^*

F

w_{hk} $\qquad$ w_{hk^*}

$k \overset{F}{\leadsto}_h k^*$

$\mathsf{wl}_h(k)$ $\qquad$ $\mathsf{wl}_h(k^*)$

F

Worldline Relocation

$\forall h$

$\forall k$ $\qquad$ $\forall R$ $\qquad$ $\exists k^*$

$k \overset{R}{\leadsto}_h k^*$
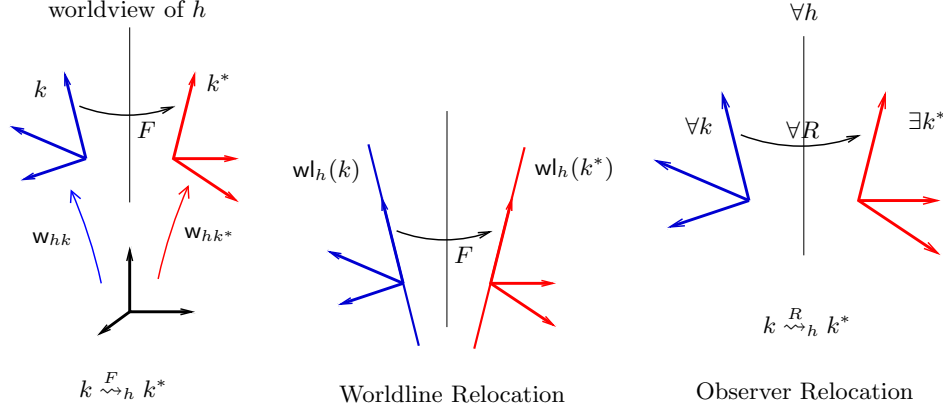
Observer Relocation

FIGURE 4. F-transforms (left) describe how h can transform what it considers to be k's worldview — and worldline (middle) — into k^*'s (Definition 6.1.1, Lemma 6.1.3 (Worldline Relocation)). Lemma 6.1.4 (Observer Rotation) tells us that all spatial rotations can be interpreted as F-transforms (right).

6.1.1. *Supporting lemmas.* Some of these initial lemmas are quite elementary, but they form the bedrock of what follows, and we need to prove them formally to ensure they definitely follow from our somewhat restricted first-order axiom set. The supporting lemmas can be informally described as follows:

Lemma 6.1.2 (WVT):
This describes various elementary properties concerning worldview transformations. We often use these results without further mention.

Lemma 6.1.3 (Worldline Relocation):
If h can F-transform k into k^*, then that transformation maps k's worldline into k^*'s.

Lemma 6.1.4 (Observer Rotation):
Every spatial rotation can be interpreted as an F-transform.

Lemma 6.1.5 (Transformed Observer Lines):
If ℓ is an observer line for k, then $\mathsf{w}_{hk}[\ell]$ is an observer line for h.

Lemma 6.1.6 (Rotated Observer Lines):
If ℓ is an observer line for k, so is any spatially rotated copy of ℓ.

Lemma 6.1.7 (Horizontal Rotation):
This is a technical lemma telling us when one pair of mutually orthogonal horizontal vectors can be spatially rotated into another (where "horizontal" means "orthogonal to the time-axis").

Lemma 6.1.8 (Same-Slope Rotation):
If two lines have the same slope and both pass through the origin, it is possible to spatially rotate one into the other.

Lemma 6.1.9 (Observer Line Intersections):
   Suppose two intersecting lines have the same slope. If one of them is an observer line for k, then so is the other.

Lemma 6.1.10 (Triangulation):
   Suppose $\mathbf{t}'$ is a line parallel to the time-axis, $\mathbf{t}$, and that $\vec{p}$ is not on $\mathbf{t}'$. Given any positive λ we can find lines ℓ_1 and ℓ_2 which intersect at $\vec{p}$, meet $\mathbf{t}'$ at different points, and have the same slope, λ. In other words, we can find an isosceles triangle whose base is along $\mathbf{t}'$ and vertex at $\vec{p}$, and whose equal non-base sides both have slope λ.

6.1.2. *Proofs of the supporting lemmas.*

Lemma 6.1.2 (WVT)**.** Assume AxWvt. Then, for every $k, h, m \in IOb$,

   (i) $\mathsf{wl}_k(k) = \mathbf{t}$;
   (ii) $\mathsf{w}_{hk}\left[\mathsf{wl}_k(m)\right] = \mathsf{wl}_h(m)$;
   (iii) $\mathsf{w}_{hk} : Q^4 \to Q^4$ is a bijection from Q^4 onto itself;
   (iv) $\mathsf{w}_{hk}^{-1} = \mathsf{w}_{kh}$.

Proof. (i) $\mathsf{wl}_k(k) = \mathsf{w}_{kk}[\mathbf{t}] = \mathsf{Id}[\mathbf{t}] = \mathbf{t}$.
   (ii) Since $\mathsf{wl}_k(m) = \mathsf{w}_{km}[\mathbf{t}]$, we have $\mathsf{w}_{hk}\left[\mathsf{wl}_k(m)\right] = \mathsf{w}_{hk}\left[\mathsf{w}_{km}[\mathbf{t}]\right] = \mathsf{w}_{hm}[\mathbf{t}] = \mathsf{wl}_h(m)$, as required.
   (iii), (iv): It follows from $\mathsf{w}_{kh} \circ \mathsf{w}_{hk} = \mathsf{w}_{kk} = \mathsf{Id}$ and $\mathsf{w}_{hk} \circ \mathsf{w}_{kh} = \mathsf{w}_{hh} = \mathsf{Id}$ that w_{kh} and w_{hk} are mutual inverses, and hence that they are both bijections. $\square$

Lemma 6.1.3 (Worldline Relocation)**.** Assume AxWvt, and suppose $k \overset{F}{\rightsquigarrow}_h k^*$ for some bijection $F : Q^4 \to Q^4$. Then F maps $\mathsf{wl}_h(k)$ onto $\mathsf{wl}_h(k^*)$; see Fig. 4 (middle).

Proof. Recall that $k \overset{F}{\rightsquigarrow}_h k^*$ means $\mathsf{w}_{hk^*} = F \circ \mathsf{w}_{hk}$. So

$$\mathsf{wl}_h(k^*) = \mathsf{w}_{hk^*}[\mathbf{t}] = (F \circ \mathsf{w}_{hk})[\mathbf{t}] = F\left[\mathsf{wl}_h(k)\right].$$

$\square$

Lemma 6.1.4 (Observer Rotation)**.** Assume AxEField, AxWvt, AxRelocate and AxIsotropy. Then given any spatial rotation $R \in \mathsf{SRot}$ and $k, h \in IOb$, there exists an observer k^* such that $k \overset{R}{\rightsquigarrow}_h k^*$; see Fig. 4 (right).

Proof. By AxRelocate, there exists an observer h^* for which $\mathsf{w}_{hh^*} = R$. Because h and h^* are related via a spatial rotation, AxIsotropy tells us there exists $k^* \in IOb$ which is related to h^* the same way k is related to h, i.e. $\mathsf{w}_{h^*k^*} = \mathsf{w}_{hk}$. It follows immediately that $\mathsf{w}_{hk^*} = \mathsf{w}_{hh^*} \circ \mathsf{w}_{h^*k^*} = R \circ \mathsf{w}_{hk}$, i.e. $k \overset{R}{\rightsquigarrow}_h k^*$, as claimed. $\square$

Lemma 6.1.5 (Transformed Observer Lines)**.** Assume AxWvt. Then $\ell \in \mathsf{ObLines}(k)$ iff $\mathsf{w}_{hk}[\ell] \in \mathsf{ObLines}(h)$.

Proof. This follows immediately from Lemma 6.1.2 (WVT), since all k-observer lines are worldlines. $\square$

Lemma 6.1.6 (Rotated Observer Lines)**.** Assume AxEField, AxWvt, AxRelocate and AxIsotropy. If $\ell \in \mathsf{ObLines}(k)$ and $R \in \mathsf{SRot}$ is any spatial rotation, then $R[\ell] \in \mathsf{ObLines}(k)$.

Proof. Choose $h \in IOb$ such that $\ell = \mathsf{wl}_k(h)$. By Lemma 6.1.4 (Observer Rotation), there is some $h^* \in IOb$ for which $h \overset{R}{\leadsto}_k h^*$, i.e. $\mathsf{w}_{kh^*} = R \circ \mathsf{w}_{kh}$. By Lemma 6.1.3 (Worldline Relocation), we have that $\mathsf{wl}_k(h^*) = R[\mathsf{wl}_k(h)] = R[\ell]$, and this worldline is in $\mathsf{ObLines}(k)$, as required. $\qquad\square$
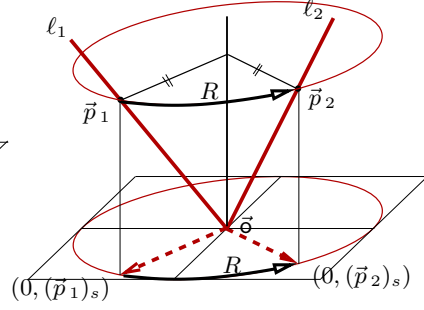
Lemma 6.1.7 (Horizontal Rotation). Let $(Q, +, \cdot, 0, 1, \leq)$ be an ordered field and suppose $\vec{p}_1, \vec{q}_1, \vec{p}_2, \vec{q}_2 \in Q^4$ satisfy:

(a) $\vec{p}_1$ and $\vec{p}_2$ have the same length, as do $\vec{q}_1$ and $\vec{q}_2$:
$|\vec{p}_1|^2 = |\vec{p}_2|^2$ and $|\vec{q}_1|^2 = |\vec{q}_2|^2$;

(b) $\vec{p}_1$ and $\vec{q}_1$ are horizontal and mutually orthogonal:
$\vec{p}_1 \cdot \vec{t} = \vec{q}_1 \cdot \vec{t} = \vec{p}_1 \cdot \vec{q}_1 = 0$; and

(c) $\vec{p}_2$ and $\vec{q}_2$ are horizontal and mutually orthogonal:
$\vec{p}_2 \cdot \vec{t} = \vec{q}_2 \cdot \vec{t} = \vec{p}_2 \cdot \vec{q}_2 = 0$.

Then there exists a spatial rotation $R \in \mathsf{SRot}$ such that $R(\vec{p}_1) = \vec{p}_2$ and $R(\vec{q}_1) = \vec{q}_2$; see the left-hand side of Figure 5.
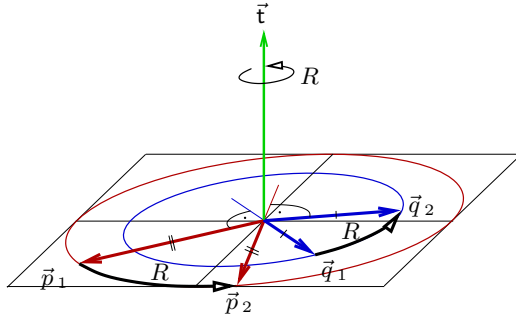
Lemma 6.1.7 (Horizontal Rotation) Lemma 6.1.8 (Same-Slope Rotation)

FIGURE 5. Illustrations for Lemma 6.1.7 (Horizontal Rotation) and Lemma 6.1.8 (Same-Slope Rotation).

Proof. Consider the linear map that takes $\alpha\vec{t} + \beta\vec{p}_1 + \gamma\vec{q}_1$ to $\alpha\vec{t} + \beta\vec{p}_2 + \gamma\vec{q}_2$. It is easy to see that this map is a linear Euclidean isometry between two subspaces of Q^4 which are each at most three-dimensional. Hence, by the refinement of Witt's theorem [ST71, Thm 234.1, p.234] there is an extension $R : Q^4 \to Q^4$ which is a linear Euclidean isometry with determinant 1. This R must be a spatial rotation, because $R(\vec{t}) = \vec{t}$. $\qquad\square$

Lemma 6.1.8 (Same-Slope Rotation). Let $(Q, +, \cdot, 0, 1, \leq)$ be a Euclidean field. Assume ℓ_1 and ℓ_2 are lines such that $\mathsf{slope}\,(\ell_1) = \mathsf{slope}\,(\ell_2)$ and $\vec{o} \in \ell_1 \cap \ell_2$. Then there exists $R \in \mathsf{SRot}$ such that $R[\ell_1] = \ell_2$.

Proof. Let $\vec{p}_1 \in \ell_1$ and $\vec{p}_2 \in \ell_2$ be such that $\vec{p}_1 \neq \vec{o} \neq \vec{p}_2$ and $(\vec{p}_1)_t = (\vec{p}_2)_t$, see the right-hand side of Figure 5. Then $|(0, (\vec{p}_1)_s)|^2 = |(0, (\vec{p}_2)_s)|^2$. Taking $\vec{q}_1 = \vec{q}_2 = \vec{o}$, Lemma 6.1.7 (Horizontal Rotation) now tells us there exists a spatial rotation R that takes $(0, (\vec{p}_1)_s)$ to $(0, (\vec{p}_2)_s)$ and leaves $\vec{o}$ fixed. Since spatial rotations leave time coordinates unchanged, this R takes $\vec{p}_1$ to $\vec{p}_2$, and since it also fixes the origin it must take ℓ_1 to ℓ_2. $\qquad\square$

Lemma 6.1.9 (Observer Line Intersections). Assume AxEField, AxWvt, AxLine, AxRelocate, and AxIsotropy. If two lines ℓ_1, ℓ_2 intersect one another and have equal slope, then for any $k \in IOb$ we have $\ell_1 \in \mathsf{ObLines}(k)$ iff $\ell_2 \in \mathsf{ObLines}(k)$.
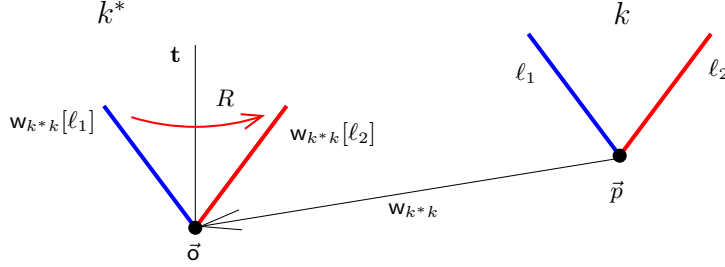

FIGURE 6. Illustration for the proof of Lemma 6.1.9 (Observer Line Intersections).

Proof. Let $\vec{p}$ be the point of intersection of ℓ_1 and ℓ_2, and let T be the translation taking $\vec{p}$ to the origin, $\vec{\mathsf{o}}$. By AxRelocate, there exists some $k^* \in IOb$ such that $\mathsf{w}_{k^*k} = T$; see Figure 6.
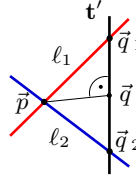
Note first that the images of ℓ_1 and ℓ_2 under w_{k^*k} are lines of equal slope because $\mathsf{w}_{k^*k} = T$ is a translation, and translations map lines to lines and leave slopes unchanged. Moreover, both of these lines pass through $T(\vec{p}) = \vec{\mathsf{o}}$, so Lemma 6.1.8 (Same-Slope Rotation) tells us there exists a spatial rotation R taking $\mathsf{w}_{k^*k}[\ell_1]$ to $\mathsf{w}_{k^*k}[\ell_2]$.

The claim now follows. For suppose ℓ_1 is a k-observer line; we have to show that ℓ_2 is also a k-observer line. Since $\mathsf{w}_{k^*k}[\ell_1] \in \mathsf{ObLines}(k^*)$ by Lemma 6.1.5 (Transformed Observer Lines), it follows that $\mathsf{w}_{k^*k}[\ell_2] \in \mathsf{ObLines}(k^*)$ as well, by Lemma 6.1.6 (Rotated Observer Lines). Applying Lemma 6.1.5 (Transformed Observer Lines) in the opposite direction now tells us that $\ell_2 \in \mathsf{ObLines}(k)$, as required.

The converse follows by symmetry. $\square$

Lemma 6.1.10 (Triangulation). Assume AxEField. Let $\mathbf{t}'$ be a line parallel to the time-axis and let $\vec{p}$ be any point not on $\mathbf{t}'$. Given any positive $\lambda \in Q$, there exist lines ℓ_1, ℓ_2 with

(i) $\mathsf{slope}(\ell_1) = \mathsf{slope}(\ell_2) = \lambda$,
(ii) $\vec{p} \in \ell_1 \cap \ell_2$,
(iii) $\ell_1 \cap \mathbf{t}' \neq \varnothing$,
(iv) $\ell_2 \cap \mathbf{t}' \neq \varnothing$,
(v) $\ell_1 \cap \ell_2 \cap \mathbf{t}' = \varnothing$.

Proof. Let $\vec{q} \in \mathbf{t}'$ be the point on $\mathbf{t}'$ with $\vec{q}_t = \vec{p}_t$. We know that $\vec{p}_s \neq \vec{q}_s$ because $\vec{p} \notin \mathbf{t}'$. Consider the points

$$\vec{q}_1 := \vec{q} + (|\vec{p}_s - \vec{q}_s| / \lambda, 0, 0, 0) \quad \text{and} \quad \vec{q}_2 := \vec{q} - (|\vec{p}_s - \vec{q}_s| / \lambda, 0, 0, 0)$$

and let ℓ_1 be the line passing through $\vec{p}$ and $\vec{q}_1$, and ℓ_2 the line passing through $\vec{p}$ and $\vec{q}_2$. Then direct calculation shows that ℓ_1 and ℓ_2 have the required properties. $\square$

6.1.3. *Main proof.* We now complete the proof of Theorem 6.1 (Observer Lines Lemma).

We use the word *plane* in the usual Euclidean sense to mean a 2-dimensional slice of Q^4, and refer to 3-dimensional 'slices' as *hyperplanes*. Formally, a subset $P \subseteq Q^4$ is a **plane** iff there are linearly independent vectors $\vec{v}, \vec{w} \neq \vec{o} \in Q^4$ and a point $\vec{p} \in Q^4$, such that $P = \{\vec{p} + \lambda\vec{v} + \mu\vec{w} \ : \ \lambda, \mu \in Q\}$ (hyperplanes are defined analogously). By AxEField, the usual properties of Euclidean planes hold. In particular, a plane P can be specified by giving a line $\ell \subseteq P$ and a point $\vec{p} \in P \backslash \ell$, or three distinct non-collinear points $\vec{p}, \vec{q}, \vec{r} \in P$, or two distinct but intersecting lines in P. Moreover, given a line $\ell \subseteq P$ and a point $\vec{p} \in P \setminus \ell$, there is exactly one line ℓ_p through $\vec{p}$ that is parallel to ℓ (indeed, if we assume AxEField, the way in which we have defined *line* and *plane* allows us to uniquely determine ℓ_p in the usual way once $\vec{p}$ and ℓ are specified).

Proof of Theorem 6.1 (Observer Lines Lemma). Let ℓ, ℓ' be lines of equal slope: $\mathsf{slope}(\ell) = \mathsf{slope}(\ell')$. If $\ell = \ell'$, there is nothing to prove, so assume that $\ell \neq \ell'$. Also, if $\mathsf{slope}(\ell) = \mathsf{slope}(\ell') = 0$, then ℓ and ℓ' are both parallel to the time-axis, and it follows easily from AxRelocate that $\ell, \ell' \in \mathsf{ObLines}(k)$.

Suppose, therefore, that $\mathsf{slope}(\ell) = \mathsf{slope}(\ell') \neq 0$.

Note first that there exist $\vec{p}, \vec{q} \in Q^4$ such that

$$\vec{p} \in \ell, \quad \vec{q} \in \ell', \quad \vec{p} \neq \vec{q}, \quad \text{and} \quad \vec{p}_t = \vec{q}_t.$$

This is true by assumption for case (b), where $\mathsf{slope}(\ell) = \mathsf{slope}(\ell') = \infty$, and it is easy to see that such $\vec{p}, \vec{q}$ also exist in case (a) where $\mathsf{slope}(\ell) = \mathsf{slope}(\ell')$ is finite.[10]

Let $\hat{\ell}$ be the line containing $\vec{p}$ and $\vec{q}$. Because $\vec{p}, \vec{q}$ have the same time coordinate, $\mathsf{slope}(\hat{\ell}) = \infty$; see Figure 7.

(a) finite slopes   (b) infinite slopes
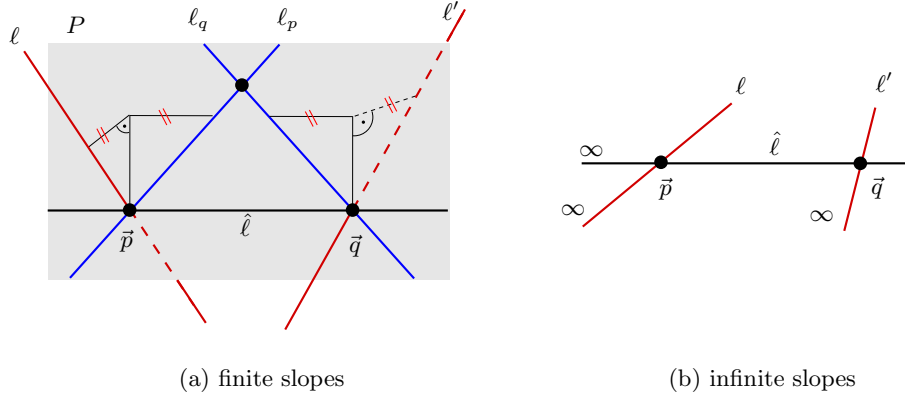
FIGURE 7. Illustration for the proof of Theorem 6.1 (Observer Lines Lemma)

We now consider cases (a) and (b) in turn.

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[10] Pick any point $\vec{p}$ on ℓ that isn't on ℓ' and consider the 'horizontal time slice' containing it; because ℓ' has finite slope, it must also pass through this time slice. Take $\vec{q}$ to be the corresponding point of intersection on ℓ'.

Case (a): finite slopes. By assumption, $0 < \mathsf{slope}(\ell) = \mathsf{slope}(\ell') \neq \infty$ and $\mathsf{slope}(\hat{\ell}) = \infty$. Let P be the plane containing $\hat{\ell}$ and parallel to $\mathbf{t}$.[11]

Let $\mathbf{t}_p$ be the line parallel to $\mathbf{t}$ which passes through $\vec{p}$, and notice that this line lies in P. Choose any point $\vec{p}\,' \in P \setminus \mathbf{t}_p$ and let $\lambda = \mathsf{slope}(\ell) = \mathsf{slope}(\ell')$. Then Lemma 6.1.10 (Triangulation) tells us that we can find two distinct lines which pass through $\vec{p}\,'$, lie in P (because they meet both $\vec{p}\,'$ and $\mathbf{t}_p$), and have slope λ. Applying the translation taking $\vec{p}\,'$ to $\vec{p}$, the images of those two lines will still lie in P and still have slope λ, but will intersect at $\vec{p}$. Similarly, we can find two distinct lines of slope λ which lie in P and pass through $\vec{q}$. Pick one of the lines passing through $\vec{q}$, and call it ℓ_q. Since the two lines through $\vec{p}$ are distinct, they cannot both be parallel to ℓ_q — let ℓ_p be one that isn't. Since ℓ_p and ℓ_q are non-parallel lines lying in the same plane, they must intersect.

The claim now follows. For suppose $\ell \in \mathsf{ObLines}(k)$. Then ℓ and ℓ_p are lines of equal slope which intersect at $\vec{p}$, so Lemma 6.1.9 (Observer Line Intersections) tells us that ℓ_p is also in $\mathsf{ObLines}(k)$, whence (applying the same argument twice more) so are ℓ_q (because it meets ℓ_p) and ℓ' (since it meets ℓ_q).

Case (b): infinite slopes. If $\mathsf{slope}(\ell) = \infty$, then ℓ and $\hat{\ell}$ are two lines of infinite slope which intersect at $\vec{p}$. Likewise, ℓ' and $\hat{\ell}$ are lines of infinite slope that intersect at $\vec{q}$. As before it now follows by Lemma 6.1.9 (Observer Line Intersections) that

$$\ell \in \mathsf{ObLines}(k) \iff \hat{\ell} \in \mathsf{ObLines}(k) \iff \ell' \in \mathsf{ObLines}(k).$$

In both cases, therefore, we have $\ell \in \mathsf{ObLines}(k) \iff \ell' \in \mathsf{ObLines}(k)$, as required. $\qquad\square$

6.2. **Line-to-Line Lemma.**

Theorem 6.2 (Line-to-Line Lemma)**.** Assume AxEField, AxWvt, AxLine, AxRelocate, AxIsotropy and ∃MovingIOb. Then given any $k, h \in IOb$, the worldview transformation w_{hk} is a bijection that takes lines to lines, planes to planes, and hyperplanes to hyperplanes.

6.2.1. *Supporting lemmas.* A number of the supporting lemmas refer to the concept of an *observer line triad*:

Definition 6.2.1 (Observer Line Triads)**.** If $\ell_1, \ell_2, \ell_3 \in \mathsf{ObLines}(k)$ are three (necessarily coplanar) lines, each pair of which intersect in a point, and whose pairwise intersections are not collinear, we shall call the set $\{\ell_1, \ell_2, \ell_3\}$ an *observer line triad for k*, or simply a *k-triad*. $\qquad\square$

The lemmas can be described informally as follows:

Lemma 6.2.4 (Speed):
Speeds are well-defined, and the terms *at rest* and *in motion* have their expected meanings.

Lemma 6.2.5 (Triads):
If one observer considers that three worldlines form a triad, all other observers agree.

Lemma 6.2.6 (Plane-to-Plane):
Suppose plane P contains a k-triad whose slopes are either all finite or else all infinite. Then $\mathsf{w}_{hk}[P]$ is contained in a plane.

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[11]P is parallel to $\mathbf{t}$ iff P contains a line parallel to $\mathbf{t}$.

Lemma 6.2.7 (Infinite Speeds $\Rightarrow$ Lines are Observer Lines):
If infinite speeds occur, then all lines are observer lines.

6.2.2. *Proofs of the supporting lemmas.*

Definition 6.2.2. Suppose AxEField and AxLine holds. If $\ell = \mathsf{wl}_k(h)$, we call the slope, $\mathsf{slope}(\ell)$, of line ℓ the **speed** *of h according to k*, i.e.

$$\mathsf{speed}_k(h) \stackrel{\text{def}}{=} \mathsf{slope}(\mathsf{wl}_k(h)).$$

Definition 6.2.3. Recall that observer $k \in IOb$ is moving according to observer $m \in IOb$ iff $\mathsf{w}_{mk}(\vec{\mathsf{t}})_s \neq \mathsf{w}_{mk}(\vec{\mathsf{o}})_s$ and at rest according to m otherwise. We say that *observer* $k \in IOb$ **is moving instantaneously according to** *observer* m iff $\mathsf{w}_{mk}(\vec{\mathsf{t}})_t = \mathsf{w}_{mk}(\vec{\mathsf{o}})_t$.

Lemma 6.2.4 (Speed)**.** Assume AxWvt, AxEField and AxLine. Then for every $m, k \in IOb$, $\mathsf{speed}_m(k)$ is well-defined, and

- k is at rest according to m iff $\mathsf{speed}_m(k) = 0$,
- k is moving according to m iff $\mathsf{speed}_m(k) \neq 0$, and
- k is moving instantaneously according to m iff $\mathsf{speed}_m(k) = \infty$.

Proof. By AxEField and AxLine, it follows that $\mathsf{speed}_m(k)$ is unambiguously defined for all k and m. The proof is straightforward after noticing that $\mathsf{w}_{mk}(\vec{\mathsf{t}}) \neq \mathsf{w}_{mk}(\vec{\mathsf{o}})$ which holds because w_{mk} is a bijection by Lemma 6.1.2 (WVT). $\qquad\square$

Lemma 6.2.5 (Triads)**.** Suppose AxEField, AxWvt, AxLine. Let $k, h \in IOb$. If $T = \{\ell_1, \ell_2, \ell_3\}$ is a k-triad, then $\mathsf{w}_{hk}[T] := \{\mathsf{w}_{hk}[\ell_1], \mathsf{w}_{hk}[\ell_2], \mathsf{w}_{hk}[\ell_3]\}$ is an h-triad.

Proof. Each ℓ_i is a k-observer line, so by Lemma 6.1.5 (Transformed Observer Lines), each $\ell_i' = \mathsf{w}_{hk}[\ell_i]$ is an h-observer line (and hence a line). Because w_{hk} is a bijection, we know that any two of the lines in $\mathsf{w}_{hk}[T]$ has non-empty intersection, and that they have three distinct pairwise intersections in total. It follows that the three lines are coplanar and that their three pairwise intersection points are not collinear. That is, $\mathsf{w}_{hk}[T]$ is an h-triad as claimed. $\qquad\square$

Lemma 6.2.6 (Plane-to-Plane)**.** Assume AxEField, AxWvt, AxLine, AxRelocate and AxIsotropy. Choose $k, h \in IOb$, let P be a plane which contains a k-triad $\{\ell_1, \ell_2, \ell_3\}$, and suppose that the slopes of these lines are either all finite, or else all infinite. Then $\mathsf{w}_{hk}[P]$ is contained in a plane.

Proof. According to Lemma 6.2.5 (Triads), the lines $\mathsf{w}_{hk}[\ell_i]$ ($i = 1, 2, 3$) form an h-triad. We can therefore define P', the plane spanned by this triad. We will prove that $\mathsf{w}_{hk}[P] \subseteq P'$.

Choose any $\vec{p} \in P$. If $\vec{p}$ lies on any of the lines ℓ_i, then the conclusion $\mathsf{w}_{hk}(\vec{p}) \in P'$ is trivial. Suppose, then, that $\vec{p}$ does not lie on any of these lines. Because the lines form a triad we can draw a line ℓ through $\vec{p}$ which is parallel to one of the lines (wlog, ℓ_1) and which intersects the other two lines (ℓ_2 and ℓ_3) in distinct points.

We claim that $\ell \in \mathsf{ObLines}(k)$. If all three lines have finite slope, this follows from Theorem 6.1 (Observer Lines Lemma) because ℓ and ℓ_1 have equal (hence finite) slopes and ℓ_1 is a k-observer line. On the other hand, if all three lines (and hence also ℓ) have infinite slope, this means there exist t_1, t_2 and t_3 such that all points on ℓ_i ($i = 1, 2, 3$) have time component t_i. But we know that the lines intersect
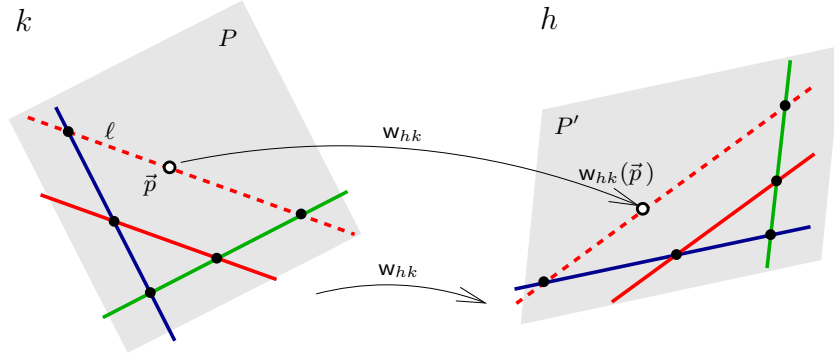
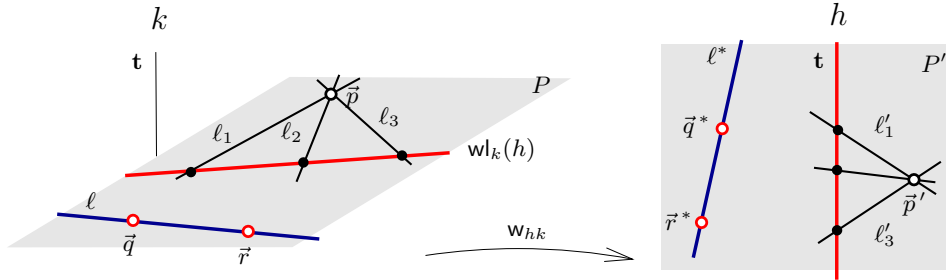FIGURE 8. Illustration for the proof of Lemma 6.2.6 (Plane-to-Plane)

one another, so we must have $t_1 = t_2 = t_3$. Since ℓ lies in the plane spanned by these lines it follows that points on ℓ share the same time component as points on ℓ_1, and we can again apply Theorem 6.1 (Observer Lines Lemma) to ℓ and ℓ_1 to deduce that $\ell \in \mathsf{ObLines}(k)$.

As claimed, therefore, ℓ is a k-observer line. Therefore, ℓ, ℓ_2 and ℓ_3 form a k-triad and Lemma 6.2.5 (Triads) tells us that $\mathsf{w}_{hk}[\ell]$, $\mathsf{w}_{hk}[\ell_2]$ and $\mathsf{w}_{hk}[\ell_3]$ form an h-triad. It follows that $\mathsf{w}_{hk}[\ell]$ lies in the same plane as $\mathsf{w}_{hk}[\ell_2]$ and $\mathsf{w}_{hk}[\ell_3]$, i.e. P', and hence $\mathsf{w}_{hk}(\vec{p}) \in \mathsf{w}_{hk}[\ell] \subseteq P'$, as required. $\qquad\square$

The following formula says that instantaneously moving observers exists.

$\underline{\exists\infty\mathsf{Speed}}$: There are observers $m, k \in IOb$ such that $\mathsf{w}_{mk}(\vec{\mathsf{o}})_t = \mathsf{w}_{mk}(\vec{\mathsf{t}})_t$.

$^{\ell}$**Lemma 6.2.7** (Infinite Speeds $\Rightarrow$ Lines are Observer Lines)**.** Assume $\mathsf{AxEField}$, AxWvt, AxLine, $\mathsf{AxRelocate}$, $\mathsf{AxIsotropy}$ and $\exists\infty\mathsf{Speed}$. Then for any observer, every $^{\mathsf{t}}$line is an observer line.

FIGURE 9.   Illustration for the proof of Lemma 6.2.7 (Infinite Speeds $\Rightarrow$ Lines are Observer Lines)

Proof. Choose $k, h \in IOb$ such that $\mathsf{speed}_k(h) = \infty$, and recall that this means that $\mathsf{slope}(\mathsf{wl}_k(h)) = \infty$. Thus, there exists some $t \in Q$ such that every point on $\mathsf{wl}_k(h)$ has time component t. Let P be any 'horizontal' plane containing $\mathsf{wl}_k(h)$, i.e. all

points in P have this same time component t. Then every line in P is in $\mathsf{ObLines}(k)$ by Theorem 6.1 (Observer Lines Lemma) because every line in P is of slope ∞.

Choose $\vec{p} \in P \setminus \mathsf{wl}_k(h)$, and notice that the plane P is determined by $\vec{p}$ and $\mathsf{wl}_k(h)$. It follows from Lemma 6.2.6 (Plane-to-Plane) that $\mathsf{w}_{hk}[P]$ is contained in a plane containing both $\mathsf{w}_{hk}(\vec{p})$ and $\mathsf{w}_{hk}[\mathsf{wl}_k(h)]$. In other words, if we define $\vec{p}\,' = \mathsf{w}_{hk}(\vec{p}\,)$, observe that $\mathsf{w}_{hk}[\mathsf{wl}_k(h)] = \mathbf{t}$, and define P' to be the plane generated by $\vec{p}\,'$ and $\mathbf{t}$, then $\mathsf{w}_{hk}[P] \subseteq P'$.

We will show first that the reverse inclusion also holds, so that $\mathsf{w}_{hk}[P]$ is the whole of P'. To this end, choose three lines ℓ_i ($i = 1, 2, 3$) in P which pass through $\vec{p}$ and whose intersections with $\mathsf{wl}_k(h)$ are three distinct points; as observed above, these are all k-observer lines. Thus, if we define, for each $i = 1, 2, 3$, $\ell_i' := \mathsf{w}_{hk}[\ell_i]$ then $\ell_1', \ell_2', \ell_3'$ and $\mathbf{t}$ ($= \mathsf{w}_{hk}[\mathsf{wl}_k(h)]$) are all h-observer lines in P'. Since w_{hk} is a bijection by Lemma 6.1.2 (WVT), all four of these lines are distinct and moreover, each ℓ_i' passes through $\vec{p}\,'$, and they meet $\mathbf{t}$ in three distinct points.

Since at most one of the lines ℓ_i' can have infinite slope (and $\mathsf{slope}(\mathbf{t}) = 0$), we have therefore shown that there exists in P' a k-triad of observer lines, all with finite slope. By Lemma 6.2.6 (Plane-to-Plane), it follows that $\mathsf{w}_{kh}[P'] \subseteq P$, and hence $P' \subseteq \mathsf{w}_{hk}[P]$. Thus, $\mathsf{w}_{hk}[P] = P'$, as claimed.

Now we will prove that every line in P' is in $\mathsf{ObLines}(h)$. Let $\ell^* \subseteq P'$ be a line and let $\vec{q}^{\,*}, \vec{r}^{\,*}$ be two distinct points on ℓ^*. Then $\vec{q} := \mathsf{w}_{kh}(\vec{q}^{\,*})$, $\vec{r} := \mathsf{w}_{kh}(\vec{r}^{\,*})$ are two distinct points in P because $\mathsf{w}_{kh}[P'] \subseteq P$ and w_{kh} is a bijection. Let ℓ be the line connecting $\vec{q}$ and $\vec{r}$. Then ℓ lies in P, and must therefore be in $\mathsf{ObLines}(k)$. Since $\mathsf{w}_{hk}[\ell] = \ell^*$, it follows by Lemma 6.1.5 (Transformed Observer Lines) that $\ell^* \in \mathsf{ObLines}(h)$ as claimed.

Now we use the fact that $\mathbf{t} \subseteq P'$ to prove that *every* line is in $\mathsf{ObLines}(h)$. Let ℓ be an arbitrary line. Then there is some $\ell^* \subseteq P'$ which has the same slope as ℓ because $\mathbf{t} \subseteq P'$ and therefore lines of every positive slope occur in P' by Lemma 6.1.10 (Triangulation), while if $\mathsf{slope}(\ell) = 0$ we can take $\ell^* = \mathbf{t}$, and if $\mathsf{slope}(\ell) = \infty$ we can take ℓ^* to be the line joining $\vec{p}\,'$ to $((\vec{p}\,')_t, \vec{0})$. Moreover, by using translations 'up or down' the time-axis as necessary, ℓ^* can be chosen such that there are $\vec{p} \in \ell$, $\vec{q} \in \ell^*$ such that $\vec{p}_t = \vec{q}_t$. We know that $\ell^* \in \mathsf{ObLines}(h)$ because every line in P' is in $\mathsf{ObLines}(h)$. But now $\ell \in \mathsf{ObLines}(h)$ by Theorem 6.1 (Observer Lines Lemma). So $\mathsf{ObLines}(h)$ is the set of all lines, as claimed.

Finally, it is easy to see that because $\mathsf{ObLines}(h)$ is the set of all lines for one observer h, the same holds for every other observer m. For suppose ℓ' is a line, and choose distinct points $\vec{p}\,', \vec{q}\,' \in \ell'$. By Lemma 6.1.2 (WVT), the points $\vec{p} := \mathsf{w}_{hm}(\vec{p}\,')$ and $\vec{q} := \mathsf{w}_{hm}(\vec{q}\,')$ are again distinct, so they define a line ℓ. As we've just seen, ℓ must be an h-observer line. It follows from Lemma 6.1.5 (Transformed Observer Lines) that $\mathsf{w}_{mh}[\ell]$ is an m-observer line, and hence a line. This means that ℓ' and $\mathsf{w}_{mh}[\ell]$ are both lines passing through the two points $\vec{p}\,' \neq \vec{q}\,'$, so they must be the same line. In other words, $\ell' = \mathsf{w}_{mh}[\ell] \in \mathsf{ObLines}(m)$, as claimed. $\qquad\square$

6.2.3. *Main proof.* We now complete the proof of Theorem 6.2 (Line-to-Line Lemma).

Definition 6.2.8 (Observer Planes)**.** Whenever a plane P contains at least one k-observer line, we shall say that P is an *observer plane for* k, or a k-observer plane. We write $\mathsf{ObPlanes}(k)$ for the set of all k-observer planes. $\qquad\square$

Proof of Theorem 6.2 (Line-to-Line Lemma). We have already noted that every worldview transformation w_{kh} is a bijection; we will show first that they also take lines to lines.

Suppose m, m' are observers in motion relative to one another, i.e. $\mathsf{speed}_m(m') > 0$ — such observers exist by $\exists\mathsf{MovingIOb}$ and Lemma 6.2.4 (Speed). There are two cases to consider, depending on whether $\mathsf{speed}_m(m')$ can or cannot be infinite.

(Case 1: $\exists\infty\mathsf{Speed}$): If m, m' can be chosen with $\mathsf{speed}_m(m') = \infty$, then Lemma 6.2.7 (Infinite Speeds $\Rightarrow$ Lines are Observer Lines) tells us that all lines belong to $\mathsf{ObLines}(h)$ and we know that w_{kh} takes observer lines to observer lines (which are again lines). So in this case, the result is immediate.

(Case 2: $\neg\exists\infty\mathsf{Speed}$): Assume, therefore, that all observers move with finite speed relative to one another (so that, given any observer o and $\ell \in \mathsf{ObLines}(o)$, we have $\mathsf{slope}(\ell) \neq \infty$); in particular, $0 < \mathsf{speed}_m(m') \neq \infty$. Our proof will be given in four stages; we will show that

(1) if a plane P contains a k-triad, then $\mathsf{w}_{hk}[P]$ is again a plane;
(2) that for every observer o there is some $\ell \in \mathsf{ObLines}(o)$ for which $\mathsf{slope}(\ell) \neq 0$;
(3) if $P \in \mathsf{ObPlanes}(k)$ there exists a k-triad lying entirely within P. Items (1) and (3) imply that w_{hk} maps k-observer planes to h-observer planes.
(4) Finally, we use this information to show that every line can be obtained as the intersection of two k-observer planes — since the images of these planes intersect in a line, the result then follows.

(1) We prove that if a plane P contains a k-triad, then $\mathsf{w}_{hk}[P]$ is a plane. Let $\{\ell_1, \ell_2, \ell_3\}$ be a k-triad contained in P, and for each $i = 1, 2, 3$ define $\ell'_i := \mathsf{w}_{hk}[\ell_i]$. Because all observer lines are assumed to have finite slopes, Lemma 6.2.6 (Plane-to-Plane) tells us that $\mathsf{w}_{hk}[P] \subseteq P'$, where P' is the plane generated by $\{\ell'_1, \ell'_2, \ell'_3\}$. Since, by Lemma 6.2.5 (Triads), $\{\ell'_1, \ell'_2, \ell'_3\}$ is likewise an h-triad contained in P' and comprising finite-slope lines, we can again apply Lemma 6.2.6 (Plane-to-Plane) to deduce that $\mathsf{w}_{kh}[P'] \subseteq P$. Consequently, $\mathsf{w}_{hk}[P] = P'$, and $\mathsf{w}_{hk}[P]$ is a plane as claimed.

(2) Next we show that for every observer o there is some $\ell \in \mathsf{ObLines}(o)$ for which $\mathsf{slope}(\ell) \neq 0$. To this end, let ℓ' be the line parallel to $\mathsf{wl}_m(m')$ which passes through the origin $\vec{\mathsf{o}}$, and note that this line cannot be the time-axis (which has slope 0). Since $\mathsf{wl}_m(m')$ is an m-observer line, so is ℓ' (by Theorem 6.1 (Observer Lines Lemma)). It follows that ℓ' and $\mathbf{t} = \mathsf{wl}_m(m)$ are non-identical intersecting m-observer lines, whence $\mathsf{w}_{om}[\ell']$ and $\mathsf{w}_{om}[\mathbf{t}]$ are non-identical intersecting o-observer lines. If these both had zero slope, they would be the same line. So at least one of them has non-zero slope and hence can be taken to be ℓ.

(3) Now we prove that for every k, if $P \in \mathsf{ObPlanes}(k)$ there exists a k-triad lying entirely in P. Suppose $P \in \mathsf{ObPlanes}(k)$, and choose some k-observer line $\ell = \mathsf{wl}_k(h) \subseteq P$ and some $\vec{p} \in P \setminus \ell$, see Figure 10. Transforming to h's worldview we have $\mathsf{w}_{hk}[\ell] = \mathsf{w}_{hk}[\mathsf{wl}_k(h)] = \mathsf{wl}_h(h) = \mathbf{t}$ and $\vec{p}' := \mathsf{w}_{hk}(\vec{p}) \notin \mathbf{t}$. By (2), we know there is some $\ell' \in \mathsf{ObLines}(h)$ for which $\mathsf{slope}(\ell') \neq 0$, and by assumption $\mathsf{slope}(\ell') \neq \infty$. Thus, by Lemma 6.1.10 (Triangulation) there exist lines ℓ'_1, ℓ'_2 passing through $\vec{p}'$ which have the same slope as ℓ', such that $\{\mathbf{t}, \ell'_1, \ell'_2\}$ is a k-triad (see Figure 10), and we know that $\ell'_1, \ell'_2 \in \mathsf{ObLines}(h)$ by Theorem 6.1 (Observer Lines Lemma). Taking $\ell_1 := \mathsf{w}_{kh}[\ell'_1]$ and $\ell_2 := \mathsf{w}_{kh}[\ell'_2]$, and recalling that $\mathsf{w}_{kh}[\mathbf{t}] = \ell$, it follows that all three lines are k-observer lines, and together they form a k-triad lying entirely

within P because their pairwise intersections comprise the point $\vec{p} \notin \ell$ together with two distinct points on ℓ.
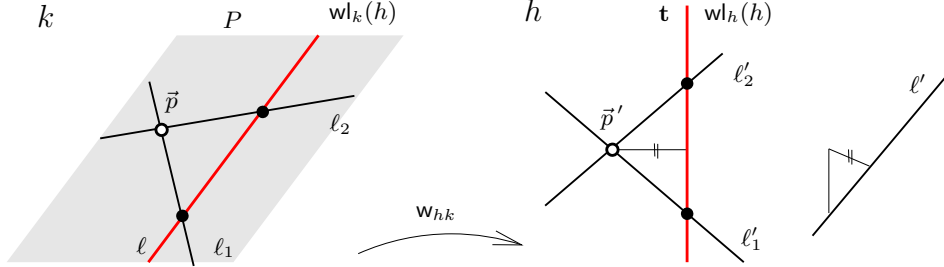

FIGURE 10. Illustration for item (3) of the proof of Theorem 6.2 (Line-to-Line Lemma).

Taken together, these results imply that whenever $P \in \mathsf{ObPlanes}(k)$, then $\mathsf{w}_{hk}[P]$ is a plane.

(4) Now let $k \in IOb$. We want to prove that any line can be obtained as the intersection of two planes in $\mathsf{ObPlanes}(k)$. To see this, let ℓ be any line, and choose any $\vec{p} \in \ell$, see Figure 11. As we have just seen, we can also choose $\ell' \in \mathsf{ObLines}(k)$ such that $\mathsf{slope}(\ell') \neq 0$ and (by assumption) $\mathsf{slope}(\ell') \neq \infty$. Let ℓ_1, ℓ_2 be lines passing through $\vec{p}$, having the same slope as ℓ', such that ℓ, ℓ_1 and ℓ_2 are not co-planar (such lines can be obtained from ℓ' by a combination of translation and spatial rotation). It follows from Theorem 6.1 (Observer Lines Lemma) that $\ell_1, \ell_2 \in \mathsf{ObLines}(k)$. For each $i = 1, 2$, let P_i be the plane containing ℓ_i and ℓ. Then P_1, P_2 are k-observer planes and their intersection is ℓ, as required.
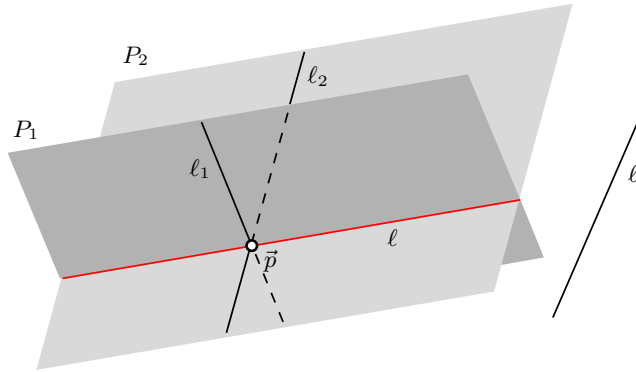
FIGURE 11. Illustration for item (4) of the proof of Theorem 6.2 (Line-to-Line Lemma).

It now follows, once again, that given any $k, h \in IOb$, the worldview transformation w_{hk} is a bijection that takes lines to lines. For if ℓ is any line, choose k-observer planes P_1, P_2 such that $\ell = P_1 \cap P_2$. Since w_{hk} is one-to-one, $\mathsf{w}_{hk}[\ell] = \mathsf{w}_{hk}[P_1] \cap \mathsf{w}_{hk}[P_2]$ and $\mathsf{w}_{hk}[P_1] \neq \mathsf{w}_{hk}[P_2]$ (as $P_1 \neq P_2$). Since $\mathsf{w}_{hk}[P_1]$ and $\mathsf{w}_{hk}[P_2]$ are distinct intersecting planes, their intersection $\mathsf{w}_{hk}[\ell]$ is a line.

This completes the proof that lines are mapped to lines. The claim for planes and hyperplanes now follows easily. Given a plane, choose three non-collinear points. These determine three distinct intersecting lines and their images determine the image plane. Likewise, we can choose four non-coplanar points in a hyperplane whose images determine the image hyperplane. □

6.3. The tx-Plane Lemma.

Definition 6.3.1 (Principal Observer)**.** We now fix one observer o for the rest of the paper (the *principal observer*) and define

$$IOb_o \overset{\text{def}}{=} \{k \in IOb : \mathsf{w}_{ko}(\vec{\mathsf{o}}) = \vec{\mathsf{o}}\}$$

to be the set of observers who agree with o (and hence each other) as to the location of the origin. □

Analogously to the definition of the time-axis $\mathbf{t}$, the three spatial axes ($\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$) are defined in the usual way as:

$$\mathbf{x} \overset{\text{def}}{=} \{(0, x, 0, 0) : x \in Q\}, \quad \mathbf{y} \overset{\text{def}}{=} \{(0, 0, y, 0) : y \in Q\}, \quad \mathbf{z} \overset{\text{def}}{=} \{(0, 0, 0, z) : z \in Q\}.$$

We write $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ for the tx-plane and $\mathsf{plane}(\mathbf{y}, \mathbf{z})$ for the yz-plane. More generally, if $\ell \neq \ell'$ are intersecting lines, then $\mathsf{plane}(\ell, \ell')$ denotes the plane containing ℓ and ℓ'.

Theorem 6.3 (tx-Plane Lemma)**.** Assume $\mathsf{KIN} + \mathsf{AxIsotropy}$. Let $m, k \in IOb_o$ such that $\mathsf{w}_{km}[\mathsf{plane}(\mathbf{t}, \mathbf{x})] = \mathsf{plane}(\mathbf{t}, \mathbf{x})$. Then

$$(6.2) \qquad \mathsf{w}_{km}[\mathsf{plane}(\mathbf{y}, \mathbf{z})] = \mathsf{plane}(\mathbf{y}, \mathbf{z})$$

and

$$(6.3) \qquad \text{if } \vec{q}, \vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z}) \text{ and } |\vec{p}| = |\vec{q}|, \text{ then } |\mathsf{w}_{km}(\vec{p})| = |\mathsf{w}_{km}(\vec{q})|.$$

Moreover, if w_{km} is also linear, then there is a positive $\lambda \in Q$ such that

$$(6.4) \qquad |\mathsf{w}_{km}(\vec{p})| = \lambda|\vec{p}|$$

for all $\vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$.

6.3.1. *Supporting lemmas.* The supporting lemmas can be informally described as:

Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$)**:**
A transformation is trivial if and only if it is a κ-isometry for at least two different choices of κ.

Lemma 6.3.3 (**IOb**$_o$)**:**
Elementary results concerning worldview transformations involving members of IOb_o.

Lemma 6.3.4 (**Affine**)**:**
Suppose f is a bijection on Q^4 taking lines to lines. Then there is an automorphism φ of Q and an affine transformation A such that $f = A \circ \widetilde{\varphi}$ (where $\widetilde{\varphi}$ is the coordinatewise extension of φ to Q^4).

Lemma 6.3.5 (**Equal Worldlines**)**:**
If any one observer considers $m, m^* \in IOb$ to have the same worldline, then all other observers do so as well.

Lemma 6.3.7 (**Colocate**)**:**
If two observers share the same worldline, the worldview transformation between them is trivial.

6.3.2. *Proofs of the supporting lemmas.*

Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$). Assume that $(Q, +, \cdot, 0, 1)$ is a field and choose $x, y \in Q$ such that $x \neq y$. Then

$$\mathsf{Triv} = {}_x\mathsf{Iso} \cap {}_y\mathsf{Iso}.$$

In particular, every trivial transformation is a Euclidean isometry.

Proof. ($\subseteq$) Choose any $x, y \in Q$, $T \in \mathsf{Triv}$ and $\vec{p} = (t, \vec{s}) \in Q^4$. We will show that $T \in {}_x\mathsf{Iso}$. Without loss of generality we can assume that T is linear (since it is the composition of a linear map with a translation, and all translations are x-isometries). It follows that $T(\vec{p}) = T(t, \vec{0}) + T(0, \vec{s})$. However, because T is trivial, we know that it fixes and preserves squared lengths in both $\mathbf{t}$ and $\mathbf{S}$, so there exist $t', \vec{s}'$ such that $T(t, \vec{0}) = (t', \vec{0})$ and $T(0, \vec{s}) = (0, \vec{s}')$, where $|t|^2 = |t'|^2$ and $|\vec{s}|^2 = |\vec{s}'|^2$. It follows immediately that $\|\vec{p}\|_x = |t|^2 - x|\vec{s}|^2 = |t'|^2 - x|\vec{s}'|^2 = \|T(\vec{p})\|_x$, i.e. T preserves squared κ-lengths. It now follows that $T \in {}_x\mathsf{Iso}$ when $x \neq 0$, and because $|\vec{s}|^2 = |\vec{s}'|^2$ no matter what the value of t, we also have $T \in {}_x\mathsf{Iso}$ when $x = 0$. Finally, because x can be any value in Q we also have $T \in {}_y\mathsf{Iso}$, and hence $\mathsf{Triv} \subseteq {}_x\mathsf{Iso} \cap {}_y\mathsf{Iso}$, as claimed.

($\supseteq$) To show the converse, choose any $x \neq y \in Q$ and any $T \in {}_x\mathsf{Iso} \cap {}_y\mathsf{Iso}$. We will show that $T \in \mathsf{Triv}$.

Assume first that T is linear. Choose any $\vec{p} = (t, \vec{s}) \in Q^4$ and suppose $T(\vec{p}) = (t', \vec{s}')$. Because T is in both ${}_x\mathsf{Iso}$ and ${}_y\mathsf{Iso}$, we have both $\|T(\vec{p})\|_x = \|\vec{p}\|_x$ and $\|T(\vec{p})\|_y = \|\vec{p}\|_y$, i.e.

$$|t'|^2 - x|\vec{s}'|^2 = |t|^2 - x|\vec{s}|^2 \text{ and} \tag{6.5}$$

$$|t'|^2 - y|\vec{s}'|^2 = |t|^2 - y|\vec{s}|^2. \tag{6.6}$$

Subtracting (6.6) from (6.5) gives

$$(x - y)|\vec{s}'|^2 = (x - y)|\vec{s}|^2$$

whence division by $(x - y) \neq 0$ gives both

$$|\vec{s}'|^2 = |\vec{s}|^2 \tag{6.7}$$

and hence (by either (6.5) or (6.6))

$$|t'|^2 = |t|^2. \tag{6.8}$$

Therefore,

$$\text{if } t = 0, \text{ then } t' = 0, \text{ and}$$

$$\text{if } \vec{s} = \vec{0}, \text{ then } \vec{s}' = \vec{0},$$

which together with (6.7) and (6.8) show that $T \in \mathsf{Triv}$.

If T is not itself linear, notice that we can write $T = L \circ \tau$ where τ is a translation and L is a linear x-isometry. Since $T \in {}_y\mathsf{Iso}$ and $L = T \circ \tau^{-1}$ differs from T only by a translation (and all translations are in ${}_y\mathsf{Iso}$), we see that L is in ${}_y\mathsf{Iso}$ too. Thus, L is a linear map in ${}_x\mathsf{Iso} \cap {}_y\mathsf{Iso}$ (in other words, the "linear" and "translation" parts of T are the same in ${}_x\mathsf{Iso}$ as in ${}_y\mathsf{Iso}$) whence it follows from what we have just shown that L is trivial. Because τ is trivial, we now conclude that $T = L \circ \tau$ is itself trivial, as claimed.

In particular, we have $\mathsf{Triv} = ({}_0\mathsf{Iso} \cap {}_{-1}\mathsf{Iso}) \subseteq {}_{-1}\mathsf{Iso}$, i.e. all trivial transformations are Euclidean isometries. $\square$

Lemma 6.3.3 (IOb$_o$). Assume AxWvt. Let $k, h \in IOb_o$ and $m \in IOb$. Then (a)–(c) below hold.

- (a) $\mathsf{w}_{kh}(\vec{\mathsf{o}}) = \vec{\mathsf{o}}$ and $\vec{\mathsf{o}} \in \mathsf{wl}_k(h)$.
- (b) If $\mathsf{w}_{km}(\vec{\mathsf{o}}) = \vec{\mathsf{o}}$, then $m \in IOb_o$.
- (c) If $R: Q^4 \to Q^4$, $R(\vec{\mathsf{o}}) = \vec{\mathsf{o}}$ and $k \overset{R}{\leadsto}_h m$, then $m \in IOb_o$.

Proof. The proof involves only straightforward applications of Lemma 6.1.2 (WVT), and we omit the details. □

Lemma 6.3.4 (Affine). Assume $Q = (Q, +, \cdot, 0, 1, \leq)$ is a Euclidean field, and suppose $f: Q^4 \to Q^4$ is a bijection taking lines to lines. Then there is an ordered-field automorphism φ of Q and an affine transformation A on Q^4 such that $f = A \circ \widetilde{\varphi}$, where $\widetilde{\varphi}: Q^4 \to Q^4$ is the map $\widetilde{\varphi}: (t, x, y, z) \mapsto (\varphi(t), \varphi(x), \varphi(y), \varphi(z))$.

Proof. By the Fundamental Theorem of Affine Geometry [Ber87, Thm. 2.6.3, p. 52], there is an automorphism φ of field $(Q, +, \cdot, 0, 1)$ and an affine transformation A such that $f = A \circ \widetilde{\varphi}$. To complete the proof of the lemma, we only have to show that φ is order preserving, i.e. $\varphi(a) \leq \varphi(b)$ iff $a \leq b$. Since $x \leq y$ iff $0 \leq y - x$, it is enough to show that $0 \leq \varphi(z)$ iff $0 \leq z$ — and this follows directly from the Euclidean property, i.e. $0 \leq d$ iff $d = c^2$ for some $c \in Q$. □

Lemma 6.3.5 (Equal Worldlines). Assume AxWvt. Suppose $m, m^* \in IOb$, and suppose $\mathsf{wl}_k(m) = \mathsf{wl}_k(m^*)$ for some $k \in IOb$. Then $\mathsf{wl}_j(m) = \mathsf{wl}_j(m^*)$ for all $j \in IOb$.

Proof. By Lemma 6.1.2 (WVT), $\mathsf{wl}_j(m) = \mathsf{w}_{jk}[\mathsf{wl}_k(m)] = \mathsf{w}_{jk}[\mathsf{wl}_k(m^*)] = \mathsf{wl}_j(m^*)$ for all $j \in IOb$. □

Definition 6.3.6. Let $m, m^* \in IOb$. If $\mathsf{wl}_k(m) = \mathsf{wl}_k(m^*)$ for some $k \in IOb$, we say that m and m^* ***share the same worldline***.

Lemma 6.3.7 (Colocate). Assume AxWvt and let $m, m^* \in IOb$. Suppose m and m^* share the same worldline. If AxColocate holds, then $\mathsf{w}_{mm^*} \in \mathsf{Triv}$.

Proof. Saying that m and m^* share the same worldline means that $\mathsf{wl}_k(m) = \mathsf{wl}_k(m^*)$ for some $k \in IOb$. By Lemma 6.3.5 (Equal Worldlines), this equation therefore holds for *all* choices of k, and in particular for $k = m$, i.e. $\mathsf{wl}_m(m) = \mathsf{wl}_m(m^*)$. The claim now follows immediately by AxColocate. □

6.3.3. *Main proof.* We now complete the proof of Theorem 6.3 (*tx*-Plane Lemma).

Proof of Theorem 6.3 (tx-Plane Lemma). Let $m, k \in IOb_o$ such that $\mathsf{w}_{km}[\mathsf{plane}(\mathbf{t}, \mathbf{x})] = \mathsf{plane}(\mathbf{t}, \mathbf{x})$. By Lemma 6.3.3 (IOb$_o$), $\mathsf{w}_{mk}(\vec{\mathsf{o}}) = \mathsf{w}_{km}(\vec{\mathsf{o}}) = \vec{\mathsf{o}}$.

Let us first prove the following claim

(6.9)   If $R \in \mathsf{SRot}$ fixes $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ pointwise, then there exists $k^* \in IOb$ such that (a) $\mathsf{w}_{kk^*} = \mathsf{w}_{km} \circ R \circ \mathsf{w}_{mk}$ and (b) $\mathsf{w}_{kk^*} \in \mathsf{Triv}$.

Proof of claim (6.9). (a) By Lemma 6.1.4 (Observer Rotation), there exists some k^* such that $k \overset{R}{\leadsto}_m k^*$, i.e. $\mathsf{w}_{mk^*} = R \circ \mathsf{w}_{mk}$. Hence, $\mathsf{w}_{kk^*} = \mathsf{w}_{km} \circ \mathsf{w}_{mk^*} = \mathsf{w}_{km} \circ R \circ \mathsf{w}_{mk}$. (b) By Lemma 6.1.3 (Worldline Relocation), we have $\mathsf{wl}_m(k^*) = R[\mathsf{wl}_m(k)]$, and because $\mathsf{wl}_m(k) = \mathsf{w}_{mk}[\mathbf{t}] \subseteq \mathsf{plane}(\mathbf{t}, \mathbf{x})$ and R leaves $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ pointwise-fixed, we have that $R[\mathsf{wl}_m(k)] = \mathsf{wl}_m(k)$. Thus, $\mathsf{wl}_m(k^*) = R[\mathsf{wl}_m(k)] = \mathsf{wl}_m(k)$, i.e. k and

k^* share the same worldline. So $\mathsf{w}_{kk^*} \in \mathsf{Triv}$ by Lemma 6.3.7 (Colocate). Thus, (6.9) holds.

Proof of statement (6.2). Choose any $\vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$ and write $\vec{p}\,' := \mathsf{w}_{km}(\vec{p})$. We have to prove that $\vec{p}\,' \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$.

We will show that $\vec{p}\,' \cdot \vec{q} = 0$ for every $\vec{q} \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$, whence it follows easily that $\vec{p}\,' \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$.

By Lemma 6.3.4 (Affine), Theorem 6.2 (Line-to-Line Lemma) and the fact that $\mathsf{w}_{km}(\vec{o}) = \vec{o}$, we know that w_{km} can be written as a composition $\mathsf{w}_{km} = L \circ \widetilde{\varphi}$ of a linear transformation, L, and a map induced by a field automorphism, φ. Therefore, $\mathsf{w}_{km}(-\vec{p}) = L(\widetilde{\varphi}(-\vec{p})) = L(-\widetilde{\varphi}(\vec{p})) = -L(\widetilde{\varphi}(\vec{p})) = -\mathsf{w}_{km}(\vec{p}) = -\vec{p}\,'$.
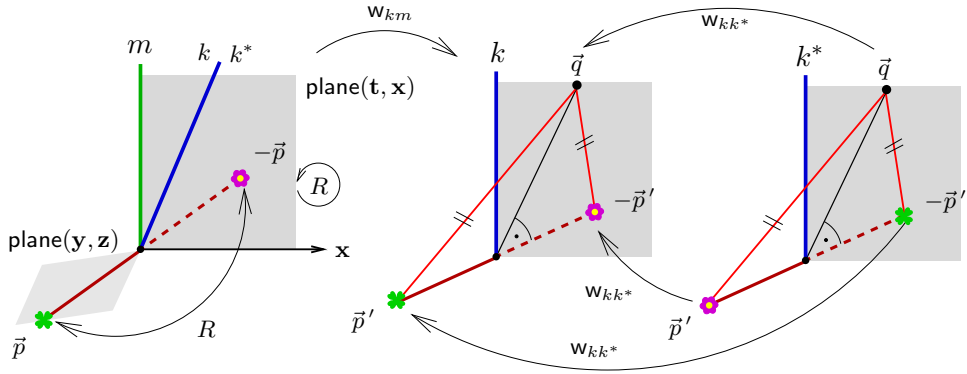

FIGURE 12. Illustration for the proof of (6.2) of Theorem 6.3 (*tx*-Plane Lemma).

Let R be the linear transformation that takes $\vec{\mathsf{t}}, \vec{\mathsf{x}}, \vec{\mathsf{y}}, \vec{\mathsf{z}}$ to $\vec{\mathsf{t}}, \vec{\mathsf{x}}, -\vec{\mathsf{y}}, -\vec{\mathsf{z}}$, respectively. Then R is a self-inverse spatial rotation that leaves $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ pointwise fixed and takes $\vec{p}$ to $-\vec{p}$, see Figure 12. So by (6.9), there is $k^* \in IOb$ such that $\mathsf{w}_{kk^*} \in \mathsf{Triv}$ and $\mathsf{w}_{kk^*} = \mathsf{w}_{km} \circ R \circ \mathsf{w}_{mk}$.

Let $\vec{q} \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$ be arbitrary. Now note that $\mathsf{w}_{mk}(\vec{q}) \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$, hence $R(\mathsf{w}_{mk}(\vec{q})) = \mathsf{w}_{mk}(\vec{q})$. Note also that $\mathsf{w}_{kk^*}(\vec{p}\,') = -\vec{p}\,'$ and $\mathsf{w}_{kk^*}(\vec{q}) = \vec{q}$ because

$$\mathsf{w}_{kk^*}(\vec{p}\,') = \mathsf{w}_{km}(R(\mathsf{w}_{mk}(\vec{p}\,'))) = \mathsf{w}_{km}(R(\vec{p})) = \mathsf{w}_{km}(-\vec{p}) = -\vec{p}\,',$$

$$\mathsf{w}_{kk^*}(\vec{q}) = \mathsf{w}_{km}(R(\mathsf{w}_{mk}(\vec{q}))) = \mathsf{w}_{km}(\mathsf{w}_{mk}(\vec{q})) = \vec{q}.$$

Now, because w_{kk^*} is trivial, we know from Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$) that it is a Euclidean isometry. Moreover, because every trivial map is the composition of a linear map and a translation, and since it fixes $\vec{o}$ (because w_{km}, w_{mk} and R all do so), w_{kk^*} must be linear.

It follows that $|\vec{q} - \vec{p}\,'| = |\mathsf{w}_{kk^*}(\vec{q} - \vec{p}\,')| = |\mathsf{w}_{kk^*}(\vec{q}) - \mathsf{w}_{kk^*}(\vec{p}\,')| = |\vec{q} + \vec{p}\,'|$, whence $(\vec{q} - \vec{p}\,') \cdot (\vec{q} - \vec{p}\,') = (\vec{q} + \vec{p}\,') \cdot (\vec{q} + \vec{p}\,')$, and so $\vec{p}\,' \cdot \vec{q} = 0$.

Since this holds for any $\vec{q} \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$, in particular it holds for both $\vec{\mathsf{t}}$ and $\vec{\mathsf{x}}$. Consequently, $\vec{p}\,' \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$ as claimed.

Proof of statement (6.3). Let $\vec{p}, \vec{q} \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$ and write $\vec{p}\,' := \mathsf{w}_{km}(\vec{p})$ and $\vec{q}\,' := \mathsf{w}_{km}(\vec{q})$. Assume $|\vec{p}| = |\vec{q}|$. We want to prove that $|\vec{p}\,'| = |\vec{q}\,'|$. By Lemma 6.1.7 (Horizontal Rotation), there is a spatial rotation that takes $\vec{\mathsf{x}}$ to $\vec{\mathsf{x}}$ and $\vec{p}$ to $\vec{q}$. Let $R' \in \mathsf{SRot}$ be such a spatial rotation. Then R' leaves $\mathsf{plane}(\mathbf{t}, \mathbf{x})$
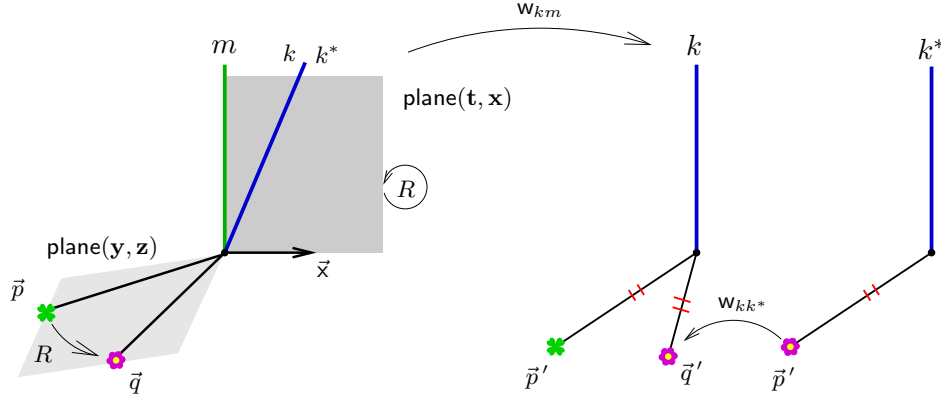
FIGURE 13.   Illustration for the proof of (6.3) of Theorem 6.3 (*tx*-Plane Lemma).

pointwise fixed and takes $\vec{p}$ to $\vec{q}$. By (6.9), there is $k^* \in IOb$ such that $\mathsf{w}_{kk^*} \in \mathsf{Triv}$ and $\mathsf{w}_{kk^*} = \mathsf{w}_{km} \circ R' \circ \mathsf{w}_{mk}$, see Figure 13. It follows that

$$\mathsf{w}_{kk^*}(\vec{p}\,') = \mathsf{w}_{km}(R'(\mathsf{w}_{mk}(\vec{p}\,'))) = \mathsf{w}_{km}(R'(\vec{p})) = \mathsf{w}_{km}(\vec{q}) = \vec{q}\,'.$$

Finally, because w_{kk^*} is trivial, Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$) tells us that it is a Euclidean isometry. It now follows that $|\vec{p}\,'| = |\mathsf{w}_{kk^*}(\vec{p}\,')| = |\vec{q}\,'|$, as claimed. Thus, (6.3) holds.

Proof of statement (6.4). Now assume that w_{km} is linear. Let $\lambda := |\mathsf{w}_{km}(\vec{y})|$. This λ is positive since $\mathsf{w}_{km}(\vec{y}) \neq \vec{\mathsf{o}}$ as $m, k \in IOb_o$. We will prove that $|\mathsf{w}_{km}(\vec{p})| = \lambda |\vec{p}|$ for every $\vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$. Clearly for $\vec{p} = \vec{\mathsf{o}}$ this holds, so assume that $\vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z}) \setminus \{\vec{\mathsf{o}}\}$, and note that

$$\left| \frac{\vec{p}}{|\vec{p}|} \right| = 1 = |\vec{y}|.$$

Then, by (6.3),

$$\left| \mathsf{w}_{km}\left(\frac{\vec{p}}{|\vec{p}|} \right) \right| = |\mathsf{w}_{km}(\vec{y})| = \lambda.$$

Therefore, by linearity of w_{km},

$$|\mathsf{w}_{km}(\vec{p})| = \left| |\vec{p}| \, \mathsf{w}_{km}\left(\frac{\vec{p}}{|\vec{p}|} \right) \right| = \lambda |\vec{p}|.$$

$\square$

6.4. The Same-Speed Lemma.

Theorem 6.4 (Same-Speed Lemma)**.** Assume KIN and $\mathsf{AxIsotropy}$, and that $k, m, h \in IOb_o$. If $\mathsf{speed}_m(k) = \mathsf{speed}_m(h)$, then

    (a) there exists κ such that w_{hk} is a κ-isometry;
    (b) $\mathsf{speed}_k(h) = \mathsf{speed}_h(k)$;
    (c) $\mathsf{speed}_h(m) = \mathsf{speed}_k(m)$.

6.4.1. *Supporting lemmas.* The supporting lemmas can be informally described as:

Lemma 6.4.1 (Translation to IOb$_o$):

Every observer can be translated into IOb_o.

Lemma 6.4.2 (Vertical Plane Rotation):

Every vertical plane can be rotated into the tx-plane.

Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed):

If w_{mm^*} is both linear and trivial, then every j agrees that m and m^* are moving at the same speed, and likewise m and m^* agree on the speed of j.

6.4.2. *Proofs of the supporting lemmas.*

Lemma 6.4.1 (Translation to IOb$_o$)**.** Assume AxWvt and AxRelocate. Given any $k \in IOb$ there exists $k^o \in IOb_o$ such that $\mathsf{w}_{k^o k}$ is a translation.

Proof. Let T be the translation taking $\mathsf{w}_{ko}(\vec{\mathsf{o}})$ to the origin and let k^o be an observer such that $\mathsf{w}_{k^o k} = T$ (such an observer exists by AxRelocate). Then $\mathsf{w}_{k^o o}(\vec{\mathsf{o}}) = (\mathsf{w}_{k^o k} \circ \mathsf{w}_{ko})(\vec{\mathsf{o}}) = T(\mathsf{w}_{ko}(\vec{\mathsf{o}})) = \vec{\mathsf{o}}$, so $k^o \in IOb_o$ as required. $\square$

Lemma 6.4.2 (Vertical Plane Rotation)**.** Assume $(Q, +, \cdot, 0, 1, \leq)$ is a Euclidean field, that P is a plane in Q^4 containing the time-axis $\mathbf{t}$, and that $\vec{p} \in P \setminus \mathbf{t}$. Then there exists a spatial rotation R that takes P and $\vec{p}$ to $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ and $(\vec{p}_t, |\vec{p}_s|, 0, 0)$, respectively.
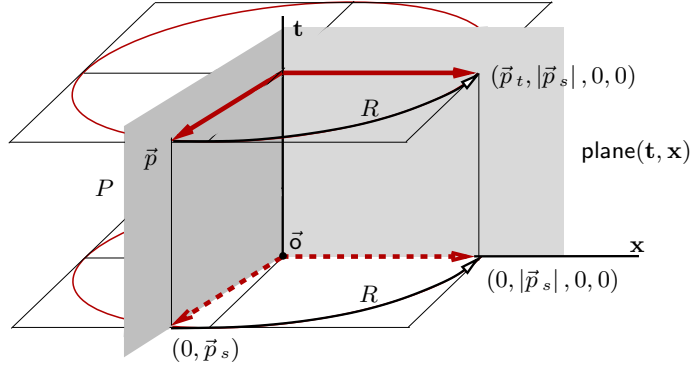

FIGURE 14. Illustration for Lemma 6.4.2 (Vertical Plane Rotation).

Proof. By Lemma 6.1.7 (Horizontal Rotation), there is $R \in \mathsf{SRot}$ which takes $(0, \vec{p}_s)$ to $(0, |\vec{p}_s|, 0, 0)$ and $\vec{\mathsf{o}}$ to $\vec{\mathsf{o}}$; see Figure 14. It is easy to see that this R has the desired properties. $\square$

Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed)**.** Assume AxWvt and AxEField and suppose $m, m^* \in IOb$ and w_{m^*m} is a linear trivial transformation. Then $\mathsf{wl}_j(m) = \mathsf{wl}_j(m^*)$ for every observer $j \in IOb$. Furthermore, if AxLine is assumed, then $\mathsf{speed}_j(m) = \mathsf{speed}_j(m^*)$ and $\mathsf{speed}_m(j) = \mathsf{speed}_{m^*}(j)$ for every $j \in IOb$.

Proof. Recall that $\mathsf{wl}_{m^*}(m) = \mathsf{w}_{m^*m}[\mathbf{t}]$. Since w_{m^*m} is a linear trivial transformation, we have $\mathsf{w}_{m^*m}[\mathbf{t}] = \mathbf{t} = \mathsf{wl}_{m^*}(m^*)$. Thus, $\mathsf{wl}_{m^*}(m) = \mathsf{wl}_{m^*}(m^*)$. Hence, for every $j \in IOb$, $\mathsf{wl}_j(m) = \mathsf{wl}_j(m^*)$ by Lemma 6.3.5 (Equal Worldlines).

Now, assume AxLine and let $j \in IOb$. Then $\mathsf{speed}_j(m) = \mathsf{speed}_j(m^*)$ since $\mathsf{wl}_j(m) = \mathsf{wl}_j(m^*)$. It is easy to see that $\mathsf{slope}(\ell) = \mathsf{slope}(f[\ell])$ holds for every trivial transformation f and line ℓ. Therefore, $\mathsf{speed}_m(j) = \mathsf{slope}(\mathsf{wl}_m(j)) = \mathsf{slope}(\mathsf{w}_{m^*m}[\mathsf{wl}_m(j)]) = \mathsf{slope}(\mathsf{wl}_{m^*}(j)) = \mathsf{speed}_{m^*}(j)$. $\qquad\square$

6.4.3. *Main proof.* We now complete the proof of Theorem 6.4 (Same-Speed Lemma).

Proof of Theorem 6.4 (Same-Speed Lemma). Suppose $\mathsf{speed}_m(k) = \mathsf{speed}_m(h)$, where $m, k, h \in IOb_o$.

(a) If $\mathsf{wl}_m(k) = \mathsf{wl}_m(h)$, then w_{hk} is a trivial transformation by Lemma 6.3.7 (Colocate), hence it is a κ-isometry by Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$).

Assume, therefore, that $\mathsf{wl}_m(k) \neq \mathsf{wl}_m(h)$. Because k and h have the same speed in m's worldview, their worldlines have the same slope according to m. By Lemma 6.3.3 (IOb_o), $\vec{\mathsf{o}} \in \mathsf{wl}_m(k) \cap \mathsf{wl}_m(h)$ because $m, k, h \in IOb_o$.

Let $\vec{p}_1 \in \mathsf{wl}_m(k)$ and $\vec{p}_2 \in \mathsf{wl}_m(h)$ be such that $\vec{p}_1 \neq \vec{\mathsf{o}} \neq \vec{p}_2$ and $(\vec{p}_1)_t = (\vec{p}_2)_t$, see Figure 15. Let $t^* := (\vec{p}_1)_t$ be the common time component of $\vec{p}_1$ and $\vec{p}_2$.

Let $\vec{s}_1 := (\vec{p}_1)_s$ and $\vec{s}_2 := (\vec{p}_2)_s$. Then $|\vec{s}_1| = |\vec{s}_2|$ because lines $\mathsf{wl}_m(k)$ and $\mathsf{wl}_m(h)$ are of same slope. Thinking of $\vec{s}_1$ and $\vec{s}_2$ as points in Q^3, let $\vec{s}^*$ be the point mid-way between them, i.e. $\vec{s}^* = (\vec{s}_1 + \vec{s}_2)/2$, and let ℓ be a line in Q^3 passing through $\vec{0}$ and $\vec{s}^*$. If we now define ρ to be the map which rotates Q^3 through $180°$ about axis ℓ, then the map R given by $R(t, \vec{s}) := (t, \rho(\vec{s}))$ is a self-inverse spatial rotation.[12]

We claim that $R(\vec{p}_1) = \vec{p}_2$. To see this, notice that the points $\vec{s}_1$ and $\vec{s}_2$ form the base of an isosceles triangle in Q^3 whose vertex is $\vec{0}$; it follows easily that the line ℓ bisects and is orthogonal to the line joining $\vec{s}_1$ to $\vec{s}_2$, whence the rotation ρ about ℓ maps $\vec{s}_1$ to $\vec{s}_2$ (and vice versa) in Q^3. Thus, $R(\vec{p}_1) = R(t^*, \vec{s}_1) = (t^*, \rho(\vec{s}_1)) = (t^*, \vec{s}_2) = \vec{p}_2$. Since R also fixes $\vec{\mathsf{o}}$, it must take $\mathsf{wl}_m(k)$ to $\mathsf{wl}_m(h)$. Point $\vec{s}^*$ is fixed by ρ because this point is on ρ's axis of rotation. Therefore, $(0, \vec{s}^*)$ is fixed by R.

So we have $R \in \mathsf{SRot}$, $R[\mathsf{wl}_m(k)] = \mathsf{wl}_m(h)$, $R^{-1} = R$ and $R(0, \vec{s}^*) = (0, \vec{s}^*)$.

Choose $h' \in IOb_o$ such that $k \stackrel{R}{\leadsto}_m h'$. Such h' exists by Lemma 6.1.4 (Observer Rotation) and Lemma 6.3.3 (IOb_o). By Lemma 6.1.3 (Worldline Relocation), we have $\mathsf{wl}_m(h') = R[\mathsf{wl}_m(k)]$, and since $R[\mathsf{wl}_m(k)] = \mathsf{wl}_m(h)$, we must have

$$\mathsf{wl}_m(h) = \mathsf{wl}_m(h'), \tag{6.10}$$

i.e. h and h' share the same worldline. It follows, by Lemma 6.3.7 (Colocate) and $h, h' \in IOb_o$, that

$$\mathsf{w}_{hh'} \text{ is a linear trivial transformation.} \tag{6.11}$$

Our goal is to prove that $\mathsf{w}_{hk} \in {}_\kappa\mathsf{Iso}$ for some κ. Since $\mathsf{w}_{hk} = \mathsf{w}_{hh'} \circ \mathsf{w}_{h'k}$ and (as we have just seen) $\mathsf{w}_{hh'}$ is trivial, it is enough to prove that $\mathsf{w}_{h'k} \in {}_\kappa\mathsf{Iso}$ for some κ.

By $k \stackrel{R}{\leadsto}_m h'$, we have $\mathsf{w}_{mh'} = R \circ \mathsf{w}_{mk}$. Thus,

$$\mathsf{w}_{kh'} = \mathsf{w}_{km} \circ \mathsf{w}_{mh'} = \mathsf{w}_{km} \circ R \circ \mathsf{w}_{mk} \tag{6.12}$$

and

$$\mathsf{w}_{h'k} = (\mathsf{w}_{km} \circ R \circ \mathsf{w}_{mk})^{-1} = \mathsf{w}_{km} \circ R^{-1} \circ \mathsf{w}_{mk}$$

---

[12] We can define ρ in the usual way. Given any $\vec{s}$ we decompose it into a sum $\vec{s} = \vec{s}_\parallel + \vec{s}_\perp$ of components parallel and perpendicular to ℓ, respectively, and then $\rho(\vec{s}) = \vec{s}_\parallel - \vec{s}_\perp$.
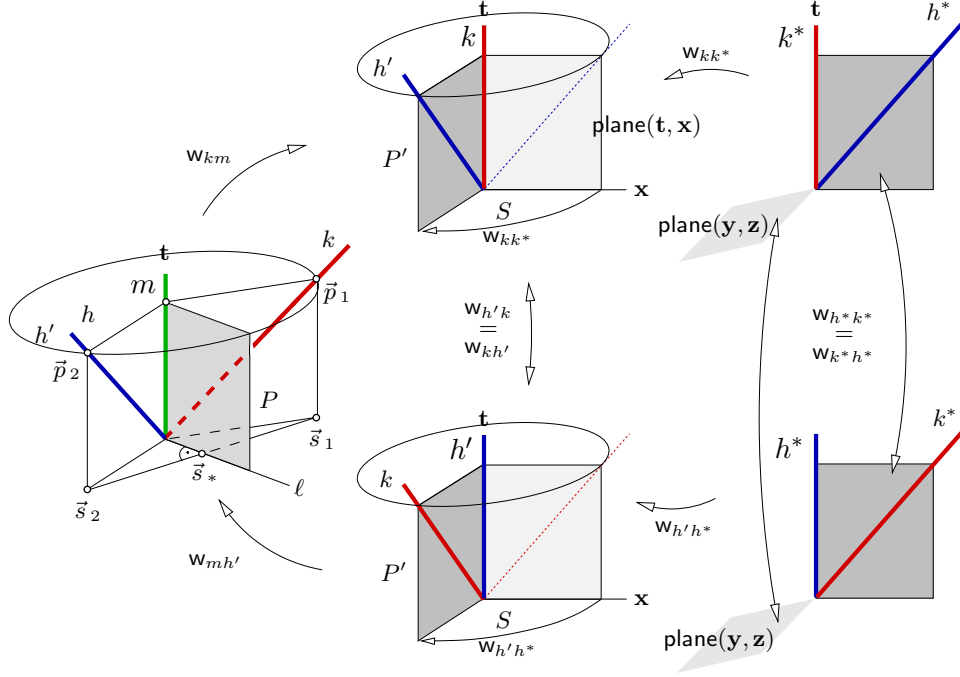
FIGURE 15. Illustration for the proof of Theorem 6.4 (Same-Speed Lemma)

whence (as $R^{-1} = R$)

$$(6.13) \qquad \mathsf{w}_{h'k} = \mathsf{w}_{kh'}, \text{ and thus } \mathsf{wl}_k(h') = \mathsf{wl}_{h'}(k).$$

Let P be the plane containing $(0, \vec{s}^*)$ and $\mathbf{t}$. Since $(0, \vec{s}^*)$ and $\mathbf{t}$ are pointwise fixed by R, it follows that the whole of P is likewise fixed pointwise by R; see Figure 15.

We claim that $\mathsf{w}_{h'k}$ ($= \mathsf{w}_{kh'}$) leaves the plane $\mathsf{w}_{km}[P]$ pointwise fixed. To see this, choose any $\vec{p} \in \mathsf{w}_{km}[P]$. By (6.12), $\mathsf{w}_{kh'}(\vec{p}) = (\mathsf{w}_{km} \circ R \circ \mathsf{w}_{mk})(\vec{p})$. But $\mathsf{w}_{mk}(\vec{p}) \in \mathsf{w}_{mk}[\mathsf{w}_{km}[P]] = P$, so $R(\mathsf{w}_{mk}(\vec{p})) = \mathsf{w}_{mk}(\vec{p})$. It follows that

$$\mathsf{w}_{kh'}(\vec{p}) = (\mathsf{w}_{km} \circ \mathsf{w}_{mk})(\vec{p}) = \vec{p}$$

as stated.

We know that $\mathsf{w}_{h'k}$ is a bijective collineation by Theorem 6.2 (Line-to-Line Lemma) and that it leaves $\vec{\mathsf{o}}$ fixed by Lemma 6.4.1 (Translation to IOb_o) because $h', k \in IOb_o$. So, by Lemma 6.3.4 (Affine), $\mathsf{w}_{h'k}$ is a linear transformation composed with a map induced by a field automorphism. But since $\mathsf{w}_{h'k}$ leaves the plane $\mathsf{w}_{km}[P]$ pointwise fixed, the automorphism component must be the identity, and we deduce that $\mathsf{w}_{h'k}$ is a linear transformation.

By $\mathsf{wl}_m(k) \neq \mathsf{wl}_m(h) = \mathsf{wl}_m(h')$ and Lemma 6.3.5 (Equal Worldlines), we have that $\mathsf{wl}_{h'}(k) \neq \mathsf{wl}_{h'}(h') = \mathbf{t}$. By Lemma 6.3.3 ($IOb_o$), we have that $\vec{\mathsf{o}} \in \mathsf{wl}_k(h')$. Let P' be the plane determined by the time-axis and $\mathsf{wl}_k(h')$ ($= \mathsf{wl}_{h'}(k)$) and let S be a spatial rotation that takes the tx-plane to P', see Figure 15. Such a rotation exists by Lemma 6.4.2 (Vertical Plane Rotation). Choose k^*, h^* such that

$\mathsf{w}_{kk^*} = \mathsf{w}_{h'h^*} = S$ (these exist by AxRelocate). Then

$$(6.14) \qquad \mathsf{w}_{h^*k^*} = \mathsf{w}_{h^*h'} \circ \mathsf{w}_{h'k} \circ \mathsf{w}_{kk^*} = S^{-1} \circ \mathsf{w}_{h'k} \circ S$$

and hence

$$\mathsf{w}_{k^*h^*} = (S^{-1} \circ \mathsf{w}_{h'k} \circ S)^{-1} = S^{-1} \circ \mathsf{w}_{h'k} \circ S$$

because $\mathsf{w}_{h'k} = \mathsf{w}_{kh'}$. Therefore, $\mathsf{w}_{h^*k^*} = \mathsf{w}_{k^*h^*}$ and $\mathsf{w}_{h^*k^*}$ is a linear transformation since S^{-1}, $\mathsf{w}_{h'k}$, and S are linear.

To prove that there is κ such that $\mathsf{w}_{h'k} \in {}_\kappa\mathsf{Iso}$, it is therefore enough to show that there is κ such that $\mathsf{w}_{h^*k^*} \in {}_\kappa\mathsf{Iso}$, because spatial rotations $S, S^{-1} \in {}_\kappa\mathsf{Iso}$ for every κ.

The worldview transformation $\mathsf{w}_{h'k}$ leaves plane P' fixed because it takes $\mathbf{t}$ and $\mathsf{wl}_k(h')$ to $\mathsf{wl}_{h'}(k)$ and $\mathbf{t}$, respectively, and P' is the unique plane that contains $\mathbf{t}$ and $\mathsf{wl}_k(h') = \mathsf{wl}_{h'}(k)$. By this and (6.14), we have that $\mathsf{w}_{h^*k^*}$ maps the tx-plane to itself. Hence, by Theorem 6.3 (tx-Plane Lemma) $\mathsf{w}_{h^*k^*}$ also takes the yz-plane to itself and there is $\lambda > 0$ such that for every $\vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$, $|\mathsf{w}_{h^*k^*}(\vec{p})| = \lambda\,|\vec{p}\,|$. But now, for every $\vec{p} \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$, we have

$$|\vec{p}\,| = |(\mathsf{w}_{k^*h^*} \circ \mathsf{w}_{h^*k^*})(\vec{p}\,)| = |(\mathsf{w}_{h^*k^*} \circ \mathsf{w}_{h^*k^*})(\vec{p}\,)| = \lambda^2\,|\vec{p}\,|.$$

Thus, $\lambda^2 = 1$, whence $\lambda = 1$ (as $\lambda > 0$).

This means that $\mathsf{w}_{h^*k^*}$ preserves Euclidean length in $\mathsf{plane}(\mathbf{y}, \mathbf{z})$.

We have proven so far that $\mathsf{w}_{h^*k^*} = \mathsf{w}_{k^*h^*}$, that $\mathsf{w}_{h^*k^*}$ is a linear transformation taking $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ to $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ and $\mathsf{plane}(\mathbf{y}, \mathbf{z})$ to $\mathsf{plane}(\mathbf{y}, \mathbf{z})$, and that it preserves Euclidean length in $\mathsf{plane}(\mathbf{y}, \mathbf{z})$. It remains to show that $\mathsf{w}_{h^*k^*} \in {}_\kappa\mathsf{Iso}$.

We have already seen that $\vec{\mathsf{o}} \in \mathsf{wl}_{h'}(k) \neq \mathbf{t}$. Thus, $\mathsf{speed}_{h'}(k) \neq 0$. By Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed) and the fact that $\mathsf{w}_{h'h^*}$ and w_{kk^*} are spatial rotations (hence linear trivial transformations), we have that $\mathsf{speed}_{h^*}(k^*) = \mathsf{speed}_{h'}(k^*) = \mathsf{speed}_{h'}(k)$. Thus, $\mathsf{speed}_{h^*}(k^*) \neq 0$.

We will choose κ so that

$$\left\|\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}})\right\|_\kappa^2 = 1.$$

We can do this because we know that $\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$, so we can write $\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) = (t_e, x_e, 0, 0)$ for some t_e and x_e, and we know that $x_e \neq 0$ because $\mathsf{speed}_{h^*}(k^*) \neq 0$ and $\vec{\mathsf{o}}, \mathsf{w}_{h^*k^*}(\mathbf{t}) \in \mathsf{wl}_{h^*}(k^*)$. So we can take $\kappa := (t_e^2 - 1)/x_e^2$, because then

$$\left\|\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}})\right\|_\kappa^2 = t_e^2 - \kappa x_e^2 = t_e^2 - \frac{(t_e^2 - 1)}{x_e^2}x_e^2 = 1,$$

as required.

It follows that $\left\|\mathsf{w}_{h^*k^*}(\vec{p}\,)\right\|_\kappa^2 = \|\vec{p}\,\|_\kappa^2$ for every $\vec{p} \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$, i.e. $\mathsf{w}_{h^*k^*}$ preserves κ-length in the tx-plane. To see why, let $\vec{p} \in \mathsf{plane}(\mathbf{t}, \mathbf{x})$. Notice that $\vec{p}$ can be written as some linear combination $\vec{p} = \lambda\vec{\mathsf{t}} + \mu\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}})$. From this and the fact that $\mathsf{w}_{h^*k^*} = \mathsf{w}_{k^*h^*}$ is a linear transformation, we have

$$\mathsf{w}_{h^*k^*}(\vec{p}\,) = \mathsf{w}_{h^*k^*}(\lambda\vec{\mathsf{t}} + \mu\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}})) = \lambda\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) + \mu\vec{\mathsf{t}}.$$

Writing $\vec{p}_\dagger = \mathsf{w}_{h^*k^*}(\vec{p}\,)$ and recalling that $\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) = (t_e, x_e, 0, 0)$, we have

$$\vec{p} = \lambda(1, 0, 0, 0) + \mu(t_e, x_e, 0, 0) \qquad \text{and} \qquad \vec{p}_\dagger = \lambda(t_e, x_e, 0, 0) + \mu(1, 0, 0, 0)$$

and now direct calculation (using $\kappa = (t_e^2 - 1)/x_e^2$) shows that

$$\|\vec{p}\,\|_\kappa^2 = (\lambda + \mu t_e)^2 - \frac{(t_e^2 - 1)}{x_e^2}\mu^2 x_e^2 = \lambda^2 + 2t_e\lambda\mu + \mu^2$$

and likewise

$$\|\vec{p}\,_\dagger\|_\kappa^2 = (\lambda t_e + \mu)^2 - \frac{(t_e^2 - 1)}{x_e^2}\lambda^2 x_e^2 = \lambda^2 + 2t_e\lambda\mu + \mu^2,$$

whence $\left\|\vec{p}\,^2\right\|_\kappa = \|\vec{p}\,_\dagger\|_\kappa^2 = \|\mathsf{w}_{h^*k^*}(\vec{p})\|_\kappa$ as claimed.

Next, we are going to prove that $\mathsf{w}_{k^*h^*}$ preserves the κ-length. To prove this, let $\vec{p} = (t, x, y, z)$ be an arbitrary point in Q^4 and let $(\hat{t}, \hat{x}, \hat{y}, \hat{z}) = \mathsf{w}_{h^*k^*}(\vec{p})$. By linearity, we have

$$(\hat{t}, \hat{x}, \hat{y}, \hat{z}) = \mathsf{w}_{h^*k^*}(t, x, y, z) = \mathsf{w}_{h^*k^*}(t, x, 0, 0) + \mathsf{w}_{h^*k^*}(0, 0, y, z),$$

whence $(\hat{t}, \hat{x}, 0, 0) = \mathsf{w}_{h^*k^*}(t, x, 0, 0)$ and $(0, 0, \hat{y}, \hat{z}) = \mathsf{w}_{h^*k^*}(0, 0, y, z)$, because $\mathsf{w}_{h^*k^*}$ preserves both the tx- and yz-planes. We also have that

$$\hat{t}^2 - \kappa\hat{x}^2 = t^2 - \kappa x^2 \qquad \text{and} \qquad \hat{y}^2 + \hat{z}^2 = y^2 + z^2$$

because $\mathsf{w}_{h^*k^*}$ preserves the κ-length in the tx-plane and preserves the Euclidean length in the yz-plane. It follows immediately that

$$(\hat{t}^2 - \kappa\hat{x}^2) - \kappa(\hat{y}^2 + \hat{z}^2) = (t^2 - \kappa x^2) - \kappa(y^2 + z^2),$$

or in other words, $\|\vec{p}\|_\kappa^2 = \|\mathsf{w}_{h^*k}(\vec{p})\|_\kappa$, and so $\mathsf{w}_{h^*k^*}$ preserves the κ-length.

Therefore, if $\kappa \neq 0$, then $\mathsf{w}_{h^*k^*}$ is a linear κ-isometry, so $\mathsf{w}_{h^*k^*} \in {}_\kappa\mathsf{Iso}$, and we are done.

Suppose, finally, that $\kappa = 0$. We will prove that $\mathsf{w}_{h^*k^*}$ is a linear 0-isometry. Recall that $\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) = (t_e, x_e, 0, 0)$ and $\kappa = (t_e^2 - 1)/x_e^2$. Since $\kappa = 0$, we have $t_e = \pm 1$, and hence $\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) = (\pm 1, x_e, 0, 0)$. Thus, $(0, x_e, 0, 0) = \mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) \mp \vec{\mathsf{t}}$. This and the fact that $\mathsf{w}_{h^*k^*}$ is both linear and self-inverse now yields

$$
\begin{aligned}
(6.15) \qquad \mathsf{w}_{h^*k^*}(0, x_e, 0, 0) \;&=\; \mathsf{w}_{h^*k^*}(\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) \mp \vec{\mathsf{t}}) \\
&=\; \mathsf{w}_{h^*k^*}(\mathsf{w}_{h^*k^*}(\vec{\mathsf{t}})) \mp \mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) \\
&=\; \vec{\mathsf{t}} \mp \mathsf{w}_{h^*k^*}(\vec{\mathsf{t}}) \\
&=\; \mp (0, x_e, 0, 0).
\end{aligned}
$$

Writing $f := \mathsf{w}_{h^*k^*}$ we have already shown that f preserves κ-length, so for $\kappa = 0$ we have $f(\vec{p})_t^2 = \|f(\vec{p})\|_0^2 = \|\vec{p}\|_0^2 = \vec{p}_t^2$ for every $\vec{p} \in Q^4$. By (5.1), it only remains to show that $|f(\vec{p})_s|^2 = |\vec{p}_s|^2$ when $\vec{p}_t = 0$. However, we know that f maps the yz-plane to itself and preserves Euclidean length in that plane, and that it simply reverses or preserves x-coordinates by (6.15). Hence, f also preserves Euclidean length in the xyz-hyperplane. Thus, $\mathsf{w}_{h^*k^*}$ is a linear 0-isometry.

This completes the proof of (a).

Proof of (b). By (6.11) (which says that $\mathsf{w}_{hh'}$ is a linear trivial transformation) and by Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed), for every $j \in IOb$, we have that

$$(6.16) \qquad\qquad \mathsf{speed}_j(h) \;=\; \mathsf{speed}_j(h') \text{ and}$$

$$(6.17) \qquad\qquad \mathsf{speed}_h(j) \;=\; \mathsf{speed}_{h'}(j),$$

and so

$$\mathsf{speed}_k(h) \overset{(6.16)}{=\!=} \mathsf{speed}_k(h') \overset{(6.13)}{=\!=} \mathsf{speed}_{h'}(k) \overset{(6.17)}{=\!=} \mathsf{speed}_h(k)$$

as required.

Proof of (c). First we show that

$$(6.18) \qquad\qquad \mathsf{wl}_k(m) = \mathsf{wl}_{h'}(m).$$

To do so, recall that $\mathsf{w}_{mh'} = R \circ \mathsf{w}_{mk}$ (by $k \overset{R}{\leadsto}_m h'$). It follows that $\mathsf{w}_{h'm} = \mathsf{w}_{km} \circ R^{-1}$, and hence (because the time-axis $\mathbf{t}$ is fixed under spatial rotations),

$$\mathsf{wl}_{h'}(m) = \mathsf{w}_{h'm}[\mathbf{t}] = (\mathsf{w}_{km} \circ R^{-1})[\mathbf{t}] = \mathsf{w}_{km}[\mathbf{t}] = \mathsf{wl}_k(m)$$

as claimed. Consequently,

$$\mathsf{speed}_k(m) \overset{(6.18)}{=\!=\!=} \mathsf{speed}_{h'}(m) \overset{(6.17)}{=\!=\!=} \mathsf{speed}_h(m).$$

This completes the proof. □

6.5. Fundamental Lemma.

Theorem 6.5 (Fundamental Lemma). Assume $\mathsf{KIN} + \mathsf{AxIsotropy} + \neg\exists\infty\mathsf{Speed}$. Then for every $k, m \in IOb_o$ with $\mathsf{speed}_k(m) > 0$, there is a positive $\varepsilon \in Q$ such that for every non-negative $v \leq \mathsf{speed}_k(m) + \varepsilon$, there is some $h \in IOb_o$ with $\mathsf{speed}_k(h) = v$ and $\mathsf{speed}_m(k) = \mathsf{speed}_m(h)$.
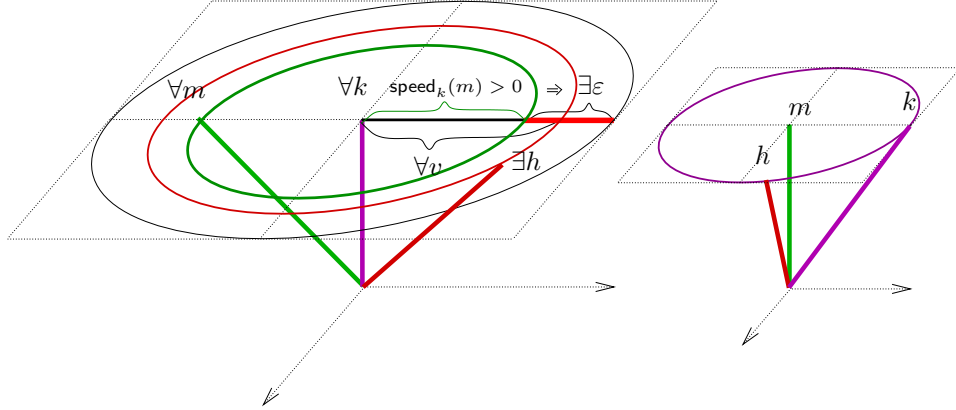

FIGURE 16. Figure illustrating Theorem 6.5 (Fundamental Lemma).

We first show that observers can be found which satisfy certain standard configurations; see Figure 17.

6.5.1. *Supporting lemmas.* The supporting lemmas can be informally described as:

Lemma 6.5.1 (Configuration):

If two observers k and m are moving at any speed $u > 0$ relative to one another, there are 'rotated versions' k^* and m^* of those observers which agree with each other as to where the tx-plane and the y-axis are located. Moreover, if u is finite, then m^* considers k^* to be moving in the positive direction of the x-axis.

Lemma 6.5.2 (Quadratic IVT):

This is a purely technical lemma stating that the Intermediate Value Theorem holds for functions of the form $f(x) = \sqrt{F(x)/G(x)}$ where F and G are quadratic polynomials over Q.
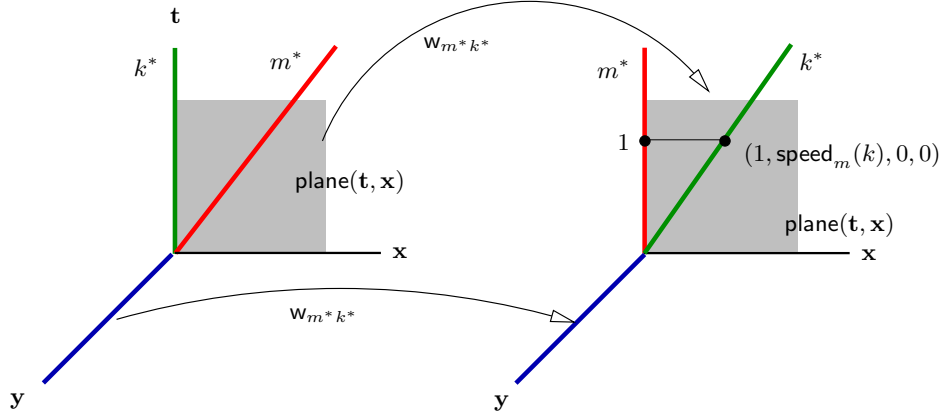
FIGURE 17. Illustration for Lemma 6.5.1 (Configuration).

6.5.2. *Proofs of the supporting lemmas.*

Lemma 6.5.1 (Configuration). Assume $\mathsf{KIN} + \mathsf{AxIsotropy}$. Given any $k, m \in IOb_o$ satisfying $\mathsf{speed}_m(k) \neq 0$, there exist $k^*, m^* \in IOb_o$ such that

(a) w_{k^*k} and w_{m^*m} are spatial rotations, hence[13]
$\mathsf{speed}_{m^*}(k^*) = \mathsf{speed}_m(k)$,
$\mathsf{speed}_{k^*}(h) = \mathsf{speed}_k(h)$ and $\mathsf{speed}_{m^*}(h) = \mathsf{speed}_m(h)$ for every $h \in IOb$;
(b) $\mathsf{w}_{k^*m^*}[\mathsf{plane}(\mathbf{t}, \mathbf{x})] = \mathsf{plane}(\mathbf{t}, \mathbf{x})$;
(c) $\mathsf{w}_{k^*m^*}[\mathbf{y}] = \mathbf{y}$;
(d) k^* moves in the positive direction of the x-axis according to m^*, i.e.
$\left(1, \mathsf{speed}_{m^*}(k^*), 0, 0\right) \in \mathsf{wl}_{m^*}(k^*)$ and $\vec{\mathsf{o}} \in \mathsf{wl}_{m^*}(k^*)$ if $\mathsf{speed}_{m^*}(k^*) \neq \infty$.

Proof. Let us recall that, by Theorem 6.2 (Line-to-Line Lemma), worldview transformations are bijections taking lines to lines and planes to planes.

We know that $\mathsf{wl}_k(m)$ and $\mathbf{t}$ are distinct lines, because $\mathsf{speed}_k(m) \neq 0$. Since, by Lemma 6.3.3 (IOb_o), they meet at the origin, we know that $\mathsf{plane}(\mathbf{t}, \mathsf{wl}_k(m))$ is a well-defined plane, and because this plane contains the time-axis, by Lemma 6.4.2 (Vertical Plane Rotation) there must exist a spatial rotation about $\mathbf{t}$ which takes $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ to $\mathsf{plane}(\mathbf{t}, \mathsf{wl}_k(m))$. By $\mathsf{AxRelocate}$ and (b) of Lemma 6.3.3 (IOb_o), there is some $k^* \in IOb_o$ for which this rotation equals w_{kk^*}, so that

$$(6.19) \qquad \mathsf{w}_{kk^*}[\mathsf{plane}(\mathbf{t}, \mathbf{x})] = \mathsf{plane}(\mathbf{t}, \mathsf{wl}_k(m)),$$

see the left-top of Figure 18.

According to Lemma 6.4.2 (Vertical Plane Rotation) there is also a spatial rotation R that takes $\mathsf{plane}(\mathbf{t}, \mathsf{wl}_m(k))$ to $\mathsf{plane}(\mathbf{t}, \mathbf{x})$; moreover, if $\mathsf{speed}_m(k) \neq \infty$, we can choose $\vec{p} \in \mathsf{wl}_m(k)$ such that $\vec{p}_t = 1$ and require of R that $R(\vec{p}) = (1, |\vec{p}_s|, 0, 0)$. In this case, because $\vec{\mathsf{o}}, \vec{p} \in \mathsf{wl}_m(k)$ and $\vec{p}_t = 1$, we have $\mathsf{speed}_m(k) = \mathsf{slope}(\mathsf{wl}_m(k)) = |\vec{p}_s|$, and so

$$(6.20) \qquad R(\vec{p}) = (1, \mathsf{speed}_m(k), 0, 0).$$

---

[13] by Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed)

Now let $m' \in IOb_o$ be such that $\mathsf{w}_{m'm} = R$ (such an m' exists by AxRelocate and (b) of Lemma 6.3.3 (IOb$_o$)). We will show that $\mathsf{w}_{m'k^*}$ fixes both the tx-plane and the yz-plane. By definition,

$$(6.21) \qquad \mathsf{w}_{m'm}[\mathsf{plane}(\mathbf{t}, \mathsf{wl}_m(k))] = \mathsf{plane}(\mathbf{t}, \mathbf{x})$$

see the left-bottom of Figure 18. If $\mathsf{speed}_m(k) \neq \infty$, by $\vec{p} \in \mathsf{wl}_m(k)$, we have that $\mathsf{w}_{m'm}(\vec{p}) \in \mathsf{wl}_{m'}(k)$. Combining this with (6.20) tells us that

$$(6.22) \qquad (1, \mathsf{speed}_m(k), 0, 0) \in \mathsf{wl}_{m'}(k) \text{ if } \mathsf{speed}_m(k) \neq \infty.$$
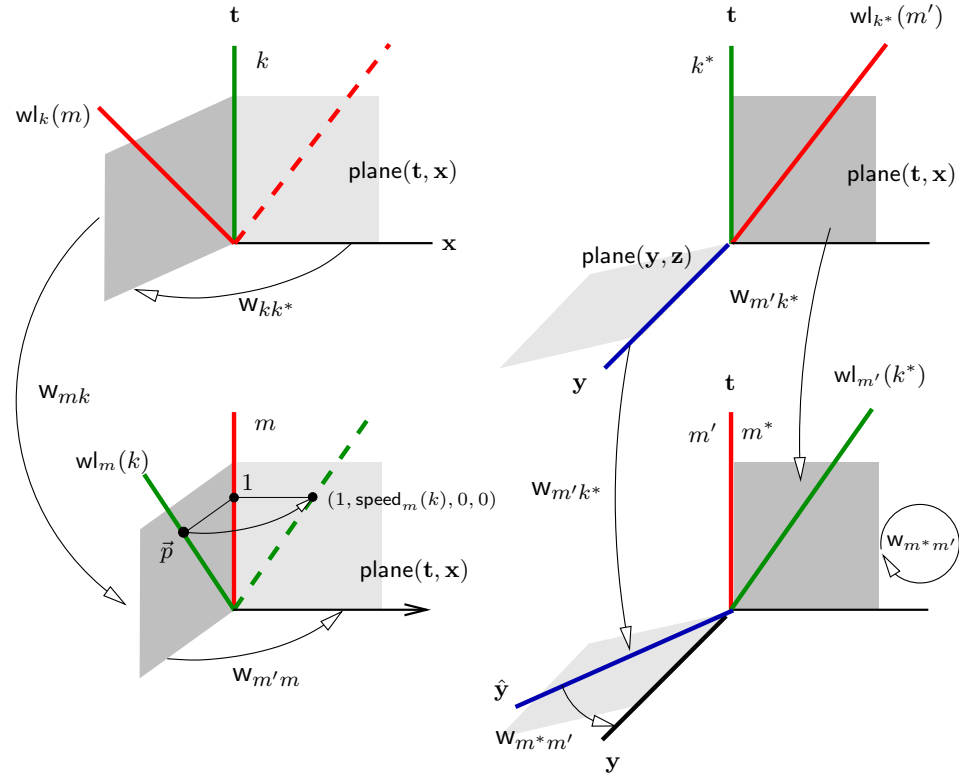

FIGURE 18.   Illustration for the proof of Lemma 6.5.1 (Configuration)

Notice next that the world-view transformation w_{mk} takes $\mathbf{t}$ to $\mathsf{wl}_m(k)$ and $\mathsf{wl}_k(m)$ to $\mathbf{t}$, respectively. Therefore,

$$(6.23) \qquad \mathsf{w}_{mk}[\mathsf{plane}(\mathbf{t}, \mathsf{wl}_k(m))] = \mathsf{plane}(\mathbf{t}, \mathsf{wl}_m(k)),$$

see the left-hand side of Figure 18. By (6.19), (6.23), (6.21), and the fact that $\mathsf{w}_{m'k^*} = \mathsf{w}_{m'm} \circ \mathsf{w}_{mk} \circ \mathsf{w}_{kk^*}$, we have that

$$\mathsf{w}_{m'k^*}[\mathsf{plane}(\mathbf{t}, \mathbf{x})] = \mathsf{plane}(\mathbf{t}, \mathbf{x}).$$

By Theorem 6.3 (tx-Plane Lemma), it follows that $\mathsf{w}_{m'k^*}[\mathsf{plane}(\mathbf{y}, \mathbf{z})] = \mathsf{plane}(\mathbf{y}, \mathbf{z})$. Thus, $\mathsf{w}_{m'k^*}$ fixes both the tx-plane and the yz-plane, as claimed.

Now write $\hat{\mathbf{y}} := \mathsf{w}_{m'k^*}[\mathbf{y}]$, and note that $\hat{\mathbf{y}} \subseteq \mathsf{plane}(\mathbf{y}, \mathbf{z})$ because $\mathsf{w}_{m'k^*}$ preserves this plane. We can find a spatial rotation which fixes the tx-plane pointwise and takes $\hat{\mathbf{y}}$ to $\mathbf{y}$ because of the following. Let $\vec{q} \in \hat{\mathbf{y}}$ and $\vec{q}\,' \in \mathbf{y}$ be such that $|\vec{q}| = |\vec{q}\,'| \neq 0$. Then $\vec{q} \cdot \vec{\mathsf{t}} = \vec{\mathsf{x}} \cdot \vec{q} = \vec{q}\,' \cdot \vec{\mathsf{t}} = \vec{\mathsf{x}} \cdot \vec{q}\,' = 0$ because $\vec{q}, \vec{q}\,' \in \mathsf{plane}(\mathbf{y}, \mathbf{z})$. Therefore, by Lemma 6.1.7 (Horizontal Rotation) there is a spatial rotation that takes $\vec{q}$ to $\vec{q}\,'$ and $\vec{\mathsf{x}}$ to itself. By AxRelocate and Lemma 6.3.3 (IOb$_o$), there is some $m^* \in IOb_o$ such that $\mathsf{w}_{m^*m'}$ is this spatial rotation, see the right-bottom of Figure 18.

Notice that $\mathsf{w}_{m^*m'}$ maps $\hat{\mathbf{y}}$ to $\mathbf{y}$ (because it fixes $\vec{\mathsf{o}}$ and maps $\vec{q} \in \hat{\mathbf{y}}$ to $\vec{q}\,' \in \mathbf{y}$) and fixes $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ pointwise because it fixes $\vec{\mathsf{t}}$ and $\vec{\mathsf{x}}$.

In summary, we have so far shown that $\mathsf{w}_{m^*m'}$ and $\mathsf{w}_{m'm}$ are spatial rotations; and that $\mathsf{w}_{m^*m'}$ and $\mathsf{w}_{m'k^*}$ both fix the tx-plane and the yz-plane.

Proof of (a). The transformation w_{k^*k} is a spatial rotation by definition. Since $\mathsf{w}_{m^*m} = \mathsf{w}_{m^*m'} \circ \mathsf{w}_{m'm}$ is a composition of two spatial rotations, it is also a spatial rotation. By Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed), $\mathsf{speed}_{m^*}(k^*) = \mathsf{speed}_m(k^*) = \mathsf{speed}_m(k)$, $\mathsf{speed}_{k^*}(h) = \mathsf{speed}_k(h)$ and $\mathsf{speed}_{m^*}(h) = \mathsf{speed}_m(h)$ for every $h \in IOb$.

Proof of (b). Since $\mathsf{w}_{m^*k^*} = \mathsf{w}_{m^*m'} \circ \mathsf{w}_{m'k^*}$ and both $\mathsf{w}_{m^*m'}$ and $\mathsf{w}_{m'k^*}$ fix the tx-plane, $\mathsf{w}_{m^*k^*}$ and its inverse $\mathsf{w}_{k^*m^*}$ also fix the tx-plane.

Proof of (c). We have $\mathbf{y} = \mathsf{w}_{m^*m'}[\hat{\mathbf{y}}] = \mathsf{w}_{m^*m'}[\mathsf{w}_{m'k^*}[\mathbf{y}]] = \mathsf{w}_{m^*k^*}[\mathbf{y}]$, so $\mathsf{w}_{m^*k^*}$ and its inverse $\mathsf{w}_{k^*m^*}$ fix the y-axis.

Proof of (d). It is already clear that $\vec{\mathsf{o}} \in \mathsf{wl}_{m^*}(k^*)$, by Lemma 6.3.3 (IOb$_o$). We need to show that $(1, \mathsf{speed}_{m^*}(k^*), 0, 0) \in \mathsf{wl}_{m^*}(k^*)$ as well.

By (6.21) and $\mathsf{wl}_{m'}(k) = \mathsf{w}_{m'm}[\mathsf{wl}_m(k)]$, we have that $\mathsf{wl}_{m'}(k) \subseteq \mathsf{plane}(\mathbf{t}, \mathbf{x})$. Because $\mathsf{w}_{m^*m'}$ fixes $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ pointwise and takes $\mathsf{wl}_{m'}(k)$ to $\mathsf{wl}_{m^*}(k)$, we therefore have $\mathsf{wl}_{m^*}(k) = \mathsf{wl}_{m'}(k)$. By Lemma 6.4.3 (LinTriv $\Rightarrow$ Same Speed), $\mathsf{wl}_{m^*}(k^*) = \mathsf{wl}_{m^*}(k)$ because $\mathsf{w}_{k^*k} \in \mathsf{SRot}$ is a linear trivial transformation. Therefore, $\mathsf{wl}_{m^*}(k^*) = \mathsf{wl}_{m^*}(k) = \mathsf{wl}_{m'}(k)$. Now assume that $\mathsf{speed}_m(k) \neq \infty$. Then (6.22) tells us that $(1, \mathsf{speed}_m(k), 0, 0) \in \mathsf{wl}_{m'}(k) = \mathsf{wl}_{m^*}(k^*)$. By (a), $\mathsf{speed}_m(k) = \mathsf{speed}_{m^*}(k^*)$. Therefore, $(1, \mathsf{speed}_{m^*}(k^*), 0, 0) \in \mathsf{wl}_{m^*}(k^*)$, as required.

This completes the proof. $\qquad\qquad\square$

Remark 6.2. Using the fact that any real-closed field is elementarily equivalent to the field of real numbers (i.e. they satisfy the same first-order logic formulas), it is easy to show that an ordered field is real-closed iff it satisfies the Intermediate Value Theorem for every polynomial function. However, for arbitrary ordered fields (e.g., the field $\mathbb{Q}$ of rationals) the Intermediate Value Theorem can fail even for quadratic functions: if $F(x) = x^2 - 2$, then despite the fact that $F(0) < 0 < F(2)$ there is no $c \in \mathbb{Q}$ for which $F(c) = 0$.

In the proof of Theorem 6.5 (Fundamental Lemma) below, we will need the following lemma stating that the Intermediate Value Theorem holds for a specific class of algebraic functions defined over Euclidean fields.

Lemma 6.5.2 (Quadratic IVT)**.** Assume AxEField, and let F and G be quadratic functions on Q.[14] Let $a < b$ be values in Q and suppose $F(x) \geq 0$ and $G(x) > 0$

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[14]$F : Q \rightarrow Q$ is called a ***quadratic function*** if there are $p, q, r \in Q$ such that $F(x) = px^2 + qx + r$ for every $x \in Q$.

for all $x \in [a, b]$. Let $g : [a, b] \to Q$ be the function $g(x) := \sqrt{F(x)/G(x)}$. Then given any y between $g(a)$ and $g(b)$, there exists $c \in [a, b]$ such that $g(c) = y$.

Proof. If $g(a) = y$ or $g(b) = y$ the proof is trivial, so suppose that y lies strictly between $g(a)$ and $g(b)$ and consider the quadratic function $p(x) = F(x) - y^2 G(x) \equiv [g(x)^2 - y^2]G(x)$. Because y^2 lies strictly between $g(a)^2$ and $g(b)^2$, the values $p(a) = [g(a)^2 - y^2]G(a)$ and $p(b) = [g(b)^2 - y^2]G(b)$ are both non-zero and have opposite signs.

We will show that there exists some $c \in (a, b)$ for which $p(c) = 0$. Because p is quadratic, it can be written in the form $p(x) = \alpha x^2 + \beta x + \gamma$. We know that p is not constant because $p(a) \neq p(b)$, so α and β cannot both be zero. If $\alpha = 0$, then $\beta \neq 0$ and $p(x) = \beta x + \gamma$ is a linear function for which a suitable c can trivially be found. Suppose, then, that $\alpha \neq 0$. Then we can rewrite p as $p(x) = \alpha \left[(x + \beta/2\alpha)^2 - (\beta^2 - 4\alpha\gamma)/4\alpha^2) \right]$, and now the fact that $p(x)$ can be both positive and negative implies immediately that the discriminant $\Delta := (\beta^2 - 4\alpha\gamma)$ is positive, whence p can be factorised over Q with the usual quadratic roots $x_1 := (-\beta + \sqrt{\Delta})/2\alpha$ and $x_2 := (-\beta - \sqrt{\Delta})/2\alpha$. Writing $p(x) = \alpha(x - x_1)(x - x_2)$ it is now easy to see from $p(a)p(b) < 0$ that at least one of these roots must lie strictly between a and b, and we set c equal to this root.

Given the definition of p it now follows from $p(c) = 0$ that $0 = [g(c)^2 - y^2]G(c)$. Because G is positive on $[a, b]$ we can divide through by $G(c)$, whence $g(c)^2 = y^2$. By construction, however, we know that $g(x) \geq 0$ for all $x \in [a, b]$, so both $g(c)$ and y (which lies between $g(a)$ and $g(b)$) are non-negative. We have therefore found a value $c \in (a, b)$ satisfying $g(c) = y$, as required.

$\square$

6.5.3. *Main proof.* We now complete the proof of Theorem 6.5 (Fundamental Lemma).

Proof of Theorem 6.5 (Fundamental Lemma). Choose any $k, m \in IOb_o$ satisfying $\mathsf{speed}_k(m) > 0$. Then $\mathbf{t}$ and $\mathsf{wl}_k(m)$ are distinct lines intersecting in $\vec{o}$. Therefore, their w_{mk}-images, $\mathsf{wl}_m(k)$ and $\mathbf{t}$, are distinct intersecting lines. Hence, $\mathsf{speed}_m(k) > 0$. By Lemma 6.5.1 (Configuration) and $\neg\exists\infty\mathsf{Speed}$, we can assume that

- $\mathsf{w}_{km}[\mathsf{plane}(\mathbf{t}, \mathbf{x})] = \mathsf{plane}(\mathbf{t}, \mathbf{x})$;
- $\mathsf{w}_{km}[\mathbf{y}] = \mathbf{y}$; and
- k moves in the positive direction of the x-axis according to m, i.e.
  $(1, \mathsf{speed}_m(k), 0, 0) \in \mathsf{wl}_m(k)$ and $\vec{o} \in \mathsf{wl}_m(k)$.

Let $r := \mathsf{speed}_m(k)$, and note that $r \neq \infty$ by $\neg\exists\infty\mathsf{Speed}$. Then $(1, r, 0, 0) \in \mathsf{wl}_m(k)$.

For each $x \in [0, r]$, let ℓ_x be the line containing $\vec{o}$ and the point $(1, x, \sqrt{r^2 - x^2}, 0)$. Observe that $\mathsf{slope}(\ell_x) = r$ for all such x, and that $\ell_r = \mathsf{wl}_m(k)$; see Figure 19. Since $\mathsf{wl}_m(k)$ is an m-observer line, by Theorem 6.1 (Observer Lines Lemma) every ℓ_x is an m-observer line, hence by Lemma 6.1.5 (Transformed Observer Lines) every $\mathsf{w}_{km}[\ell_x]$ is a k-observer line. It follows from $\neg\exists\infty\mathsf{Speed}$ that the function $f : [0, r] \to Q$ given by

$$f(x) := \mathsf{slope}(\mathsf{w}_{km}[\ell_x]).$$

is well-defined, and it is easy to see that

$$f(r) = \mathsf{slope}(\mathsf{w}_{km}[\ell_r]) = \mathsf{slope}(\mathsf{w}_{km}[\mathsf{wl}_m(k)]) = \mathsf{slope}[\mathbf{t}] = 0.$$

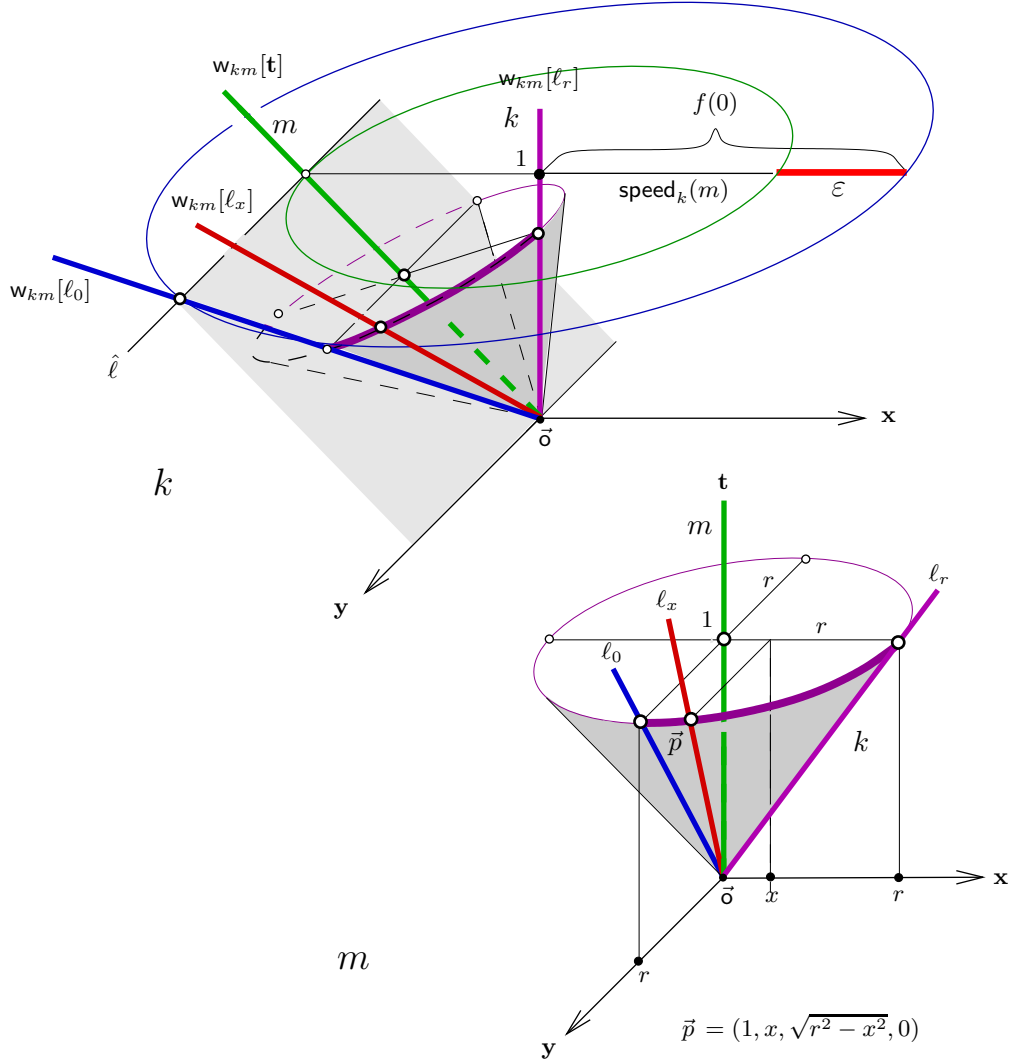We will prove that $f(0) > \mathsf{speed}_k(m)$.

FIGURE 19. Illustration for the proof of Theorem 6.5 (Fundamental Lemma)

Recall that w_{km} is a bijection taking planes to planes by Theorem 6.2 (Line-to-Line Lemma). Since $\ell_0 \subseteq \mathsf{plane}(\mathbf{t}, \mathbf{y})$ and w_{km} fixes the y-axis, we have

$$\mathsf{w}_{km}[\ell_0] \subseteq \mathsf{plane}(\mathsf{w}_{km}[\mathbf{t}], \mathsf{w}_{km}[\mathbf{y}]) = \mathsf{plane}(\mathsf{w}_{km}[\mathbf{t}], \mathbf{y}).$$

Let us write $\hat{P} := \mathsf{plane}(\mathsf{w}_{km}[\mathbf{t}], \mathbf{y})$.

Because $\mathsf{slope}(\mathsf{w}_{km}[\mathbf{t}]) = \mathsf{speed}_k(m)$ cannot be infinite (by $\neg\exists\infty\mathsf{Speed}$), there exists some $\vec{s} \in Q^3$ and such that $(1, \vec{s}) \in \mathsf{w}_{km}[\mathbf{t}]$. And because $\mathbf{t}$ is a subset of the tx-plane (which is fixed by w_{km}), we know that $\mathsf{w}_{km}[\mathbf{t}] \subset \mathsf{plane}(\mathbf{t}, \mathbf{x})$. Thus, the y- and z-components of $\vec{s}$ must both be zero, and there exists some $\hat{x} \in Q$ with $(1, \hat{x}, 0, 0) \in \mathsf{w}_{km}[\mathbf{t}]$.

By Lemma 6.3.3 (IOb$_o$), we have $\mathsf{w}_{km}(\vec{\mathsf{o}}) = \vec{\mathsf{o}}$, so we know that $\vec{\mathsf{o}} \in \mathsf{w}_{km}[\mathbf{t}] \subseteq \hat{P}$. It follows that $\hat{P} = \mathsf{plane}(\mathsf{w}_{km}[\mathbf{t}], \mathbf{y})$ is the unique plane containing both the origin and the line $\hat{\ell} := \{(1, \hat{x}, y, 0) : y \in Q\}$, and every line in this plane which has finite slope and passes through the origin must intersect $\hat{\ell}$ at some point $(1, \hat{x}, y, 0)$ where $y \in Q$. The line of this form with the smallest slope is the one which minimises the value of $\hat{x}^2 + y^2$, and since this is minimal precisely when $y = 0$ the line in this plane through the origin which has the least slope is $\mathsf{w}_{km}[\mathbf{t}]$. At the same time, we know that $\mathsf{w}_{km}[\ell_0]$ is a line in this plane, and that $\mathsf{w}_{km}[\ell_0] \neq \mathsf{w}_{km}[\mathbf{t}]$ because w_{km} is a bijection and $\ell_0 \neq \mathbf{t}$. Hence, $\mathsf{slope}(\mathsf{w}_{km}[\ell_0]) > \mathsf{slope}(\mathsf{w}_{km}[\mathbf{t}])$. Therefore, we have

$$f(0) = \mathsf{slope}(\mathsf{w}_{km}[\ell_0]) > \mathsf{slope}(\mathsf{w}_{km}[\mathbf{t}]) = \mathsf{speed}_k(m).$$

Thus, $f(0) > \mathsf{speed}_k(m)$ as claimed.

Let $\varepsilon = f(0) - \mathsf{speed}_k(m)$. We will prove that for this choice of ε the conclusion of the lemma holds, i.e. that for every non-negative $v \leq \mathsf{speed}_k(m) + \varepsilon$ there is $h \in IOb_o$ such that $\mathsf{speed}_k(h) = v$ and $\mathsf{speed}_m(k) = \mathsf{speed}_m(h)$.

To prove this, choose any $v \in Q$ satisfying $0 \leq v \leq \mathsf{speed}_k(m) + \varepsilon = f(0)$, and recall that $f(r) = 0$. Thus,

(6.24) $$f(0) \geq v \geq f(r).$$

We will use Lemma 6.5.2 (Quadratic IVT) to prove that

(6.25) $$\text{there is } x \in [0, r] \text{ such that } f(x) = v.$$

We know from Theorem 6.2 (Line-to-Line Lemma) that w_{km} is a bijection taking lines to lines. It also preserves the origin since $m, k \in IOb_o$. Hence, by Lemma 6.3.4 (Affine), there exists some linear transformation L and automorphism φ of $(Q, +, \cdot, 0, 1, \leq)$ for which $\mathsf{w}_{km} = L \circ \widetilde{\varphi}$.

By construction, $\widetilde{\varphi}$ maps each coordinate axis to itself, so it takes $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ to $\mathsf{plane}(\mathbf{t}, \mathbf{x})$ and $\mathbf{y}$ to $\mathbf{y}$. We have already seen that w_{km} does likewise, and so the same must be true of L.

We can therefore find $a, b, c, d, \lambda \in Q$ with $\lambda \neq 0$ such that, for every $t, x, y \in Q$,

$$\mathsf{w}_{km}(t, x, y, 0) = (a\varphi(t) + b\varphi(x), c\varphi(t) + d\varphi(x), \lambda\varphi(y), 0).$$

As φ is an automorphism of $(Q, +, \cdot, 0, 1, \leq)$, it follows that $\varphi(1) = 1$; that for every $x \in [0, r]$ we have $\varphi(x) \leq \varphi(r)$; and that

$$\mathsf{w}_{km}\left(1, x, \sqrt{r^2 - x^2}, 0\right) = \left(a + b\varphi(x), c + d\varphi(x), \lambda\sqrt{\varphi(r)^2 - \varphi(x)^2}, 0\right).$$

By definition, for every $x \in [0, r]$, ℓ_x is the line containing $\vec{\mathsf{o}}$ and $\left(1, x, \sqrt{r^2 - x^2}, 0\right)$; therefore, $\mathsf{w}_{km}[\ell_x]$ is the line containing $\vec{\mathsf{o}}$ and $\mathsf{w}_{km}\left(1, x, \sqrt{r^2 - x^2}, 0\right)$, and $f(x) \in Q$ is the slope of this line. Since this slope cannot be infinite we have, for all $x \in [0, r]$, that

(6.26) $$a + b\varphi(x) \neq 0$$

and hence

$$f(x) = \sqrt{\frac{(c + d\varphi(x))^2 + \lambda^2\left(\varphi(r)^2 - \varphi(x)^2\right)}{(a + b\varphi(x))^2}}.$$

Let $F : [0, \varphi(r)] \to Q$ and $G : [0, \varphi(r)] \to Q$ be the quadratic functions defined by

$$
\begin{aligned}
F(y) &:= (c + dy)^2 + \lambda^2 \left(\varphi(r)^2 - y^2 \right) \\
G(y) &:= (a + by)^2,
\end{aligned}
$$

and consider any $y \in [0, \varphi(r)]$. Because $(\varphi(r)^2 - y^2) \geq 0$, it follows immediately that $F(y) \geq 0$. Moreover, $G(y) > 0$, because φ is an ordered-field automorphism, whence $\varphi^{-1}(y) \in [0, r]$, and so by (6.26) we have $a + by = a + b\varphi(\varphi^{-1}(y)) \neq 0$. So, if we now define $g(y) = \sqrt{F(y)/G(y)}$, then g is of the correct form for Lemma 6.5.2 (Quadratic IVT) to be applied over the interval $[0, \varphi(r)]$.

Because $f = g \circ \varphi$, it follows from (6.24) and $\varphi(0) = 0$ that

$$
g(0) \geq v \geq g(\varphi(r)).
$$

By Lemma 6.5.2 (Quadratic IVT), there therefore exists some $y \in [0, \varphi(r)]$ with $g(y) = v$. Taking $x = \varphi^{-1}(y)$ now shows that there exists $x \in [0, r]$ satisfying $f(x) = v$, and (6.25) holds as claimed.

Accordingly, let $\tilde{x} \in [0, r]$ be such that $f(\tilde{x}) = \mathsf{slope}(\mathsf{w}_{km}[\ell_{\tilde{x}}]) = v$. Then $\ell_{\tilde{x}}$ is a line satisfying $\mathsf{slope}(\ell_{\tilde{x}}) = r = \mathsf{speed}_m(k)$ and $\mathsf{slope}(\mathsf{w}_{km}[\ell_{\tilde{x}}]) = f(\tilde{x}) = v$. Since $\ell_{\tilde{x}}$ is an m-observer line, there exists $h \in IOb_o$ with $\mathsf{wl}_m(h) = \ell_{\tilde{x}}$, and hence

- $\mathsf{wl}_k(h) = \mathsf{w}_{km}[\ell_{\tilde{x}}]$,
- $\mathsf{speed}_m(h) = \mathsf{slope}(\mathsf{wl}_m(h)) = \mathsf{slope}(\ell_{\tilde{x}}) = r = \mathsf{speed}_m(k)$, and
- $\mathsf{speed}_k(h) = \mathsf{slope}(\mathsf{wl}_k(h)) = \mathsf{slope}(\mathsf{w}_{km}[\ell_{\tilde{x}}]) = v$.

This is exactly what we had to prove, viz. there exists some h with $\mathsf{speed}_k(h) = v$ and $\mathsf{speed}_m(k) = \mathsf{speed}_m(h)$. $\qquad \square$

6.6. Main Lemma.

Theorem 6.6 (Main Lemma). Assume $\mathsf{KIN} + \mathsf{AxIsotropy}$. Then there is $k \in IOb_o$ and $\kappa \in Q$ such that

$$(6.27) \qquad\qquad \{\mathsf{w}_{mk} \; : \; m \in IOb_o\} \subseteq {}_\kappa\mathsf{Iso}.$$

6.6.1. *Supporting lemmas.* The supporting lemmas can be informally described as:

Lemma 6.6.1 (Same Speed Easy):
If m considers k and h to be moving at the same speed and w_{mk} is a κ-isometry, then so is w_{mh}.

Lemma 6.6.2 (Rest):
Two observers are at rest with respect to one another if and only if the transformation between them is trivial.

Lemma 6.6.3 (Observer Origin):
Given any point on an observer's worldline, we can find an observer with the same worldline which regards that point as its origin.

Lemma 6.6.4 (Median Observer):
Given any two observers, there is a third observer which sees them both moving with the same speed.

Lemma 6.6.5 (κ is unique):
If two observers are moving relative to one another, there exists a unique value κ for which the transformation between them is a κ-isometry.

6.6.2. *Proofs of the supporting lemmas.*

Lemma 6.6.1 (Same Speed Easy). Assume $\mathsf{KIN} + \mathsf{AxIsotropy}$, and let $k, h, m \in IOb_o$. If $\mathsf{speed}_m(k) = \mathsf{speed}_m(h)$ and $\mathsf{w}_{mk} \in {}_\kappa\mathsf{Iso}$, then $\mathsf{w}_{mh} \in {}_\kappa\mathsf{Iso}$.

Proof. By Lemma 6.1.8 (Same-Slope Rotation), there exists a spatial rotation R taking $\mathsf{wl}_m(k)$ to $\mathsf{wl}_m(h)$, and by Lemma 6.1.4 (Observer Rotation) there is some observer k^* satisfying $k \overset{R}{\rightsquigarrow}_m k^*$. Since $\mathsf{w}_{mk^*} = R \circ \mathsf{w}_{mk}$ and $R[\mathsf{wl}_m(k)] = \mathsf{wl}_m(h)$, it follows that $\mathsf{wl}_m(k^*) = R[\mathsf{wl}_m(k)] = \mathsf{wl}_m(h)$, so that k^* and h share the same worldline. By Lemma 6.3.7 (Colocate), w_{k^*h} is therefore trivial, and hence a κ-isometry. It now follows that $\mathsf{w}_{mh} = \mathsf{w}_{mk^*} \circ \mathsf{w}_{k^*h} = R \circ \mathsf{w}_{mk} \circ \mathsf{w}_{k^*h}$ is a composition of κ-isometries, so $\mathsf{w}_{mh} \in {}_\kappa\mathsf{Iso}$ as claimed. $\square$

Lemma 6.6.2 (Rest). Assume KIN. For all observers $k, m \in IOb$, we have

$$k \text{ is at rest according to } m \quad \text{iff} \quad \mathsf{w}_{mk} \in \mathsf{Triv}.$$

Proof. ($\Rightarrow$) Suppose first that k is at rest according to m, i.e. $\mathsf{w}_{mk}(\vec{\mathsf{o}})_s = \mathsf{w}_{mk}(\vec{\mathsf{t}})_s$. We will show that $\mathsf{w}_{mk} \in \mathsf{Triv}$.

Recall that $\mathsf{wl}_m(k)$ is a line (by AxLine) and notice that $\mathsf{w}_{mk}(\vec{\mathsf{o}}), \mathsf{w}_{mk}(\vec{\mathsf{t}}) \in \mathsf{w}_{mk}[\mathbf{t}] = \mathsf{wl}_m(k)$. Hence, $\mathsf{wl}_m(k)$ is parallel to $\mathbf{t}$ (because it is a line containing two distinct points, $\mathsf{w}_{mk}(\vec{\mathsf{o}})$ and $\mathsf{w}_{mk}(\vec{\mathsf{t}})$, whose spatial components are identical), and it passes through $\mathsf{w}_{mk}(\vec{\mathsf{o}})$.

Next, according to $\mathsf{AxRelocate}$ we can find an observer $m' \in IOb$ for which $\mathsf{w}_{mm'}$ is the translation taking $\vec{\mathsf{o}}$ to $\mathsf{w}_{mk}(\vec{\mathsf{o}})$. Because it is a translation, $\mathsf{w}_{mm'}$ necessarily takes $\mathbf{t}$ to a line parallel to $\mathbf{t}$; and because this line is $\mathsf{w}_{mm'}[\mathbf{t}] = \mathsf{wl}_m(m')$, we see that $\mathsf{wl}_m(m')$ is parallel to $\mathbf{t}$. Moreover, because $\mathbf{t}$ contains $\vec{\mathsf{o}}$, we know that $\mathsf{w}_{mk}(\vec{\mathsf{o}}) = \mathsf{w}_{mm'}(\vec{\mathsf{o}}) \in \mathsf{wl}_m(m')$, whence $\mathsf{wl}_m(m')$ is also a line parallel to $\mathbf{t}$ that passes through $\mathsf{w}_{mk}(\vec{\mathsf{o}})$.

Since $\mathsf{wl}_m(k)$ and $\mathsf{wl}_m(m')$ are parallel lines which share a common point, they must be the same (world)line, so $\mathsf{w}_{m'k} \in \mathsf{Triv}$ by Lemma 6.3.7 (Colocate). At the same time we know that $\mathsf{w}_{mm'} \in \mathsf{Triv}$, because it is a translation. It therefore follows by composition that $\mathsf{w}_{mk} = \mathsf{w}_{mm'} \circ \mathsf{w}_{m'k} \in \mathsf{Triv}$, as claimed.

($\Leftarrow$) To prove the converse, suppose that $\mathsf{w}_{mk} \in \mathsf{Triv}$. We need to show that k is at rest according to m, i.e. $\mathsf{w}_{mk}(\vec{\mathsf{t}})_s = \mathsf{w}_{mk}(\vec{\mathsf{o}})_s$. But this is obvious because every trivial transformation maps $\mathbf{t}$ to a line parallel to $\mathbf{t}$. $\square$

Remark 6.3. It follows easily from Lemma 6.6.2 (Rest) and the fact that Triv is a group under composition that "being at rest according to" is an equivalence relation on observers, and "moving according to" is a symmetric relation.

Lemma 6.6.3 (Observer Origin). Assume $\mathsf{AxEField}$, AxWvt and $\mathsf{AxRelocate}$. If $\ell \in \mathsf{ObLines}(k)$ and $\vec{p} \in \ell$, then there exists some $h \in IOb$ for which $\mathsf{w}_{kh}(\vec{\mathsf{o}}) = \vec{p}$ and $\mathsf{wl}_k(h) = \ell$.

Proof. Choose $h' \in IOb$ such that $\mathsf{wl}_k(h') = \ell$. By $\vec{p} \in \mathsf{wl}_k(h')$, we have $\mathsf{w}_{h'k}(\vec{p}) \in \mathsf{w}_{h'k}[\mathsf{wl}_k(h')] = \mathsf{wl}_{h'}(h') = \mathbf{t}$. Let $h \in IOb$ be such that $\mathsf{w}_{h'h}$ is the translation by vector $\mathsf{w}_{h'k}(\vec{p})$. Such h exists by $\mathsf{AxRelocate}$. Translation $\mathsf{w}_{h'h}$ fixes $\mathbf{t}$ because $\mathsf{w}_{h'k}(\vec{p}) \in \mathbf{t}$. Then $\mathsf{wl}_k(h) = \mathsf{w}_{kh}[\mathbf{t}] = \mathsf{w}_{kh'}[\mathsf{w}_{h'h}[\mathbf{t}]] = \mathsf{w}_{kh'}[\mathbf{t}] = \mathsf{wl}_k(h') = \ell$ and $\mathsf{w}_{kh}(\vec{\mathsf{o}}) = \mathsf{w}_{kh'}(\mathsf{w}_{h'h}(\vec{\mathsf{o}})) = \mathsf{w}_{kh'}(\mathsf{w}_{h'k}(\vec{p})) = \vec{p}$ as claimed. $\square$

Lemma 6.6.4 (Median Observer). Assume KIN, $\mathsf{AxIsotropy}$, and $\neg\exists\infty\mathsf{Speed}$. Then given any $k, m \in IOb_o$, there exists some $h \in IOb_o$ for which $\mathsf{speed}_h(k) = \mathsf{speed}_h(m)$.

Proof. If $\mathsf{speed}_k(m) = 0$, the result follows trivially by choosing $h = k$, so suppose $\mathsf{speed}_k(m) > 0$. By applying Theorem 6.5 (Fundamental Lemma) choosing $v = \mathsf{speed}_k(m)$, there exists $h \in IOb_o$ such that

$$(6.28) \qquad\qquad \mathsf{speed}_k(h) \;\; = \;\; \mathsf{speed}_k(m)$$

$$(6.29) \qquad\qquad \mathsf{speed}_m(k) \;\; = \;\; \mathsf{speed}_m(h).$$

Applying Theorem 6.4 (Same-Speed Lemma) to (6.29) tells us that

$$(6.30) \qquad\qquad \mathsf{speed}_k(h) \;\; = \;\; \mathsf{speed}_h(k)$$

$$(6.31) \qquad\qquad \mathsf{speed}_h(m) \;\; = \;\; \mathsf{speed}_k(m)$$

and so

$$\mathsf{speed}_h(k) \overset{(6.30)}{=\!=\!=} \mathsf{speed}_k(h) \overset{(6.28)}{=\!=\!=} \mathsf{speed}_k(m) \overset{(6.31)}{=\!=\!=} \mathsf{speed}_h(m)$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 6.6.5 (κ is unique)**.** Assume AxEField and let $m, k \in IOb$ be observers such that k is moving according to m and $\mathsf{w}_{mk} \in {}_\kappa\mathsf{Iso}$. Then κ is uniquely determined by:

$$(6.32) \qquad\qquad \kappa = \frac{\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right|^2 - 1}{\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s \right|^2}.$$

Proof. Let $f : Q^4 \to Q^4$ be the linear part of w_{mk}, i.e. $f(\vec{p}) := \mathsf{w}_{mk}(\vec{p}) - \mathsf{w}_{mk}(\vec{\mathsf{o}})$. Then f is a linear κ-isometry, so it preserves κ-length. Hence, $1 = \left\| \vec{\mathsf{t}} \right\|_\kappa = \left\| f(\vec{\mathsf{t}}) \right\|_\kappa = f(\vec{\mathsf{t}})_t^2 - \kappa |f(\vec{\mathsf{t}})_s|^2 = |\mathsf{w}_{mk}(\vec{\mathsf{t}})_t - \mathsf{w}_{mk}(\vec{\mathsf{o}})_t|^2 - \kappa |\mathsf{w}_{mk}(\vec{\mathsf{t}})_s - \mathsf{w}_{mk}(\vec{\mathsf{o}})_s|^2$. We have that $\mathsf{w}_{mk}(\vec{\mathsf{t}})_s \neq \mathsf{w}_{mk}(\vec{\mathsf{o}})_s$ because k is moving according to m. Thus, (6.32) follows by reorganizing the equality above. $\qquad\qquad$ $\square$

6.6.3. *Main proof.* We now complete the proof of Theorem 6.6 (Main Lemma).

Proof of Theorem 6.6 (Main Lemma). There are two cases to consider: Case 1: $\neg \exists \infty \mathsf{Speed}$ holds. Case 2: $\exists \infty \mathsf{Speed}$ holds.

Proof of Case 1: Assume $\neg \exists \infty \mathsf{Speed}$.

Suppose $\hat{k}, \hat{m}$ are any observers in IOb_o. According to Lemma 6.6.4 (Median Observer), there is some $\hat{h} \in IOb_o$ such that $\mathsf{speed}_{\hat{h}}(\hat{k}) = \mathsf{speed}_{\hat{h}}(\hat{m})$. By Theorem 6.4 (Same-Speed Lemma), $\mathsf{w}_{\hat{m}\hat{k}}$ is a κ-isometry for some $\kappa \in Q$. This shows that every worldview transformation between two observers in IOb_o is a κ-isometry for some κ, and by Lemma 6.6.5 (κ is unique), this κ is unique if the two observers are moving relative to each other (however, even this unique κ may vary with the choice of the two observers.)

Suppose, then, that $k \in IOb_o$. We will show that κ can be found such that (6.27) holds.

Notice first that if any observer $m \in IOb_o$ is at rest relative to k, then Lemma 6.6.2 (Rest) tells us that w_{mk} is trivial, thus it is a κ-isometry for every $\kappa \in Q$ by Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$). So we only need to consider observers which are moving relative to k.

Suppose, therefore, that $m_1, m_2 \in IOb_o$ are two observers, and that at least one is moving according to k. Without loss of generality we can assume that $0 < \mathsf{speed}_k(m_1)$ and $\mathsf{speed}_k(m_2) \leq \mathsf{speed}_k(m_1)$. We have already seen that

$$(6.33) \qquad\qquad\qquad \mathsf{w}_{m_1 k} \in {}_{\tilde{\kappa}}\mathsf{Iso}$$

for some unique $\tilde{\kappa}$. We will show that $\mathsf{w}_{m_2 k} \in {}_{\tilde{\kappa}}\mathsf{Iso}$ as well. We have already seen that this is the case if $\mathsf{speed}_k(m_2) = 0$, so we can assume that $0 < \mathsf{speed}_k(m_2)$.

By Theorem 6.5 (Fundamental Lemma), choosing $v = \mathsf{speed}_k(m_2)$ and $m = m_1$, there exists $h \in IOb_o$ such that

$$(6.34) \qquad \mathsf{speed}_k(h) = \mathsf{speed}_k(m_2)$$

$$(6.35) \qquad \mathsf{speed}_{m_1}(k) = \mathsf{speed}_{m_1}(h)$$

It follows from Lemma 6.6.1 (Same Speed Easy) with (6.33) and (6.35) that

$$(6.36) \qquad \mathsf{w}_{m_1 h} \in {}_{\tilde{\kappa}}\mathsf{Iso}$$

and hence (by (6.33)) that

$$(6.37) \qquad \mathsf{w}_{kh} = \mathsf{w}_{km_1} \circ \mathsf{w}_{m_1 h} = \mathsf{w}_{m_1 k}^{-1} \circ \mathsf{w}_{m_1 h} \in {}_{\tilde{\kappa}}\mathsf{Iso}.$$

Applying Lemma 6.6.1 (Same Speed Easy) with (6.34) and (6.37) now tells us that $\mathsf{w}_{km_2} \in {}_{\tilde{\kappa}}\mathsf{Iso}$. But then

$$\mathsf{w}_{m_2 k} \in {}_{\tilde{\kappa}}\mathsf{Iso}$$

as claimed.

Finally, let $m \in IOb_o$ be arbitrary. As we have shown, no matter whether m is at rest or in motion relative to k, there is some κ_m such that $\mathsf{w}_{m_1 k}$ and w_{mk} are both in ${}_{\kappa_m}\mathsf{Iso}$. But because m_1 is moving relative to k this κ_m is unique for m_1, so we must have $\kappa_m = \tilde{\kappa}$. Thus, taking $\kappa := \tilde{\kappa}$ ensures that (6.27) holds as claimed.

Proof of Case 2: Assume $\exists \infty \mathsf{Speed}$.

By Lemma 6.2.7 (Infinite Speeds $\Rightarrow$ Lines are Observer Lines), every observer considers every line to be the worldline of an observer, so in particular any 'horizontal' line through $\vec{o}$ is an observer line. By Lemma 6.6.3 (Observer Origin), therefore, there exists $h \in IOb_o$ satisfying $\mathsf{speed}_o(h) = \infty$.

Recall that $\mathbf{S}$ is the spatial hyperplane $\{(0, x, y, z) : x, y, z \in Q\}$; let us consider $\mathsf{w}_{oh}[\mathbf{S}]$. By Theorem 6.2 (Line-to-Line Lemma), this is a 3-dimensional subspace of Q^4 which contains $\mathsf{w}_{oh}(\vec{o}) = \vec{o}$ (because $h \in IOb_o$). It follows that the subspace formed by the intersection of $\mathbf{S}$ with $\mathsf{w}_{oh}[\mathbf{S}]$ must be at least 1-dimensional and so there is some line ℓ such that $\vec{o} \in \ell \subseteq \mathsf{w}_{oh}[\mathbf{S}] \cap \mathbf{S}$. See Figure 20.
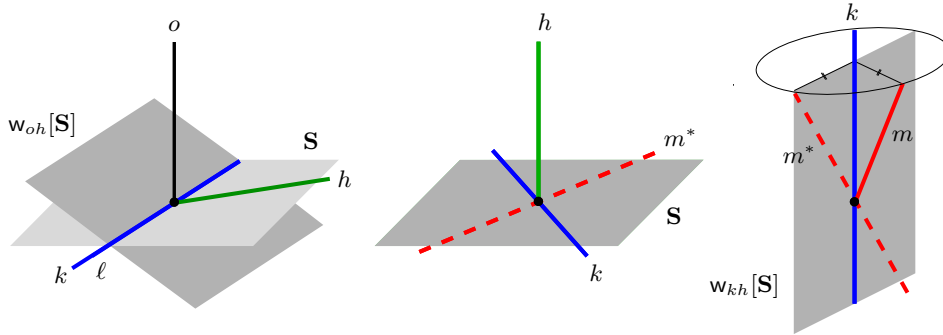

FIGURE 20.   Illustration for the proof of Theorem 6.6 (Main Lemma) if $\exists \infty \mathsf{Speed}$ is assumed.

Because every observer considers every line to be an observer line, o considers ℓ to be an observer line, so there exists some k such that $\ell = \mathsf{wl}_o(k)$. By Lemma 6.6.3

(Observer Origin), we can choose this k to be in IOb_o. Since $\ell \subseteq \mathbf{S}$, we have $\mathsf{speed}_o(k) = \infty$. It follows that $\mathsf{speed}_o(k) = \mathsf{speed}_o(h)$ (both are infinite), whence Theorem 6.4 (Same-Speed Lemma) tells us that $\mathsf{w}_{hk} \in {}_\kappa\mathsf{Iso}$ for some κ. Let us fix such a κ. We will prove that (6.27) holds for this κ.

To do this, we first switch from o's worldview to h's. By construction, we know that $\mathsf{wl}_o(k) = \ell \subseteq \mathsf{w}_{oh}[\mathbf{S}]$, so by applying w_{ho}, we have

$$(6.38) \qquad\qquad \mathsf{wl}_h(k) \subseteq \mathbf{S},$$

and hence $\mathsf{speed}_h(k) = \infty$.

Now let m be any observer $m \in IOb_o$.

In the particular case when $\mathsf{wl}_h(m) \subseteq \mathbf{S}$, we must have $\mathsf{speed}_h(m) = \infty$ because all points in $\mathbf{S}$ have the same time coordinate. In this case, we have $\mathsf{speed}_h(k) = \mathsf{speed}_h(m)$, and since we know that $\mathsf{w}_{hk} \in {}_\kappa\mathsf{Iso}$, Lemma 6.6.1 (Same Speed Easy) tells us that w_{hm} (hence also w_{mh}) is a κ-isometry as well. It now follows by composition, in this special case, that $\mathsf{w}_{mk} = \mathsf{w}_{mh} \circ \mathsf{w}_{hk}$ is a κ-isometry, as required.

Now consider things more generally from k's point of view. As before, $\mathsf{w}_{kh}[\mathbf{S}]$ is a hyperplane, and we know from (6.38) that $\mathsf{wl}_h(k) \subseteq \mathbf{S}$. It follows that

$$\mathbf{t} = \mathsf{wl}_k(k) = \mathsf{w}_{kh}[\mathsf{wl}_k(h)] \subseteq \mathsf{w}_{kh}[\mathbf{S}]$$

so $\mathsf{w}_{kh}[\mathbf{S}]$ contains the time-axis $\mathbf{t}$.

We can therefore find a line ℓ such that $\vec{\mathsf{o}} \in \ell \subseteq \mathsf{w}_{kh}[\mathbf{S}]$ and $\mathsf{slope}(\ell) = \mathsf{speed}_k(m)$. For if $\mathsf{speed}_k(m) = \infty$ we can choose the line through $\vec{\mathsf{o}}$ in $\mathsf{w}_{kh}[\mathbf{S}]$ that is perpendicular to $\mathbf{t}$, and if $\mathsf{speed}_k(m) = 0$ we can take $\ell = \mathbf{t}$. For the remaining case, where $0 < \mathsf{speed}_k(m) < \infty$, choose any point $\vec{p} \in \mathsf{w}_{kh}[\mathbf{S}] \setminus \mathbf{t}$. By Lemma 6.1.10 (Triangulation), we can find a line of slope $\mathsf{speed}_k(m)$ in $\mathsf{w}_{kh}[\mathbf{S}]$ which meets $\mathbf{t}$, and a translation along $\mathbf{t}$ can then be applied to find a parallel line (also in $\mathsf{w}_{kh}[\mathbf{S}]$) that passes through $\vec{\mathsf{o}}$.

Because all lines are observer lines, ℓ is an observer line; and by Lemma 6.6.3 (Observer Origin) there is some $m^* \in IOb_o$ for which $\mathsf{wl}_k(m^*) = \ell \subseteq \mathsf{w}_{kh}[\mathbf{S}]$. But this means that $\mathsf{wl}_h(m^*) = \mathsf{w}_{hk}[\mathsf{wl}_k(m^*)] \subseteq \mathsf{w}_{hk}[\mathsf{w}_{kh}[\mathbf{S}]] = \mathbf{S}$ and hence, as we saw in the special case above, $\mathsf{w}_{m^*k} \in {}_\kappa\mathsf{Iso}$. But now Lemma 6.6.1 (Same Speed Easy) tells us that from $\mathsf{speed}_k(m) = \mathsf{slope}(\ell) = \mathsf{speed}_k(m^*)$ and $\mathsf{w}_{km^*} \in {}_\kappa\mathsf{Iso}$ we can deduce $\mathsf{w}_{km} \in {}_\kappa\mathsf{Iso}$. Therefore, for arbitrary $m \in IOb_o$, $\mathsf{w}_{mk} \in {}_\kappa\mathsf{Iso}$, i.e. (6.27) holds. $\qquad\square$

7. Proofs of the main theorems

Proof of Theorem 5.1 (Characterisation). If $\neg\exists\mathsf{MovingIOb}$ is assumed, then $\mathbb{W} \subseteq \mathsf{Triv}$ by Lemma 6.6.2 (Rest), hence $\mathbb{W} \subseteq {}_\kappa\mathsf{Iso}$ for every κ by Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa\mathsf{Iso}$).

Assume $\exists\mathsf{MovingIOb}$. Let $k \in IOb_o$ and κ be such that (6.27) in Theorem 6.6 (Main Lemma) holds, i.e. $\{\mathsf{w}_{mk} : m \in IOb_o\} \subseteq {}_\kappa\mathsf{Iso}$. Then by Lemma 6.6.5 ($\kappa$ is unique) it is enough to prove that the worldview transformations are κ-isometries.

To prove that worldview transformations are κ-isometries, choose any observers $m_1, m_2 \in IOb$. By Lemma 6.4.1 (Translation to IOb_o), we can find $m_1^o, m_2^o \in IOb_o$ for which $\mathsf{w}_{m_1 m_1^o}$ and $\mathsf{w}_{m_2^o m_2}$ are translations and hence κ-isometries. As $\mathsf{w}_{m_1^o k}$ and $\mathsf{w}_{m_2^o k}$ are also κ-isometries, so it follows that

$$(7.1) \quad \mathsf{w}_{m_1 m_2} = \mathsf{w}_{m_1 m_1^o} \circ \mathsf{w}_{m_1^o k} \circ \mathsf{w}_{k m_2^o} \circ \mathsf{w}_{m_2^o m_2} = \mathsf{w}_{m_1 m_1^o} \circ \mathsf{w}_{m_1^o k} \circ \mathsf{w}_{m_2^o k}^{-1} \circ \mathsf{w}_{m_2^o m_2}$$

is a κ-isometry. □

Proof of Theorem 5.2 (Satisfaction). Let us first prove that

$$(7.2) \qquad\qquad \mathbb{W}_k = \mathsf{G}, \text{ for every } k \in IOb.$$

To do so, let $k \in IOb$. Then, by the definition of $\mathbb{W}_k$ and the construction of $\mathcal{M}_G$,

$$\mathbb{W}_k = \{\mathsf{w}_{kh} : h \in IOb\} = \{k \circ h^{-1} : h \in \mathsf{G}\} = k \circ \mathsf{G}^{-1} = \mathsf{G}$$

because G is a group. Thus, (7.2) holds.

(a) By construction of $\mathcal{M}_G$, we have $\mathsf{w}_{kk} = k \circ k^{-1} = Id$ and $\mathsf{w}_{mh} \circ \mathsf{w}_{hk} = m \circ h^{-1} \circ h \circ k^{-1} = m \circ k^{-1} = \mathsf{w}_{mk}$ for every $m, k, h \in IOb = \mathsf{G}$. Thus, AxWvt holds. By (7.2), we have that $\mathbb{W}_k = \mathbb{W}_h$ for every $k, h \in IOb$, which is a trivial reformulation of AxSPR. Finally, also by (7.2), we have $\mathbb{W} = \bigcup_{k \in IOb} \mathbb{W}_k = \mathsf{G}$.

(b) A trivial reformulation of AxRelocate is that $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathbb{W}_k$ for all $k \in IOb$, which, by (7.2), is equivalent to $\mathsf{SRot} \cup \mathsf{Trans} \subseteq \mathsf{G}$ in $\mathcal{M}_G$.

(c) By definition of worldline, a trivial reformulation of AxLine is that $g[\mathbf{t}]$ is a line for every $g \in \mathbb{W}$. We know from (a) that $\mathbb{W} = \mathsf{G}$, hence the statement holds.

(d) We know from (a) that AxWvt holds, hence by Lemma 6.1.2 (WVT), $\mathsf{wl}_k(k) = \mathbf{t}$ for every $k \in IOb$. Recall that by definition of worldline $\mathsf{wl}_k(k') := \mathsf{w}_{kk'}[\mathbf{t}]$ for every $k, k' \in IOb$. Hence, for every $k, k' \in IOb$, $\mathsf{wl}_k(k) = \mathsf{wl}_k(k')$ is equivalent to $\mathsf{w}_{kk'}[\mathbf{t}] = \mathbf{t}$ in $\mathcal{M}_\mathsf{G}$. Therefore, AxColocate holds in $\mathcal{M}_\mathsf{G}$ iff $g \in \mathsf{Triv}$ whenever $g \in \mathbb{W}$ and $g[\mathbf{t}] = \mathbf{t}$. We know from (a) that $\mathbb{W} = \mathsf{G}$, hence the statement holds. □

Proof of Theorem 5.3 (Model Construction). From Lemma 5.2 (Satisfaction)(a-c), it is clear that AxWvt, AxSPR, AxLine and AxRelocate all hold, and that $\mathbb{W} = \mathsf{G}$. To see that AxColocate also holds, suppose $g \in c\mathsf{Poi} \cup c\mathsf{Eucl} \cup \mathsf{Gal}$ satisfies $g[\mathbf{t}] = \mathbf{t}$. We will show that $g \in \mathsf{Triv}$, whence the result follows by Lemma 5.2 (Satisfaction)(d).

To this end, write $g = T \circ L$ as a composition of a translation T and linear κ-isometry L, and recall that a linear map is trivial if and only if it fixes (setwise) both the time-axis and the present simultaneity, and preserves squared lengths in both. We will show that L has these properties.

To see that $L[\mathbf{t}] = \mathbf{t}$, note that $T(\vec{\mathsf{o}}) = T(L(\vec{\mathsf{o}})) = g(\vec{\mathsf{o}}) \in \mathbf{t}$, whence T must be a translation along the t-axis. Thus, g and T both fix $\mathbf{t}$ setwise, whence so does $L = T^{-1} \circ g$.

To see that L preserves squared length in $\mathbf{t}$, choose arbitrary $t \in Q$. Since $L[\mathbf{t}] = \mathbf{t}$ there is some $t' \in Q$ such that $L(t, \vec{0}) = (t', \vec{0})$, and now $\left\| L(t, \vec{0}) \right\|_\kappa^2 = \left\| (t, \vec{0}) \right\|_\kappa^2$ forces $t' = \pm t$. Thus, L preserves squared lengths in $\mathbf{t}$.

If $\kappa = 0$, then L fixes the present simultaneity $\mathbf{S}$ and preserves the square lengths in it by definition. To see that the same statement holds if $\kappa \neq 0$, choose arbitrary $\vec{s} \in Q^3$ and define $t^* \in Q$ and $\vec{s}^* \in Q^3$ by $(t^*, \vec{s}^*) := L(0, \vec{s})$. Then by linearity

$$L(1, \vec{s}) = (\pm 1 + t^*, \vec{s}^*) \qquad \text{and} \qquad L(1, -\vec{s}) = (\pm 1 - t^*, -\vec{s}^*).$$

Because $\|(1, \vec{s})\|_\kappa^2 = \|(1, -\vec{s})\|_\kappa^2$ and L is a linear κ-isometry, we have that $\|L(1, \vec{s})\|_\kappa^2 = \|L(1, -\vec{s})\|_\kappa^2$, which implies that $(1 + t^*)^2 = (1 - t^*)^2$ and hence $t^* = 0$. Thus, $L(0, \vec{s}) = (0, \vec{s}^*)$, i.e. L maps $\mathbf{S}$ to itself. If $\kappa \neq 0$, $\|(0, \vec{s})\|_\kappa^2 = \|L(0, \vec{s})\|_\kappa^2 = \|(0, \vec{s}^*)\|_\kappa$ implies that $|\vec{s}|^2 = |\vec{s}^*|^2$. Hence, L preserves the square lengths in $\mathbf{S}$.

As claimed, therefore, L is a linear map which fixes both the time-axis and the present simultaneity, and preserves squared lengths in both, whence it is linear

trivial and $g = T \circ L$ is trivial. As outlined above, it now follows that AxColocate also holds, and that hence $\mathcal{M}_\mathsf{G}$ is a model in which $\mathsf{KIN} + \mathsf{AxSPR}$ holds and $\mathbb{W} = \mathsf{G}$. $\quad \square$

Proof of Theorem 5.4 (Determination). Assume that G is a group satisfying the conditions. We will prove that statements (i) and (ii) are equivalent.

Assume that (i) holds. By Theorem 5.2 (Satisfaction), $\mathcal{M}_\mathsf{G}$ is a model of $\mathsf{KIN} + \mathsf{AxSPR}$ (and hence also $\mathsf{KIN} + \mathsf{AxIsotropy}$) for which $\mathbb{W} = \mathsf{G}$. Then (ii) follows by Theorem 5.1 (Characterisation).

Assume that (ii) holds. Then by Theorem 5.3 (Model Construction) $\mathcal{M}_\mathsf{G}$ is a model of $\mathsf{KIN} + \mathsf{AxSPR}$ for which $\mathbb{W} = \mathsf{G}$. Then (i) follows by Theorem 5.2 (Satisfaction). $\quad \square$

Proof of Theorem 5.5 (Classification). Assume $\mathsf{KIN} + \mathsf{AxIsotropy}$. It is clear that at least one of cases (1)-(4) holds. First we show the consequences of the cases and then from those we show that they are mutually exclusive.

(Cases 1-3) If $k, m \in IOb$ are at rest relative to each other, then because w_{mk} is trivial by Lemma 6.6.2 (Rest), it is also a Euclidean isometry by Lemma 6.3.2 ($\mathsf{Triv} = \bigcap_\kappa \mathsf{Iso}$). Thus, for all observers k and m we have

(7.3) $\qquad$ if $\mathsf{w}_{mk}(\vec{\mathsf{t}})_s = \mathsf{w}_{mk}(\vec{\mathsf{o}})_s$, then $\left| \mathsf{w}_{mk}(\vec{\mathsf{t}})_t - \mathsf{w}_{mk}(\vec{\mathsf{o}})_t \right| = 1$.

We claim we can choose k^* and m^* such that $\mathsf{w}_{m^*k^*}(\vec{\mathsf{t}})_s \neq \mathsf{w}_{m^*k^*}(\vec{\mathsf{o}})_s$. This is true by definition if $\exists \mathsf{MovingAccurateClock}$ holds, and follows from (7.3) if either $\exists \mathsf{SlowClock}$ or $\exists \mathsf{FastClock}$ holds because in each of these cases we can choose m^*, k^* such that $\left| \mathsf{w}_{m^*k^*}(\vec{\mathsf{t}})_t - \mathsf{w}_{m^*k^*}(\vec{\mathsf{o}})_t \right| \neq 1$.

It follows that $\exists \mathsf{MovingIOb}$ holds in all three cases, and so by Theorem 5.1 (Characterisation), there is a unique κ such that $\mathbb{W} \subseteq {}_\kappa \mathsf{Iso}$. Recall from (6.32) that κ can be determined from the motion of any two observers moving relative to one another by

$$\kappa = \frac{\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_t - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_t \right|^2 - 1}{\left| \mathsf{w}_{mk}\left(\vec{\mathsf{t}}\right)_s - \mathsf{w}_{mk}\left(\vec{\mathsf{o}}\right)_s \right|^2}.$$

So, given our choice of m^*, k^* (and the definitions of $\exists \mathsf{FastClock}$, $\exists \mathsf{SlowClock}$ and $\exists \mathsf{MovingAccurateClock}$) we have

$$\exists \mathsf{SlowClock} \Rightarrow \left| \mathsf{w}_{m^*k^*}(\vec{\mathsf{t}})_t - \mathsf{w}_{m^*k^*}(\vec{\mathsf{o}})_t \right|^2 > 1 \qquad \Rightarrow \kappa > 0$$

$$\exists \mathsf{FastClock} \Rightarrow \left| \mathsf{w}_{m^*k^*}(\vec{\mathsf{t}})_t - \mathsf{w}_{m^*k^*}(\vec{\mathsf{o}})_t \right|^2 < 1 \qquad \Rightarrow \kappa < 0$$

$$\exists \mathsf{MovingAccurateClock} \Rightarrow \left| \mathsf{w}_{m^*k^*}(\vec{\mathsf{t}})_t - \mathsf{w}_{m^*k^*}(\vec{\mathsf{o}})_t \right|^2 = 1 \qquad \Rightarrow \kappa = 0$$

Because (6.32) holds for any two relatively moving observers it now follows from the uniqueness of κ that $\exists \mathsf{SlowClock} \Rightarrow \forall \mathsf{MovingClockSlow}$, $\exists \mathsf{FastClock} \Rightarrow \forall \mathsf{MovingClockFast}$ and $\exists \mathsf{MovingAccurateClock} \Rightarrow \forall \mathsf{ClockAccurate}$.

Finally, to complete the proof of cases (1-3) it is enough to note that

$$\kappa > 0 \qquad \Rightarrow \qquad {}_\kappa \mathsf{Iso} = c\mathsf{Poi} \text{ where } c = \sqrt{1/\kappa};$$

$$\kappa < 0 \qquad \Rightarrow \qquad {}_\kappa \mathsf{Iso} = c\mathsf{Eucl} \text{ where } c = \sqrt{-1/\kappa};$$

$$\kappa = 0 \qquad \Rightarrow \qquad {}_\kappa \mathsf{Iso} = \mathsf{Gal}.$$

(Case 4). If $\neg \exists \mathsf{MovingIOb}$ holds, then all worldview transformations are trivial by Lemma 6.6.2 (Rest), so $\mathbb{W} \subseteq \mathsf{Triv}$ as claimed.

The four cases are clearly mutually exclusive, because the situations

$$(\forall \mathsf{MovingClockSlow} + \exists \mathsf{MovingIOb}), \quad (\forall \mathsf{MovingClockFast} + \exists \mathsf{MovingIOb}),$$

$$\exists \mathsf{MovingAccurateClock} \quad \text{and} \quad \neg \exists \mathsf{MovingIOb}$$

are mutually exclusive. □

Proof of Theorem 5.6 (Consistency). (Cases 1-3) By Theorem 5.3 (Model Construction) and (5.2), there are models $\mathcal{M}_P$, $\mathcal{M}_E$ and $\mathcal{M}_G$ of $\mathsf{KIN} + \mathsf{AxSPR}$ such that the set of worldview transformations are respectively Poi, Eucl and Gal. In all three models, there are $m, k \in IOb$ such that $\mathsf{w}_{mk} \left(\vec{\mathsf{t}} \right)_s \neq \mathsf{w}_{mk} \left(\vec{\mathsf{o}} \right)_s$ because if $\mathbb{W} = \mathsf{Poi}$ or $\mathbb{W} = \mathsf{Eucl}$ or $\mathbb{W} = \mathsf{Gal}$, then it can be easily seen that there is $f \in \mathbb{W}$ such that $f \left(\vec{\mathsf{t}} \right)_s \neq f \left(\vec{\mathsf{o}} \right)_s$. Let such m and k be fixed. Then $\exists \mathsf{MovingIOb}$ holds. Thus, by Theorem 5.1 (Characterisation), there is a unique κ such that the set of worldview transformations is a subset of $_\kappa \mathsf{Iso}$. This κ is positive ($\kappa = 1$) in $\mathcal{M}_P$, negative ($\kappa = -1$) in $\mathcal{M}_E$ and 0 in $\mathcal{M}_G$. Then by equation (6.32) in Lemma 6.6.5 (κ is unique) it can be seen that $\exists \mathsf{SlowClock}$ holds in $\mathcal{M}_P$, $\exists \mathsf{FastClock}$ holds in $\mathcal{M}_E$ and $\exists \mathsf{MovingAccurateClock}$ holds in $\mathcal{M}_G$.

(Case 4) It remains to prove that $\mathsf{KIN} + \mathsf{AxSPR} + \neg \exists \mathsf{MovingIOb}$ has a model. Let $\mathcal{M}_T$ be a model of $\mathsf{KIN} + \mathsf{AxSPR}$ such that $\mathbb{W} = \mathsf{Triv}$. Such $\mathcal{M}_T$ exists by Theorem 5.3 (Model Construction) and (5.2). Let us notice that for any $f \in \mathsf{Triv}$, $f \left(\vec{\mathsf{t}} \right)_s = f \left(\vec{\mathsf{o}} \right)_s$. Therefore, for every $m, k \in IOb$, $\mathsf{w}_{mk} \left(\vec{\mathsf{t}} \right)_s = \mathsf{w}_{mk} \left(\vec{\mathsf{o}} \right)_s$, and this means that $\neg \exists \mathsf{MovingIOb}$ holds in $\mathcal{M}_T$. □

8. Discussion

In this paper, we have presented an essentially elementary description of what can be deduced about the geometry of $(1+3)$-dimensional spacetime from isotropy if we restrict ourselves to first-order logic and make as few background assumptions as reasonably possible. Nonetheless, there is potential to go further, as even our own very simple assumptions can potentially be weakened while still providing a physically relevant description. The history of the field has shown repeatedly that authors have inadvertently made unconscious, and sometimes unnecessary, assumptions, and it would be foolish to assume that we are necessarily immune to this problem. We have accordingly started a programme of painstakingly machine-verifying our results using interactive theorem provers [SN14], but this programme remains very much in its infancy. In the meantime, therefore, we have been as explicit as possible at all stages of our proofs.

We began by noting that, in the elementary framework advocated in this paper there are reasons why it is no longer appropriate to assume that the ordered field Q of numbers used when recording physical measurements is the field $\mathbb{R}$ of real numbers. Partly this is because practical measurements can never achieve more than a few decimal points of accuracy, and partly because the field $\mathbb{R}$ cannot be uniquely characterised in terms of the first-order sentences it satisfies. But as we have also shown, it is simply not necessary to make the assumption. As long as Q allows the taking of square roots of non-negative values, all of our results hold.

Our results tell us, subject to a small number of very basic axioms, that the worldview transformations that characterise kinematics in isotropic spacetime form a group $\mathbb{W}$ of κ-isometries for some κ. In contrast to earlier studies, we have not needed to assume the full special principle of relativity, but have shown instead that the strictly weaker assumption that space is isotropic is already enough to

entail these results. We accordingly obtain four basic possibilities: the universe is not static (there are moving observers) and $\mathbb{W}$ is a subgroup of either Poi, Eucl or Gal, or the universe is static (all observers are at rest with respect to one another) and $\mathbb{W} \subseteq$ Triv.

As usual (if moving observers exist) we can identify which kind of spacetime we are in by considering whether moving clocks run slow or fast or remain accurate. But because we have not restricted ourselves to $Q = \mathbb{R}$, we have allowed for the possibility that the structure of Q may be somewhat more complicated than usually assumed (for example, there is no reason why Q should not contain infinite or infinitesimal values). This in turn means that the topological structure of Q^4 need not satisfy the usual theorems of $\mathbb{R}^4$, nor the symmetry group $\mathsf{Sym}(Q^4)$ those of $\mathsf{Sym}(\mathbb{R}^4)$. Even so, we have shown that all 'reasonable' subgroups G of $\mathsf{Sym}(Q^4)$ can occur as the transformation group $\mathbb{W}$ in some associated model $\mathcal{M}_\mathsf{G}$. In other words, assuming that $Q = \mathbb{R}$ has inadvertently imposed severe and unnecessary limitations on the set of models investigated in earlier papers.

Nonetheless, many questions remain to be answered. Which of our results still hold, for example, if we remove the requirement for Q to be Euclidean? Are square roots essential, and if not, how can this be interpreted physically? For example, when $\kappa > 0$ the value κ corresponds to a model in which the speed of light is given by $c = \sqrt{1/\kappa}$, but what happens if κ has no square root? Presumably this would be a model in which light signals cannot exist, since they would need to travel with non-existent speed. Some familiar expressions might still be meaningful, for example $\sqrt{1 - v^2/c^2}$ can be rewritten as $\sqrt{1 - \kappa v^2}$, but even so, how does time dilation 'work' if v is a value for which $\sqrt{1 - \kappa v^2}$ is undefined?

There is also the issue of dimensionality. Our initial investigations suggest that all of the proofs presented here go through for dimensions $d \geq (1 + 3)$, but can fail for $d = (1 + 1)$. But do they hold for $d = (1 + 2)$? The answer appears to be yes if we allow trivial transformations to reverse the direction of time — but is this inclusion of reflections essential? We simply do not know.

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