The method of multiple Richardson extrapolation

Teshome Bayleyegn[1] **and Ágnes Havasi**[2]

[1]ELTE Eötvös Loránd University, 1117 Budapest, Pázmány Péter s. 1/C, Hungary
[2]ELTE Eötvös Loránd University, Institute of Mathematics and MTA-ELTE Numerical Analysis and Large Networks Research Group, 1117 Budapest, Pázmány Péter s. 1/C , Hungary
[1]`sbayleyegn130@gmail.com`, [2]`agnes.havasi@ttk.elte.hu`

1 Introduction

Ordinary and partial differential equations are the most important and frequently occurring mathematical models in several areas of applied mathematics. In order to study and understand certain physical, biological, etc. phenomena, such types of equations should be solved, often with a high accuracy and/or within a reasonable computational time. Richardson extrapolation [3] is one of the most powerful numerical techniques which can be used in the efforts to improve the accuracy and performance of underlying numerical methods to solve large and complex problems.

The procedure is based on calculating a suitable linear combination of numerical solutions obtained on two meshes by the same underlying numerical method of order p. This original version of the method is called classical Richardson extrapolation (CRE). The CRE increases the order of accuracy by one if the right-hand side function of the ordinary differential equation to be solved is sufficiently smooth. However, this accuracy is not always sufficient in the applications. The question arises naturally: how can we increase the order even further?

We present a possible generalization of the CRE, which we call multiple Richardson extrapolation (MRE). This method can be combined with any one-step numerical method, e.g., with some Runge–Kutta method, both explicit and implicit. When stiff systems are solved, which frequently arise, e.g., in chemical models, the numerical method should have favourable stability properties on a fixed mesh. In search for an accurate scheme with good absolute stability properties, we will study the absolute stability of the MRE for the simplest Runge–Kutta methods, namely, the first order explicit Euler (EE) and implicit Euler (IE) methods and analyse their stability regions.

2 Classical and multiple Richardson extrapolation

The classical Richardson extrapolation (CRE) method allows us to increase the order p of the underlying method by one. Consider the Cauchy problem for a system of ODE's

$$\begin{cases} y'(t) &= f(t,y), \quad t \in [0,T] \\ y(0) &= y_0, \end{cases} \tag{1}$$

where the unknown function y is of the type $\mathbb{R} \to \mathbb{R}^d$ and $y_0 \in \mathbb{R}^d$. Solve the problem with two different time-step sizes, h and $h/2$, and denote the numerical solutions at time t_n of the coarse mesh by z_n and w_n respectively. Then the combined solution

$$y_{\mathrm{CRE},n} := \frac{2^p w_n - z_n}{2^p - 1} \tag{2}$$

which is called classical Richardson extrapolation, approximates the exact solution to the order $p+1$.

The multiple Richardson extrapolation (MRE) is a new procedure [1] obtained by applying CRE to the combined method (CRE + underlying method of order p), and it provides an order of accuracy $p+2$.

$$y_{\mathrm{MRE},n} := \frac{2^{p+1} y_{\mathrm{CRE}}^{h/2} - y_{\mathrm{CRE}}^{h}}{2^{p+1} - 1} \tag{3}$$

3 Absolute stability analysis

The absolute stability analysis is based on Dahlquist's scalar test problem

$$\begin{cases} y'(t) &= \lambda y(t), \quad t \in [0,\infty) \\ y(0) &= y_0, \end{cases} \tag{4}$$

where $y : \mathbb{R} \to \mathbb{C},\ \lambda = \alpha + \beta i \in \mathbb{C},\ y_0 \in \mathbb{C}$. The exact solution is $y(t) = y_0 \exp(\lambda t),\ t \in [0,\infty)$, which is bounded iff $\alpha \le 0$. From a numerical method for stiff problems it is required that the numerical solution of (4) remains bounded for $\alpha \le 0$ as $t_n \to \infty$ for any or at least not too small time-steps h. Let $\mu := \lambda h$, then for a one-step method $y_{n+1} = R(\mu) y_n$, where the function R, depending on μ is called stability function of the method. Clearly, the numerical solution remains bounded for the grid points of $[0,\infty)$ iff $|R(\mu)| \le 1$. The set $S := \{\mu \in \mathbb{C} : |R(\mu)| \le 1\}$ is called stability region of the method with stability function $R(\mu)$. It is desirable that S is as large as possible, and for stiff systems it should involve $\mathbb{C}^-$, i.e., the whole left half-plane with the imaginary axis. If $\mathbb{C}^- \subset S$, then the method is called A-stable.

We plotted the stability regions for the EE method as underlying method for different versions of the Richardson extrapolation in Figure 1. The CRE increases the stability region, which becomes even larger for the MRE. The figure also shows the stability region obtained for another possible generalization of CRE, called repeated Richardson extrapolation (RRE, [5]), but the MRE has a larger stability region. For more results for other explicit Runge–Kutta methods see [1]. A larger stability region allows the choice of larger time steps, which improves the efficiency. However, since for the EE method we always get bounded stability regions, the application of MRE is not very helpful when stiff problems are to be solved. Therefore, in the following we investigate the implicit Euler (IE) method as underlying method.

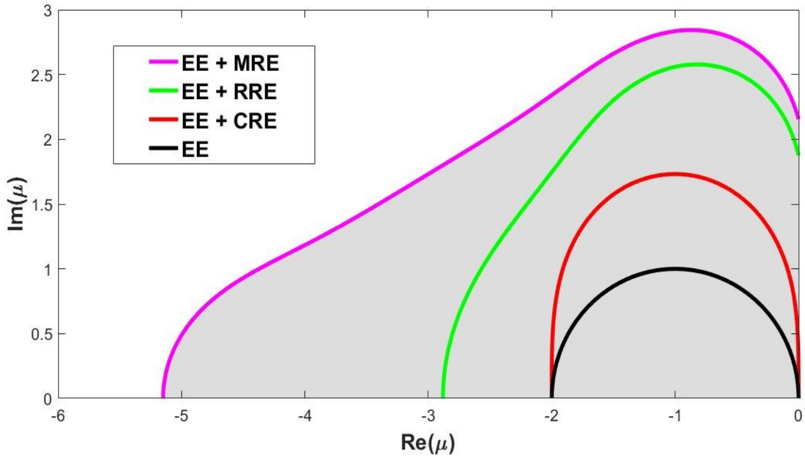

Figure 1: Stability regions for EE as underlying method.

In [2] we analysed the absolute stability function and stability region of the IE + MRE method. The IE and IE + CRE methods have been proved to be A-stable, so their stability regions include the entire left half-plane [4]. We plotted the stability region (in grey) of the IE + MRE method in Fig. 2, which suggests that this method is also A-stable. However, the zoomed picture in Fig. 3 reveals that the combined method IE + MRE is not A-stable. Since the boundary of the stability region extends to the left half-plane, we can have problems with the absolute stability when the matrix (or Jacobian matrix) of the problem to be solved has purely imaginary eigenvalues. It is not recommended to solve such problems with the IE + MRE method. For more details see [2].

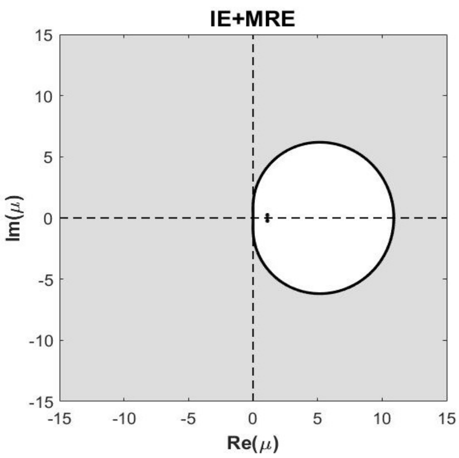

Figure 2: The stability region of IE + MRE.

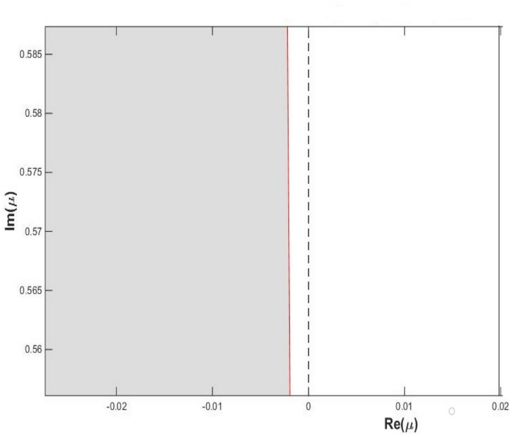

Figure 3: Zoomed detail of the stability region of IE + MRE.

Acknowledgements. "Application Domain Specific Highly Reliable IT Solutions" project has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme TKP2020-NKA-06 (National Challenges Subprogramme) funding scheme. This work was completed in the ELTE Institutional Excellence Program (TKP2020-IKA-05) financed by the Hungarian Ministry of Human Capacities. The project has been supported by the European Union, and co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017-00002), and further, it was supported by the Hungarian Scientific Research Fund OTKA SNN125119.

References

[1] Bayleyegn, T., Havasi, Á.: Multiple Richardson Extrapolation Applied to Explicit Runge–Kutta Methods, In Ivan Dimov and Stefka Fidanova, editors, Advances in High Performance Computing, Springer, Cham, (2021), pp. 262-270.

[2] Bayleyegn, T., Havasi, Á.: Multiple Richardson Extrapolation and its Combination with the Implicit Euler Method, Annales Univ. Sci. Budapest., Sect. Math., 2020 (Accepted for publication).

[3] Richardson, L. F.: The Approximate Arithmetical Solution by Finite Differences of Physical Problems Including Differential Equations, with an Application to the Stresses in a masonry dam, Philosophical Transactions of the Royal Society of London, Series A 210 (1911), 307-357.

[4] Zlatev, Z., Dimov, I., Faragó, I., Havasi, Á.: Richardson Extrapolation - Practical Aspects and Applications. De Gruyter, 2017.

[5] Zlatev, Z., Dimov, I., Faragó, I., Georgiev, K., Havasi, Á.: Stability Properties of Repeated Richardson Extrapolation Applied Together with Some Implicit Runge-Kutta Methods, Springer, Cham, vol. 11386 (2019).