Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **48**, 1–22; https://doi.org/10.14232/ejqtde.2021.1.48 www.math.u-szeged.hu/ejqtde/

Necessary and sufficient conditions for the existence of invariant algebraic curves

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Received 28 August 2020, appeared 13 July 2021
Communicated by Gabriele Villari

Abstract. We present a set of conditions enabling a polynomial system of ordinary differential equations in the plane to have invariant algebraic curves. These conditions are necessary and sufficient. Our main tools include factorizations over the field of Puiseux series near infinity of bivariate polynomials generating invariant algebraic curves. The set of conditions can be algorithmically verified. This fact gives rise to a method, which is able not only to find some irreducible invariant algebraic curves, but also to perform their classification. We study in details the problem of classifying invariant algebraic curves in the most difficult case: we consider differential systems with infinite number of trajectories passing through infinity. As an example, we find necessary and sufficient conditions such that a general polynomial Liénard differential system has invariant algebraic curves. We present a set of all irreducible invariant algebraic curves for quintic Liénard differential systems with a linear damping function. It is supposed in scientific literature that the degrees of their irreducible invariant algebraic curves are bounded by 6. While we derive irreducible invariant algebraic curves of degree 9.

Keywords: invariant algebraic curves, Darboux polynomials, Liénard differential systems, Puiseux series.

2020 Mathematics Subject Classification: 34C05, 37C80.

1 Introduction

Performing the complete classification of trajectories contained in algebraic curves or surfaces for a given polynomial system of ordinary differential equations is a very difficult problem. Such algebraic curves and surfaces producing trajectories of a differential system are called invariants. The knowledge of the set of all irreducible invariants is very important in describing dynamical properties and establishing integrability of a system under consideration. It was noted by Jean Gaston Darboux and Henri Poincaré that the main difficulty in finding irreducible invariants lies in the fact that their degrees are unknown in advance. Nowadays the problem of defining an upper bound on the degrees of irreducible invariant algebraic curves is known as the Poincaré problem. This problem is very difficult in general settings. Solutions are only available in restricted cases, for more details see [20] and references therein.

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Let us consider the following polynomial system of ordinary differential equations in the plane

$$x_t = P(x,y), \quad y_t = Q(x,y) \tag{1.1}$$

with coprime polynomials $P(x,y)$ and $Q(x,y) \in \mathbb{C}[x,y]$. By $\mathbb{C}[x,y]$ we denote the ring of bivariate polynomials with coefficients from the field of complex numbers $\mathbb{C}$. The curve $F(x,y) = 0$ with $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}$ is an invariant algebraic curves of this system whenever the following condition is valid $F_t|_{F=0} = (PF_x + QF_y)|_{F=0} = 0$. If $F(x,y)$ is irreducible in $\mathbb{C}[x,y]$, then the ideal generated by $F(x,y)$ is radical. Consequently, there exists an element $\lambda(x,y)$ of the ring $\mathbb{C}[x,y]$ such that the following linear partial differential equation $P(x,y)F_x + Q(x,y)F_y = \lambda(x,y)F$ is satisfied. The polynomial $\lambda(x,y)$ is called the cofactor of the invariant algebraic curve $F(x,y) = 0$. The degree of $\lambda(x,y)$ is at most $d-1$, where d is the maximum between the degrees of the polynomials $P(x,y)$ and $Q(x,y)$. Let the variable y be privileged with respect to the variable x, then the function $y(x)$ satisfies the following algebraic first-order ordinary differential equation

$$P(x,y)y_x - Q(x,y) = 0. \tag{1.2}$$

The aim of the present article is to present new necessary and sufficient conditions for the existence of invariant algebraic curves. Our main tools include asymptotic analysis of solutions to equation (1.2) and some results of algebraic geometry. The problem of finding a set of conditions satisfied by a polynomial system of ordinary differential equations in the plane with invariant algebraic curves was previously considered by J. Chavarriga et al. [3]. The method of article [3] also uses the local properties of solutions of differential system (1.2). The conditions obtained by J. Chavarriga et al. are necessary conditions, but not sufficient. Let us name some other works [15–17], which deal with algebraic functions, asymptotic series and their role in finding first integrals and invariant algebraic curves of system (1.1).

Puiseux (or fractional power) series generalize Laurent series and can be used if one needs to find local representations of solutions for algebraic equations of the form $F(x,y(x)) = 0$ with $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$. A Puiseux series in a neighborhood of the point $x = \infty$ reads as

$$y(x) = \sum_{l=0}^{+\infty} c_l x^{\frac{l_0 - l}{n_0}}, \tag{1.3}$$

where $l_0 \in \mathbb{Z}$, $n_0 \in \mathbb{N}$. The set of formal Puiseux series given by (1.3) produces an algebraically closed field, which we denote by $\mathbb{C}_\infty\{x\}$. In addition, we shall consider the ring $\mathbb{C}_\infty\{x\}[y]$ of polynomials in one variable with coefficients from the field $\mathbb{C}_\infty\{x\}$. It follows from the algebraic closeness of the field $\mathbb{C}_\infty\{x\}$ that every element from the ring $\mathbb{C}_\infty\{x\}[y]$ is a product of polynomials in y of degree at most one. The differentiation in the field $\mathbb{C}_\infty\{x\}$ is defined as a formal operation with most of the properties similar to those valid for convergent Puiseux series. Any bivariate polynomial $F(x,y) \in \mathbb{C}[x,y]$ can be viewed as an element of the ring $\mathbb{C}_\infty\{x\}[y]$. Consequently, for the algebraic curve $F(x,y) = 0$ given by the polynomial $F(x,y)$, we can construct a factorization into a zero-degree and first-degree factors in the ring $\mathbb{C}_\infty\{x\}[y]$, see [5,6,10,24].

All the Puiseux series solving equation (1.2) can be found using algorithms of the power geometry [1,2] and Painlevé methods [19]. After the classification of Puiseux series satisfying equation (1.2) is completed, the computation of invariant algebraic curves $F(x,y) = 0$ can be made purely algebraic. Indeed, one should require that the non-polynomial part of the factorization for the polynomial $F(x,y)$ in the ring $\mathbb{C}_\infty\{x\}[y]$ vanishes. Generally speaking, this approach gives an infinite algebraic system. Due to the Hilbert's basis theorem only finite number of equations can be considered in practice. Note that the roles of x and y can be changed.

Let us name other methods of finding invariant algebraic curves. The most commonly used methods include the method of undetermined coefficients, the method of the extactic polynomial [4, 21], and an algorithm based on decomposing the vector field related to the original differential system into weight-homogenous components [19]. The method of undetermined coefficients is able to find invariant algebraic curves of fixed degrees only. In addition, the computations may be sufficiently involved. The method of the extactic polynomial was introduced by M. N. Lagutinski [21] and further developed by C. Christopher et al. [4] This method requires calculating certain determinants that are as a rule sufficiently huge. In addition, the method needs a priori information about an upper bound on the degrees of irreducible invariant algebraic curves. The algorithm of decomposing the vector field related to the original system into weight-homogenous components gives an infinite sequence of partial differential equations. On the contrary, the second part of the method of Puiseux series is purely algebraic. Moreover, the latter method is capable to solve the Poincaré for a given polynomial differential system. This comparison shows that the method of Puiseux series presented in works [5, 6, 10] and developed in this article is a natural and visual method of finding and classifying invariant algebraic curves of polynomial differential systems in the plane (1.1). Let us mention that the problem of finding all irreducible invariant algebraic curves of differential systems (1.1) with infinite number of trajectories passing through infinity was not considered in articles [5, 6, 10]. Meanwhile this case turns out to be the most difficult. In this work our goal is to fill this gap. In other words we shall examine the situation with infinite number of Puiseux series near the point $x = \infty$ that satisfy equation (1.2).

As an application of our method we shall consider the famous Liénard differential systems. The systems of first-order ordinary differential equations given by

$$x_t = y, \quad y_t = -f(x)y - g(x) \tag{1.4}$$

are commonly referred to as Liénard differential systems. These systems are used to model different phenomena in physics, chemistry, biology, economics, etc. In this article we consider polynomial Liénard differential systems, i.e. $f(x)$ and $g(x)$ are polynomials

$$f(x) = f_0x^m + \cdots + f_m, \quad g(x) = g_0x^n + \cdots + g_n, \quad f_0g_0 \neq 0 \tag{1.5}$$

with coefficients in the field $\mathbb{C}$. K. Odani proved that Liénard systems with $n \leq m$ have no invariant algebraic curves with the exception for some trivial cases [23]. Integrability properties of these families of systems under the condition $n \leq m$ were studied by J. Llibre and C. Valls [22]. H. Żołądek considered the problem of finding limit cycles contained in the ovals of hyperelliptic invariant algebraic curves $(y - p(x))^2 - q(x) = 0$ with $p(x), q(x) \in \mathbb{C}[x]$, see [25]. The general structure of irreducible invariant algebraic curves and some other properties in the case $m < n < 2m + 1$ were investigated in articles [6, 10]. Explicit expressions of invariant algebraic curves for Liénard differential systems with $m = 1$ and $n = 2$ where presented in work [14]. This article is devoted to the leftover cases: $n \geq 2m + 1$. Let us note that the case $n = 2m + 1$ is in certain sense degenerate and the problem of classifying invariant algebraic curves for $n = 2m + 1$ is very complicated. This degeneracy can be explained analyzing properties of Puiseux series satisfying an algebraic first-order ordinary differential equation of the form (1.2) related to associated Liénard differential systems.

This article is organized as follows. In Section 2 we present and prove our main results and consider some computational aspects of solving an algebraic system resulting form our theorems. In Section 3 we study Liénard differential systems with $n \geq 2m+1$ in details. In particular, we present the general structure of their invariant algebraic curves and cofactors. Finally, in Section 4 we derive the complete classification of irreducible invariant algebraic curves of systems (1.4) with $m = 1$ ($\deg f(x) = 1$) and $n = 5$ ($\deg g(x) = 5$). In the Appendix, an algorithm of finding Puiseux series solving an algebraic first-order ordinary differential equation is described.

2 Computational aspects of the Puiseux series method

Let us begin this section with some preliminary observations resulting from a factorization of an invariant algebraic curve $F(x,y) = 0$ of differential system (1.1) in the ring $\mathbb{C}_\infty\{x\}[y]$. It is straightforward to show that invariant algebraic curves of differential system (1.1) capture Puiseux series satisfying equation (1.2).

Lemma 2.1 ([5]). *Let $y(x)$ be a Puiseux series near the point $x = \infty$ that satisfies the equation $F(x,y) = 0$ with $F(x,y) = 0$ being an invariant algebraic curve of differential system* (1.1) *such that $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$. Then the series $y(x)$ solves equation* (1.2).

Suppose $S(x,y)$ is an element of the ring $\mathbb{C}_\infty\{x\}[y]$. Let us introduce two operators of projection acting in this ring. The first operator $\{S(x,y)\}_+$ gives the sum of the monomials of $S(x,y)$ with non-negative integer powers. In other words, $\{S(x,y)\}_+$ yields the polynomial part of $S(x,y)$. Analogously, the projection $\{S(x,y)\}_- = S(x,y) - \{S(x,y)\}_+$ produces the non-polynomial part of $S(x,y)$. It is straightforward to show that these projections are linear operators. The action of the projection operators can be extended to the ring of Puiseux series in y near the point $y = \infty$ with coefficients from the field $\mathbb{C}_\infty\{x\}$.

By $\mu(x)$ we shall denote the highest-order coefficient (with respect to y) of the bivariate polynomial $F(x,y)$ producing the invariant algebraic curve $F(x,y) = 0$ of differential system (1.1). The following theorem was proved in articles [5, 10].

Theorem 2.2 ([5, 10]). *Let $F(x,y) = 0$ with $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ be an irreducible invariant algebraic curve of differential system* (1.1). *Then $F(x,y)$ and its cofactor $\lambda(x,y)$ take the form*

$$\begin{aligned} F(x,y) &= \left\{ \mu(x) \prod_{j=1}^{N} \{y - y_j(x)\} \right\}_+, \\ \lambda(x,y) &= \left\{ P(x,y) \sum_{m=0}^{\infty} \sum_{l=1}^{L} \frac{\nu_l x_l^m}{x^{m+1}} + \sum_{m=0}^{\infty} \sum_{j=1}^{N} \frac{\{Q(x,y) - P(x,y) y_{j,x}\} y_j^m}{y^{m+1}} \right\}_+, \end{aligned} \tag{2.1}$$

where $y_1(x), \ldots, y_N(x)$ are pairwise distinct Puiseux series in a neighborhood of the point $x = \infty$ that satisfy equation (1.2), *$x_1, \ldots, x_L$ are pairwise distinct zeros of the polynomial $\mu(x) \in \mathbb{C}[x]$ with multiplicities $\nu_1, \ldots, \nu_L \in \mathbb{N}$ and $L \in \mathbb{N} \cup \{0\}$. The degree of $F(x,y)$ with respect to y does not exceed the number of distinct Puiseux series of the from* (1.3) *satisfying equation* (1.2) *whenever the latter is finite. If $\mu(x) = \mu_0$, where $\mu_0 \in \mathbb{C}$, then we suppose that $L = 0$ and the first series is absent in the expression for the cofactor $\lambda(x,y)$.*

Theorem 2.2 gives rise to the following algorithm of finding invariant algebraic curves $F(x,y) = 0$ with $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$.

At *the first step* one should construct all the Puiseux series (near finite points and infinity) that satisfy equation (1.2). Algorithms of classifying Puiseux series solving an algebraic ordinary differential equation are available in the framework of the power geometry [1,2] and the Painlevé methods [19], see Appendix.

At *the second step* one uses Theorem 2.2 in order to derive the structure of an irreducible invariant algebraic curve and its cofactor, see relations (2.1). Possible zeros of the polynomial $\mu(x)$ can be obtained using Puiseux series near finite points possessing certain properties. We shall not discuss this problem here, for more details see [10]. Note that at this step all possible combinations of Puiseux series near infinity found at the first step should be considered if one wishes to classify irreducible invariant algebraic curves. Requiring that the following condition

$$\left\{\mu(x)\prod_{j=1}^{N}\{y-y_j(x)\}\right\}_{-}=0 \tag{2.2}$$

is satisfied yields a system of algebraic equations.

At *the third step* one solves the algebraic system and makes the verification substituting the resulting polynomial $F(x,y)$ related to the invariant algebraic curve and its cofactor $\lambda(x,y)$ into equation

$$P(x,y)F_x+Q(x,y)F_y=\lambda(x,y)F. \tag{2.3}$$

Interestingly, we do not need to consider the convergence of formal Puiseux series solving equation (1.2). Indeed, we perform all the steps of the method working with formal series, and finally, if some formal Puiseux series enters the factorization in the ring $\mathbb{C}_\infty\{x\}[y]$ of the resulting polynomial $F(x,y)$ giving the invariant algebraic curve $F(x,y)=0$, then this series is convergent in some domain by a Newton–Puiseux theorem.

The aim of the present article is to consider the problem of constructing and solving the system arising at the third step of the method.

Let us leave for a while the x-dependence of the elements $y_j(x)$ from the field $\mathbb{C}_\infty\{x\}$ and consider the ring $\mathrm{Sym}\subset\mathbb{C}[y_1,\ldots,y_N]$ of symmetric polynomials in N variables. It is a classical result that Sym is isomorphic to a polynomial ring with N generators. The most commonly used generators include elementary symmetric polynomials given by

$$s_k=\sum_{1\le j_1<j_2<\cdots<j_k\le N}y_{j_1}y_{j_2}\cdots y_{j_k},\quad 1\le k\le N \tag{2.4}$$

and power-sum symmetric polynomials

$$S_k=\sum_{j=1}^{N}y_j^k,\quad 1\le k\le N. \tag{2.5}$$

These generators are related via the Newton's identities of the form

$$\begin{gathered}ks_k=\sum_{j=1}^{k}(-1)^{j-1}s_{k-j}S_j,\quad 1\le k\le N;\\ S_k=(-1)^{k-1}ks_k+\sum_{j=1}^{k-1}(-1)^{k+j-1}s_{k-j}S_j,\quad 1\le k\le N,\end{gathered} \tag{2.6}$$

where additionally should be set $s_0=1$. It is not an easy problem to find the coefficients of the Puiseux series given by the elementary symmetric polynomials $s_k(y_1(x),\ldots,y_N(x))$ with

$k > 1$ if N is not known in advance. This is due to the fact that the coefficients of Puiseux series satisfying an algebraic ordinary differential equation are defined via recurrence relations. At the same time computing coefficients of symmetric polynomials $S_k(y_1(x),\ldots,y_N(x))$ is straightforward. The following theorem contains necessary and sufficient conditions enabling the existence of invariant algebraic curves.

Theorem 2.3. *The polynomial $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ of degree $N > 0$ with respect to y gives an invariant algebraic curve $F(x,y) = 0$ of differential system (1.1) if and only if there exist N Puiseux series $y_1(x),\ldots,y_N(x)$ from the field $\mathbb{C}_\infty\{x\}$ that solve equation (1.2) and satisfy the conditions*

$$\left\{\sum_{j=1}^{k}(-1)^{j-1}w_{k-j}(x)S_j(y_1(x),\ldots,y_N(x))\right\}_- = 0, \quad 1 \leq k \leq N, \tag{2.7}$$

where $w_m(x) \in \mathbb{C}[x]$ are defined as

$$w_m(x) = \left\{\frac{1}{m}\sum_{j=1}^{m}(-1)^{j-1}w_{m-j}(x)S_j(y_1(x),\ldots,y_N(x))\right\}_+, 1 \leq m \leq N \tag{2.8}$$

and $w_0(x) = \mu(x)$ with $\mu(x) \in \mathbb{C}[x]$ being the highest-order coefficient with respect to y of the polynomial $F(x,y)$.

Proof. Let us prove necessity of conditions (2.7). Factorizing the polynomial $F(x,y)$ giving an invariant algebraic curve $F(x,y) = 0$ of differential system (1.1) in the ring $\mathbb{C}_\infty\{x\}[y]$ yields

$$F(x,y) = \mu(x)\prod_{j=1}^{N}\{y - y_j(x)\}, \tag{2.9}$$

where it follows from Lemma 2.1 that the Puiseux series $y_1(x),\ldots,y_N(x)$ satisfy equation (1.2). It is straightforward to rewrite relation (2.9) in the form

$$F(x,y) = \mu(x)\sum_{j=0}^{N}(-1)^j s_j(y_1(x),\ldots,y_N(x))y^{N-j}. \tag{2.10}$$

The non-polynomial part of this expression vanishes and the elements $\mu(x)s_m(y_1(x),\ldots,y_N(x))$ should be polynomials coinciding with $w_m(x)$ given in (2.8). Considering the non-polynomial coefficients at y^{N-k}, we obtain the conditions

$$\{\mu(x)s_k(y_1(x),\ldots,y_N(x))\}_- = 0, \quad 1 \leq k \leq N. \tag{2.11}$$

Using relations (2.6), we see that conditions (2.11) are equivalent to (2.7).

In order to verify sufficiency of conditions (2.7), let us consider a formal expression (2.9) and at first prove that it is a polynomial in $\mathbb{C}[x,y]$. We need to establish that for each k from 1 to N the coefficient at y^{N-k} in expression (2.9) is a polynomial. We shall use induction on k. If $k = 1$, then condition (2.7) reads as $\{\mu(x)S_1(y_1(x),\ldots,y_N(x))\}_- = 0$ and we see that the coefficient at y^{N-1} in relations (2.9) and (2.10) is a polynomial in x taking the form $-w_1(x)$, where $w_1(x) = \{\mu(x)S_1(y_1(x),\ldots,y_N(x))\}_+$. Let us suppose that the coefficients at y^{N-k} with $1 < k \leq l$ are polynomials in x. These polynomials we denote as $(-1)^k w_k(x)$. It is straightforward to prove that they are given by relations (2.8) with $1 < k \leq l$.

The coefficient at $y^{N-(l+1)}$ in relation (2.9) is equal to $(-1)^{l+1}\mu(x)s_{l+1}(y_1(x),\ldots,y_N(x))$. Using expression (2.6), we find

$$\mu(x)s_{l+1}(y_1(x),\ldots,y_N(x)) = \frac{1}{l+1}\sum_{j=1}^{l+1}(-1)^{j-1}\mu(x)s_{l+1-j}S_j \tag{2.12}$$

According to the induction hypothesis, we see that the elements $\mu(x)s_{l+1-j}$, $1 \le j \le l+1$ are polynomials in x coinciding with $w_{l+1-j}(x)$, $1 \le j \le l+1$ and consequently it follows from condition (2.7) at $k = l+1$ that the coefficient at $y^{N-(l+1)}$ is a polynomial in x. Thus we conclude that expression (2.9) gives a bivariate polynomial $F(x,y)$ from the ring $\mathbb{C}[x,y]$.

Finally, let us establish that the polynomial $F(x,y)$ indeed gives an invariant algebraic curve $F(x,y) = 0$ of differential system (1.1). Let $f(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ be an irreducible factor of the polynomial $F(x,y)$. The element $Pf_x + Qf_y$ is also a polynomial, which we denote as $h(x,y)$, i.e. $h(x,y) = Pf_x + Qf_y$. Let us take one of the Puiseux series near infinity $y_j(x)$ that satisfies the equation $f(x,y_j(x)) = 0$. Differentiating this equation, we obtain $f_x(x,y_j(x)) + f_y(x,y_j(x))y_{j,x} = 0$. Since $f(x,y)$ divides $F(x,y)$, we see that the series $y_j(x)$ solves equation (1.2) and we get $P(x,y_j(x))y_{j,x} - Q(x,y_j(x)) = 0$. Combining the equations $f_x(x,y_j(x)) + f_y(x,y_j(x))y_{j,x} = 0$ and $P(x,y_j(x))y_{j,x} - Q(x,y_j(x)) = 0$ yields the relation $h(x,y_j(x)) = O$, where O is the zero element of the field $\mathbb{C}_\infty\{x\}$. Note that $P(x,y_j(x)) \neq O$. Indeed, assuming the converse, we find from equation (1.2) that $Q(x,y_j(x)) = O$. This fact contradicts the assumption that the polynomials $P(x,y)$ and $Q(x,y)$ are coprime in the ring $\mathbb{C}[x,y]$. It follows from the relations $f(x,y_j(x)) = 0$ and $h(x,y_j(x)) = 0$, that two algebraic curves $f(x,y) = 0$ and $h(x,y) = 0$ intersect in an infinite number of points inside the domain of convergence of the series $y_j(x)$. Using the Bézout's theorem, we see that there exists a polynomial both dividing $f(x,y)$ and $h(x,y)$. Since $f(x,y)$ is irreducible, we find that $h(x,y) = \lambda_0(x,y)f(x,y)$ with $\lambda_0(x,y) \in \mathbb{C}[x,y]$. Recalling the definition of $h(x,y)$, we conclude that the polynomial $f(x,y)$ gives an invariant algebraic curve of differential system (1.1) and the same is true for all other irreducible divisors of $F(x,y)$. Thus, so does $F(x,y)$. This completes the proof. □

If the highest-order coefficient (with respect to y) of the polynomial $F(x,y)$ is a constant, then there is no loss of generality in setting $\mu(x) = 1$. Repeating the reasoning of Theorem 2.3 for this particular case we obtain the following lemma.

Lemma 2.4. *The polynomial $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$ of degree $N > 0$ with respect to y and with $\mu(x) = 1$ gives an invariant algebraic curve $F(x,y) = 0$ of differential system* (1.1) *if and only if there exist N Puiseux series $y_1(x),\ldots,y_N(x)$ defined in a neighborhood of the point $x = \infty$ that solve equation* (1.2) *and satisfy the conditions*

$$\left\{\sum_{j=1}^{N} y_j^k(x)\right\}_- = 0, \quad 1 \le k \le N. \tag{2.13}$$

Again we remark that an algorithm of finding Puiseux series solving a first-order algebraic ordinary differential equation is presented in the Appendix. It follows from Theorem 2.2 that the Puiseux series in Theorem 2.3 and in Lemma 2.4 should be pairwise distinct whenever one wishes to find irreducible invariant algebraic curves.

If all the Puiseux series near the point $x = \infty$ satisfying equation (1.2) have uniquely determined coefficients, then the degrees with respect to y of bivariate polynomials giving irreducible invariant algebraic curves of differential system (1.1) are bounded by the number of distinct Puiseux series. This fact was established in Theorem 2.2. Consequently, the algebraic system produced by Theorem 2.3 involves only the parameters of the original system and possibly the zeros of the polynomial $\mu(x)$, which are connected with the existence of Puiseux series near finite points that solve equation (1.2) and have certain properties [10]. While if there exists a family of Puiseux series near the point $x = \infty$ solving equation (1.2) such that these series possess arbitrary coefficients resulting from the presence of a rational non-negative Fuchs index, then it is unknown in advance how many times this family should be taken in representation (2.1) of the polynomial $F(x,y)$ producing irreducible invariant algebraic curve $F(x,y) = 0$. Let us consider one of such families with an arbitrary coefficient c_m where $m \in \mathbb{N}_0$. The coefficient c_m is arbitrary in the sense that it is not provided by equation (1.2). Suppose that representation (2.1) involves this family of series M times with $M \in \mathbb{N}$. The coefficients $c_m^{(1)}, \ldots, c_m^{(M)}$ will entre the algebraic system. The problem is to find not only $c_m^{(1)}, \ldots, c_m^{(M)}$, but also the number M. Note that the coefficients $c_m^{(1)}, \ldots, c_m^{(M)}$ should be pairwise distinct whenever the resulting invariant algebraic curve is irreducible. Due to the invariance of the polynomial $F(x,y)$ with respect to permutations of the Puiseux series $y_1(x), \ldots, y_N(x)$ and the structure of recurrence relations satisfied by coefficients of a Puiseux series solving an algebraic first-order ordinary differential equation, we conclude that the polynomial $F(x,y)$ inherits the invariance with respect to the permutations of $c_m^{(1)}, \ldots, c_m^{(M)}$. Consequently, the algebraic system with the exception for some degenerate cases can be rewritten in terms of invariants

$$C_k = \sum_{j=1}^{M} \left(c_m^{(j)}\right)^k. \tag{2.14}$$

The same result follows from Theorem 2.3. In relation (2.14) we should set $k \in \mathbb{N}$ whenever the family of Puiseux series under consideration corresponds to an edge of the Newton polygon related to equation (1.2). While $k \in \mathbb{Z}$ provided that the family of Puiseux series in question corresponds to a vertex of the Newton polygon. Thus, we conclude that the variables M and $\{C_k\}$ should be added to the list of variables. Further, one needs to study the structure of the polynomial ideal generated by the algebraic system in the ring of polynomials in the variables including the parameters of the original system, possible zeroes of the polynomial $\mu(x)$, $\{C_k\}$, and M. Solutions with $M \in \mathbb{N}$ should be selected. If several families of Puiseux series near the point $x = \infty$ that have arbitrary coefficients take part in representation (2.1), then the variables $\{C_k\}$ and M should be introduced for each family of series.

It was proved in article [10] that there exists at most one irreducible invariant algebraic curve $F(x,y) = 0$ of differential system (1.1) such that a Puiseux series near the point $x = \infty$ that solves equation (1.2) and possesses uniquely determined coefficients enters the representation of the polynomial $F(x,y)$ in the field $\mathbb{C}_\infty\{x\}$. Consequently, the most difficult problem is finding irreducible invariant algebraic curves given by representation (2.1) with all the Puiseux series possessing coefficients not provided by equation (1.2).

The following theorem is very important for practical solving the algebraic system in the latter case.

Theorem 2.5. *Let us consider the algebraic system of equations*

$$\sum_{j=1}^{M} (a_j)^k = M g_k, \quad k \in \mathbb{N}, \tag{2.15}$$

where $a_1, \ldots, a_M \in \mathbb{C}$ and $M \in \mathbb{N}$ are unknown variables, $\{g_k\}$ are given complex numbers. If for some $M_0 \in \mathbb{N}$ this system has a solution $(a_1, \ldots, a_{M_0})$ with $a_{j_1} \neq a_{j_2}$ whenever $j_1 \neq j_2$, then there are *no other solutions of this system except for $M = lM_0$, where $l \in \mathbb{N} \setminus \{1\}$. The latter solutions involve l multiple roots for each element of the tuple $(a_1, \ldots, a_{M_0})$. Note that tuples obtained from each other by permutations of their elements are supposed to be equivalent. We consider only one representative from each equivalence class.*

Proof. It is straightforward to verify that there exist "multiple" solutions for any solution with pairwise distinct elements of the tuple $(a_1, \ldots, a_{M_0})$. Let us establish that there are no other solutions. The proof is by contradiction. Suppose that system (2.15) possesses a solution $(\tilde{a}_1, \ldots, \tilde{a}_{M_1})$ with $M = M_1$, where either $M_1 \neq lM_0$ or $M_1 = lM_0$ and the tuple $(\tilde{a}_1, \ldots, \tilde{a}_{M_1})$ does not coincide with that described in the statement of the theorem. We recall that the left-hand side of relations (2.15) represent power-sum symmetric polynomials in the ring $\mathbb{C}[a_1, \ldots, a_M]$:

$$p_k = \sum_{j=1}^{M} \left(a_j\right)^k. \tag{2.16}$$

Let us introduce the elementary symmetric polynomials

$$e_k = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq M} a_{j_1} a_{j_2} \cdots a_{j_k}, \tag{2.17}$$

which are uniquely expressible via power-sum symmetric polynomials. Further, we consider the following algebraic equation of degree $M_2 = M_0 M_1$

$$a^{M_2} - e_1\left(a_1, \ldots, a_{M_2}\right) a^{M_2 - 1} + e_2\left(a_1, \ldots, a_{M_2}\right) a^{M_2 - 2} + \cdots + (-1)^{M_2} e_{M_2}\left(a_1, \ldots, a_{M_2}\right) = 0 \tag{2.18}$$

It is straightforward to show that this equation possesses two distinct sets of solutions: M_1 multiple roots for each element of the tuple $(a_1, \ldots, a_{M_0})$ and M_0 multiple roots for each element of the tuple $(\tilde{a}_1, \ldots, \tilde{a}_{M_1})$. The set of solutions of a polynomial equation in one variable over the field $\mathbb{C}$ is unique up to the permutation of the roots. This contradiction completes the proof. □

If all the Puiseux series in representation (2.1) possess arbitrary coefficients, then the elements C_k given in (2.14) are of the form $C_k = Mg_k$. It follows from the fact that $F^l(x, y) = 0$ with $l \in \mathbb{N}$ is an invariant algebraic curve whenever so does $F(x, y) = 0$. Consequently, Theorem 2.5 can be used for establishing uniqueness of irreducible invariant algebraic curves. Indeed, as soon as a solution $(a_1, \ldots, a_{M_0})$ with $a_{j_1} \neq a_{j_2}$ and $M_0 \in \mathbb{N}$ is found one should stop calculations because other solutions will give reducible invariant algebraic curves. Examples will be given in Section 4.

3 Invariant algebraic curves for Liénard differential systems

Now our aim is to apply the general results of the previous section to polynomial Liénard differential systems (1.4). Supposing that the variable y is dependent and the variable x is independent, we see that the function $y(x)$ satisfies the following first-order ordinary differential equation

$$y y_x + f(x) y + g(x) = 0. \tag{3.1}$$

Let us begin with simple properties of invariant algebraic curves and their cofactors.

Lemma 3.1. *Suppose $F(x,y) = 0$ with $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}$ is an invariant algebraic curve of a Liénard differential system. The following statements are valid.*

1. *There are no invariant algebraic curves such that $F(x,y) \in \mathbb{C}[x]$.*
2. *The highest-order coefficient with respect to y of the polynomial $F(x,y)$ is a constant.*
3. *The cofactors of invariant algebraic curves are independent of y.*

Proof. Substituting $\lambda(x,y) = \lambda_0(x)y^l$, $F(x,y) = \mu(x)y^N$ with l, $N \in \mathbb{N} \cup \{0\}$ into the partial differential equation

$$yF_x - \{f(x)y + g(x)\}F_y = \lambda(x,y)F. \tag{3.2}$$

and balancing the highest-order terms with respect to y, we conclude that $\mu(x) \in \mathbb{C}$, $l = 0$, and $N \in \mathbb{N}$. This means that cofactors of invariant algebraic curves do not depend on y and there are no invariant algebraic curves independent of y. In addition, we observe that the highest-order coefficient (with respect to y) of $F(x,y)$ is a constant. Without loss of generality we set $\mu(x) = 1$. This result can be also obtained using the structure of Puiseux series near finite points that satisfy equation (3.1), for more details see [10]. □

Our next step is to establish that the necessary and sufficient conditions of Theorem 2.3 and Lemma 2.4 become very easy in the case of Liénard differential systems.

Theorem 3.2. *The polynomial $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}$ of degree $N \in \mathbb{N}$ with respect to y gives an invariant algebraic curve of a Liénard differential system if and only if there exist N Puiseux series $y_1(x), \ldots, y_N(x)$ defined in a neighborhood of the point $x = \infty$ that solve equation (3.1) and satisfy the conditions*

$$\left\{ \sum_{j=1}^{N} y_j(x) \right\}_{-} = 0. \tag{3.3}$$

Proof. It follows from Lemma 3.1 that Liénard differential systems do not have invariant algebraic curves with generating polynomials independent of y. Let us suppose that $F(x,y) = 0$ is an invariant algebraic curve of a system (1.4) such that $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}[x]$.

We shall use the results of Lemma 2.4. Let us show that if conditions (2.13) are satisfied at $k = 1$, then they are also satisfied for all $k \in \mathbb{N}$. Our proof is by induction on k. Suppose that conditions (2.13) with $k \leq m$ hold. The Puiseux series appearing in these conditions solve equation (3.1). Substituting $y(x) = y_j(x)$ into equation (3.1) and multiplying the result by y_j^{m-1}, we get

$$\frac{1}{m+1}\frac{d}{dx}\left(y_j^{m+1}\right) = -f(x)y_j^m - g(x)y_j^{m-1}. \tag{3.4}$$

Performing the summation, we obtain

$$\frac{1}{m+1}\frac{d}{dx}\left(\sum_{j=1}^{N} y_j^{m+1}\right) = -f(x)\sum_{j=1}^{N} y_j^m - g(x)\sum_{j=1}^{N} y_j^{m-1}. \tag{3.5}$$

It follows from the induction hypothesis that the right-hand side in (3.5) is a polynomial. This yields

$$\frac{1}{m+1}\left\{\frac{d}{dx}\left(\sum_{j=1}^{N} y_j^{m+1}\right)\right\}_{-} = 0 \tag{3.6}$$

It is straightforward to see that for any element $y(x)$ of the field $\mathbb{C}_\infty\{x\}$ the following relation $\{y(x)\}_- = 0$ is valid whenever $\{y_x(x)\}_- = 0$. Consequently, we get

$$\left\{\sum_{j=1}^{N} y_j^{m+1}\right\}_- = 0. \tag{3.7}$$

Finally, the necessity and sufficiency of condition (3.3) follows from the results of Lemma 2.4 and the calculations carried out above. □

In Section 4 we shall use this lemma to perform the classification of irreducible invariant cases for Liénard differential systems with $n = 1$ ($\deg f(x) = 1$) and $m = 5$ ($\deg g(x) = 5$).

Now let us present the general structure of invariant algebraic curves and their cofactors for Liénard systems satisfying the condition $n \geq 2m + 1$. Recall that systems (1.4) with $n < 2m + 1$ were considered in articles [5,10]. We begin with the case $n > 2m + 1$.

Theorem 3.3. *Let $F(x, y) = 0$ with $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ be an irreducible invariant algebraic curve of a Liénard differential system from the family (1.4) with $n > 2m + 1$. Then $F(x, y)$ and its cofactor take the form*

$$F(x,y) = \left\{\prod_{j=1}^{N_1}\left\{y - y_j^{(1)}(x)\right\}\prod_{j=1}^{N_2}\left\{y - y_j^{(2)}(x)\right\}\right\}_+, \tag{3.8}$$

$$\lambda(x,y) = -(N_1 + N_2)f - \left\{N_1 h_x^{(1)} + N_2 h_x^{(2)}\right\}_+, \tag{3.9}$$

where the Puiseux series $y_j^{(1,2)}(x)$ are given by the relations

$$y_j^{(1,2)}(x) = h^{(1,2)}(x) + \sum_{k=2(n+1)}^{\infty} c_{k,j}^{(1,2)} x^{\frac{n+1}{2} - \frac{k}{2}}, \quad h^{(1,2)}(x) = \sum_{k=0}^{2n+1} c_k^{(1,2)} x^{\frac{n+1}{2} - \frac{k}{2}} \tag{3.10}$$

and N_1, $N_2 \in \mathbb{N} \cup \{0\}$, $N_1 + N_2 \geq 1$. The coefficients $c_{2(n+1),j}^{(1,2)}$ with the same upper index are pairwise distinct and all the coefficients $c_{m,j}^{(1,2)}$ with $m > 2(n+1)$ are expressible via $c_{2(n+1),j}^{(1,2)}$. If n is an odd number, then the corresponding Puiseux series are Laurent series and $c_{2l-1}^{(1,2)} = 0$, $c_{2l-1,j}^{(1,2)} = 0$ with $l \in \mathbb{N}$. In addition, $N_k = 1$ whenever n is odd and $N_l = 0$, where k, $l = 1$, 2 and $k \neq l$. If n is an even number, then $N_1 = N_2$.

Proof. It follows from Lemma 3.1 that we can set $\mu(x) = 1$. By Theorem 2.2 Puiseux series from the field $\mathbb{C}_\infty\{x\}$ that arise in representation (2.1) are those satisfying equation (3.1). Let us perform the classification of Puiseux series near the point $x = \infty$ solving equation (3.1) with the restriction $n > 2m + 1$. For this aim we shall use the algorithm presented in the Appendix. There exists only one dominant balance that produce Puiseux series in a neighborhood of the point $x = \infty$. The ordinary differential equation related to this balance and its solutions are the following

$$yy_x + g_0 x^n = 0, \quad y^{(1,2)}(x) = c_0^{(1,2)} x^{\frac{n+1}{2}}, \quad c_0^{(1,2)} = \pm\frac{\sqrt{-2(n+1)g_0}}{(n+1)}. \tag{3.11}$$

Calculating the Gâteaux derivative of the balance at its power solutions yields the Fuchs index: $p = n + 1$. Definitions of dominant balances and Fuchs indices can be found in [1,2,5,19], see also Appendix. Thus, we conclude that the Puiseux series corresponding to asymptotics (3.11) exist and have arbitrary coefficients at $x^{-(n+1)/2}$ provided that the compatibility conditions related to the unique Fuchs are satisfied. If n is an odd number, then Puiseux series (3.10) are Laurent series.

Finding the factorization of $F(x,y)$ in the ring $\mathbb{C}_\infty\{x\}[y]$ and taking the polynomial part of this representation, we obtain (3.8). Since the polynomial $F(x,y)$ in (3.8) is irreducible, we get the condition of the theorem on the coefficients $c^{(1,2)}_{2(n+1),j}$ with the same upper index.

Now let us suppose that n is an odd number and $N_2 = 0$. Our aim is to show that $N_1 = 1$. All the Puiseux series near the point $x = \infty$ arising in expression (3.8) are Laurent series with the same initial part of the series. Further, we introduce the new variable z by the rule

$$z = y - \sum_{l=0}^{\frac{n+1}{2}} c^{(1)}_{2l} x^{\frac{n+1}{2}-l}. \tag{3.12}$$

Calculating the projection in expression (3.8) yields

$$\left\{ \prod_{j=1}^{N_1} \left(z - c^{(1)}_{n+3} x^{-1} - \ldots - c^{(1)}_{2(n+1),j} x^{-\frac{n+1}{2}} - \ldots \right) \right\}_+ = z^{N_1}. \tag{3.13}$$

Requiring that the resulting invariant algebraic curve be given by an irreducible polynomial, we get $N_1 = 1$. The same can be done if $N_1 = 0$ and n is odd.

Substituting $L = 0$ and series (3.10) into expression (2.1), we find the cofactor as given in (3.9). Finally, if n is even, we calculate the coefficient at $y^{N_1+N_2-1}x^{(n+1)/2}$. The result is $(N_1 - N_2)c^{(1)}_0$. Since $y^{N_1+N_2-1}x^{(n+1)/2}$ is not an element of the ring $\mathbb{C}[x,y]$ and $c^{(1)}_0 \neq 0$, we get $N_1 = N_2$. The proof is completed. □

Let us turn to Liénard differential systems satisfying the condition $n = 2m+1$. We shall see that the Fuchs indices of the dominant balances near the point $x = \infty$ for equation (3.1) depend on the parameters f_0 and g_0. It was proved in article [10] and in Theorem 3.3 that such a situation cannot take place for other Liénard differential systems. This fact makes classification of irreducible invariant algebraic curves sufficiently difficult in the case $n = 2m+1$. The method of Puiseux series can deal with each case of a fixed positive rational Fuchs index separately.

We shall demonstrate that the structure of polynomials producing invariant algebraic curves is in strong correlation with the properties of the following quadratic equation

$$p^2 - \varrho p + (m+1)\varrho = 0, \tag{3.14}$$

where we have introduced notation

$$\varrho = 4(m+1) - \frac{f_0^2}{g_0}. \tag{3.15}$$

The set of all positive rational numbers will be denoted as $\mathbb{Q}^+$. Let p_1 and p_2 be the roots of equation (3.14).

Theorem 3.4. *Suppose $F(x,y) = 0$ with $F(x,y) \in \mathbb{C}[x,y] \setminus \mathbb{C}$ is an irreducible invariant algebraic curve of a Liénard differential system from family (1.4) with $n = 2m+1$. One of the following statements holds.*

1. *If p_1, $p_2 \notin \mathbb{Q}^+ \cup \{0\}$, then the polynomial $F(x,y)$ is of degree at most two with respect to y and*

$$\begin{aligned} F(x,y) &= \left\{\left\{y - y^{(1)}(x)\right\}^{s_1}\left\{y - y^{(2)}(x)\right\}^{s_2}\right\}_+, \\ \lambda(x,y) &= -(s_1+s_2)f(x) - \left\{s_1 y_x^{(1)} + s_2 y_x^{(2)}\right\}_+, \\ y^{(k)}(x) &= \sum_{l=0}^{\infty} c_l^{(k)} x^{m+1-l}, \quad c_0^{(k)} = \frac{f_0}{p_k - 2(m+1)}, \quad k = 1,2, \end{aligned} \tag{3.16}$$

where s_1 and s_2 are either 0 or 1 independently, $s_1 + s_2 > 0$. The Puiseux series $y^{(k)}(x)$, $k = 1$, 2 are Laurent series and possess uniquely determined coefficients.

2. *If $p_k \in \mathbb{Q}^+$, $p_q \notin \mathbb{Q}^+$, where either $k = 1$, $q = 2$ or $k = 2$, $q = 1$, then the polynomial $F(x,y)$ and the cofactor $\lambda(x,y)$ take the form*

$$\begin{aligned} F(x,y) &= \left\{\prod_{j=1}^{N_k}\left\{y - y_j^{(k)}(x)\right\}\left\{y - y^{(q)}(x)\right\}^{s_q}\right\}_+, \\ \lambda(x,y) &= -(N_k+s_q)f(x) - \left\{\sum_{j=1}^{N_k} y_{j,x}^{(k)} + s_q y_x^{(q)}\right\}_+, \\ y_j^{(k)}(x) &= \sum_{l=0}^{\infty} c_{l,j}^{(k)} x^{m+1-\frac{l}{n_k}}, \quad y^{(q)}(x) = \sum_{l=0}^{\infty} c_l^{(q)} x^{m+1-l}, \\ c_{0,j}^{(k)} &= \frac{f_0}{p_k - 2(m+1)}, \quad c_0^{(q)} = \frac{f_0}{p_q - 2(m+1)}, \end{aligned} \tag{3.17}$$

where $N_k \in \mathbb{N} \cup \{0\}$, s_q is either 0 or 1, $N_k + s_q > 0$. The Puiseux series $y^{(q)}(x)$ is a Laurent series and possesses uniquely determined coefficients. The Puiseux series $y_j^{(k)}(x)$ have pairwise distinct coefficients $c_{n_k p_k, j}^{(k)}$. The number n_k is defined as $p_k = l_k/n_k$, where l_k and n_k are coprime natural numbers.

3. *If p_1, $p_2 \in \mathbb{Q}^+$, then the polynomial $F(x,y)$ and the cofactor $\lambda(x,y)$ take the form*

$$\begin{aligned} F(x,y) &= \left\{\prod_{j=1}^{N_1}\left\{y - y_j^{(1)}(x)\right\}\prod_{j=1}^{N_2}\left\{y - y_j^{(2)}(x)\right\}\right\}_+, \\ \lambda(x,y) &= -(N_1+N_2)f(x) - \left\{\sum_{j=1}^{N_1} y_{j,x}^{(1)} + \sum_{j=1}^{N_2} y_{j,x}^{(2)}\right\}_+, \\ y_j^{(k)}(x) &= \sum_{l=0}^{\infty} c_{l,j}^{(k)} x^{m+1-\frac{l}{n_k}}, \quad c_{0,j}^{(k)} = \frac{f_0}{p_k - 2(m+1)}, \quad k = 1,2, \end{aligned} \tag{3.18}$$

where N_1, $N_2 \in \mathbb{N} \cup \{0\}$, $N_1 + N_2 > 0$. The Puiseux series $y_j^{(k)}(x)$ possess pairwise distinct coefficients $c_{n_k p_k, j}^{(k)}$. The number n_k is defined as $p_k = l_k/n_k$, where l_k and n_k are coprime natural numbers, $k = 1, 2$.

4. *If $p_1 = p_2 = 0$, then the polynomial $F(x,y)$ and the cofactor $\lambda(x,y)$ take the form*

$$\begin{aligned} F(x,y) &= y + \frac{f_0}{2(m+1)} x^{m+1} - \sum_{l=1}^{m+1} c_l x^{m+1-l}, \\ \lambda(x,y) &= -f(x) + \frac{f_0}{2} x^m - \sum_{l=1}^{m} (m+1-l) c_l x^{m-l}, \end{aligned} \tag{3.19}$$

where the coefficients $c_1, \ldots, c_{m+1}$ *are uniquely determined. In addition, the following relation* $4(m+1)g_0 - f_0^2 = 0$ *is valid.*

There are no other irreducible invariant algebraic curves than those described above.

Proof. Again we use Theorem 2.2 and Lemma 3.1. Let us find Puiseux series near the point $x = \infty$ that satisfy equation (3.1) with the restriction $n = 2m+1$. There exists only one dominant balance producing power asymptotics near the point $x = \infty$. The ordinary differential equation related to this balance and its power solutions are of the form

$$yy_x + f_0 x^m y + g_0 x^{2m+1} = 0 : \quad y^{(k)}(x) = c_0^{(k)} x^{m+1}, \quad k = 1, 2, \tag{3.20}$$

where the coefficients $c_0^{(1,2)}$ satisfy the following equation $(m+1)c_0^2 + f_0 c_0 + g_0 = 0$. Calculating the Gâteaux derivative of the balance at its power solutions yields the following equation for the Fuchs indices p: $(2(m+1) - p)c_0 + f_0 = 0$. Expressing c_0 from this equation and substituting the result into the equation $(m+1)c_0^2 + f_0 c_0 + g_0 = 0$, we get relation (3.14). Starting from power asymptotics we can derive asymptotic series possessing these asymptotics as leading-order terms. We are interested in Puiseux asymptotic series.

If equation (3.14) does not have positive rational solutions, then both Puiseux series related to asymptotics (3.20) possess uniquely determined coefficients. Since the number of distinct Puiseux series near the point $x = \infty$ satisfying equation (3.1) is finite and equals 2, it follows from Theorem 2.2 that the degree with respect to y of the polynomial $F(x, y)$ is bounded by 2. Constructing the factorization of the polynomial $F(x, y)$ in the ring $\mathbb{C}_\infty\{x\}[y]$ yields representation (3.16).

Further, if one of the solutions of equation (3.14) defining the Fuchs indices is a positive rational number and another one is not, then the Puiseux series related to the former case possesses an arbitrary coefficient provided that the compatibility condition for this Fuchs index is satisfied. Another Puiseux series possesses uniquely determined coefficients. As a result we obtain relation (3.17). Since the polynomial giving the invariant algebraic curve under consideration is irreducible, the coefficients $c^{(k)}_{n_k p_k, j}$ corresponding to the positive rational Fuchs index should be pairwise distinct. The number n_k can be obtained from the relation $p_k = l_k / n_k$, where l_k and n_k are coprime natural numbers. For more details see the Appendix.

If both solutions of equation (3.14) are positive rational numbers, then the Puiseux series have arbitrary coefficients and exist whenever the corresponding compatibility conditions for the Fuchs indices hold. We get expression (3.18). Since polynomials generating the invariant algebraic curves in question are irreducible, we conclude that the coefficients with the same upper index $c^{(k)}_{n_k p_k, j}$, $k = 1, 2$ should be pairwise distinct. The numbers n_k, $k = 1, 2$ are found similarly to the previous case.

Finally, we need to examine the situation, when two roots of the equation $(m+1)c_0^2 + f_0 c_0 + g_0 = 0$ merge. This gives $4(m+1)g_0 - f_0^2 = 0$ and $c_0 = -f_0/(2\{m+1\})$. Substituting this relation into the equation $(2(m+1) - p)c_0 + f_0 = 0$ for the Fuchs index yields $p = 0$. Consequently, we obtain the Puiseux series with integer exponents and uniquely determined coefficients. This gives the unique irreducible invariant algebraic curve as given in (3.19).

The cofactors $\lambda(x, y)$ we find from expression (2.1). Since we have considered all possible combinations of the Puiseux series from the field $\mathbb{C}_\infty\{x\}$ that solve equation (3.1), we conclude that other irreducible invariant algebraic curves cannot exist. □

Proving the above theorem, we have also established that if the compatibility condition for the Puiseux series $y_j^{(k)}(x)$ to exist is not satisfied and $p_k \in \mathbb{N}$ in the case of representation (3.17), then the irreducible invariant algebraic curve, if exists, is given by the polynomial $F(x,y) = y - c_0^{(q)} x^{m+1} - c_1^{(q)} x^m - \ldots - c_{m+1}^{(q)}$. If a similar situation occurs for representation (3.18), then either $N_1 = 0$ or $N_2 = 0$ and the product in expression (3.18) involving the corresponding series is absent. Moreover, if $p_1, p_2 \in \mathbb{N}$ and the compatibility conditions for both Puiseux series are not satisfied, then there are no invariant algebraic curves.

Let us note that invariant algebraic curves of Theorems 3.3 and 3.4 exist under certain restrictions on the parameters of the original differential systems.

4 Examples

The most interesting families of Liénard differential systems (1.4) satisfying the condition $n \geq 2m+1$ are those with the smallest degrees of the polynomial $g(x)$. They include cubic, quartic, and quintic systems with a constant or linear damping function. In addition, a quadratic damping function is allowed if $g(x)$ is a fifth degree polynomial. According to Theorem 3.4 the cases $\deg f(x) = 1$, $\deg g(x) = 3$ and $\deg f(x) = 2$, $\deg g(x) = 5$ are degenerate. Partial results were obtained in articles [12,13].

We have studied other Liénard differential systems from those listed above. All of them with the exception for the family satisfying the conditions $\deg f(x) = 1$ and $\deg g(x) = 5$ have irreducible invariant algebraic curves given by bivariate polynomials of degrees at most 2 with respect to y. While in the case $\deg f(x) = 1$ and $\deg g(x) = 5$ there exist irreducible invariant algebraic curves of higher degrees. The aim of the present section is to perform a classification of irreducible invariant algebraic curves of Liénard differential systems (1.4) satisfying the conditions $\deg f(x) = 1$ and $\deg g(x) = 5$. We shall prove that algebraic curves of degree 3 with respect to y arise. It is sometimes supposed that Liénard systems satisfying under the restriction $\deg g \neq 2 \deg f + 1$ ($n \neq 2m+1$) do not have such invariant algebraic curves.

Liénard differential systems with $\deg f(x) = 1$ and $\deg g(x) = 5$ are of the form

$$x_t = y, \quad y_t = -(\alpha x + \beta) y - (\varepsilon x^5 + r x^4 + \nu x^3 + e x^2 + \sigma x + \delta), \quad \alpha\varepsilon \neq 0. \tag{4.1}$$

A change of variables $x \mapsto X(x + x_0)$, $y \mapsto Yy$, $T \mapsto Tt$, $XYT \neq 0$ relates systems (4.1) with their simplified version at $\alpha = 5$, $\varepsilon = -3$, $r = 0$. Thus, without loss of generality, we obtain the systems

$$x_t = y, \quad y_t = -(5x + \beta) y + (3x^5 - \nu x^3 - e x^2 - \sigma x - \delta) \tag{4.2}$$

where all the parameters are from the field $\mathbb{C}$.

Theorem 4.1. *Differential systems (4.2) admit invariant algebraic curves if and only if restrictions on the parameters given below are satisfied. Generating polynomials of irreducible algebraic invariants and their cofactors are of the form:*

invariant algebraic curves of the first degree with respect to y

1. $e = \sigma + \frac{1}{8}\nu - \frac{15}{16} + \frac{15}{8}\beta + \frac{1}{16}\nu^2 - \frac{1}{8}\beta\nu - \frac{3}{16}\beta^2,$

$$\delta = \frac{1}{192}(3\beta - \nu + 3)\left(\nu^2 - 6\nu - 2\beta\nu + 6\beta - 3\beta^2 + 9 + 16\sigma\right),$$

$$F(x,y) = y - x^3 + x^2 + \frac{1}{4}(\beta + \nu - 3)x + \frac{1}{3}\sigma + \frac{1}{48}(\beta + \nu - 3)(-3\beta + \nu - 3),$$

$$\lambda(x,y) = -3x^2 - 3x + \frac{1}{4}(\nu - 3\beta - 3);$$

2. $e=\frac{15}{16}+\frac{15}{8}\beta-\sigma-\frac{1}{8}\nu-\frac{1}{16}\nu^2-\frac{1}{8}\beta\nu+\frac{3}{16}\beta^2,$
$\delta=\frac{1}{192}(3\beta+\nu-3)\left(\nu^2-6\nu+2\beta\nu-6\beta-3\beta^2+9+16\sigma\right),$
$F(x,y)=y+x^3+x^2+\frac{1}{4}(\beta-\nu+3)x+\frac{1}{3}\sigma+\frac{1}{48}(\beta-\nu+3)(3-3\beta-\nu),$
$\lambda(x,y)=3x^2-3x+\frac{1}{4}(3-\nu-3\beta);$

invariant algebraic curves of the second degree with respect to y

3. $e=0,\quad \delta=0,\quad \sigma=\frac{1}{12}(9-\nu^2),\quad \beta=0,$
$F(x,y)=y^2+\left(2x^2+1-\frac{1}{3}\nu\right)y-x^6+\frac{1}{2}(\nu-1)x^4-\frac{1}{12}(\nu+1)(\nu-3)x^2$
$+\frac{1}{216}(\nu+3)(\nu-3)^2,\quad \lambda(x,y)=-6x;$

4. $e=\frac{3}{1024}\beta\left(512-5\beta^2\right),\ \delta=-\frac{3}{262144}\beta^3\left(\beta^2+1280\right),$
$\sigma=\frac{3}{65536}\beta^2\left(2816+15\beta^2\right),\ \nu=\frac{15}{128}\beta^2+3,$
$F(x,y)=y^2+\left(2x^2+\frac{1}{2}\beta x-\frac{5}{128}\beta^2\right)y-x^6+\left(\frac{15}{256}\beta^2+1\right)x^4-\frac{1}{512}\beta\left(5\beta^2-256\right)x^3$
$+\frac{3}{65536}\beta^2\left(512+15\beta^2\right)x^2-\frac{1}{131072}\beta^3\left(1280+3\beta^2\right)x+\frac{5}{16777216}\beta^4\left(\beta^2+1280\right),$
$\lambda(x,y)=-6x-\frac{3}{2}\beta;$

invariant algebraic curves of the third degree with respect to y

5. $e=\frac{28511847}{62500},\ \delta=-\frac{94714508889}{19531250},\ \sigma=-\frac{8628822111}{1562500},\ \nu=\frac{133188}{625},\ \beta=\frac{91}{5},$
$F(x,y)=y^3+\left(x^3+3x^2-\frac{24297}{625}x-\frac{15500849}{62500}\right)y^2+\left(2x^5-x^6+\frac{73219}{625}x^4+\frac{4316949}{31250}x^3\right.$
$\left.-\frac{11403548611}{1562500}x^2-\frac{7670383903}{19531250}x+\frac{109912617846031}{976562500}\right)y-x^9-x^8+\frac{96266}{625}x^7+\frac{36191047}{62500}x^6$
$-\frac{17544478133}{1562500}x^5-\frac{812450830009}{19531250}x^4+\frac{138358719104879}{390625000}x^3+\frac{131625246607012067}{97656250000}x^2$
$-\frac{925725907851168424}{152587890625}x-\frac{3563835411131462914069}{61035156250000},\quad \lambda(x,y)=3x^2-9x-\frac{58422}{625};$

6. $e=-\frac{28511847}{62500},\ \delta=\frac{94714508889}{19531250},\ \sigma=-\frac{8628822111}{1562500},\ \nu=\frac{133188}{625},\ \beta=-\frac{91}{5},$
$F(x,y)=y^3+\left(3x^2-x^3+\frac{24297}{625}x-\frac{15500849}{62500}\right)y^2-\left(x^6+2x^5-\frac{73219}{625}x^4+\frac{4316949}{31250}x^3\right.$
$\left.+\frac{11403548611}{1562500}x^2-\frac{7670383903}{19531250}x-\frac{109912617846031}{976562500}\right)y+x^9-x^8-\frac{96266}{625}x^7+\frac{36191047}{62500}x^6$
$+\frac{17544478133}{1562500}x^5-\frac{812450830009}{19531250}x^4-\frac{138358719104879}{390625000}x^3+\frac{131625246607012067}{97656250000}x^2$
$+\frac{925725907851168424}{152587890625}x-\frac{3563835411131462914069}{61035156250000},\quad \lambda(x,y)=-3x^2-9x+\frac{58422}{625}.$

Proof. The structure of the polynomials $F(x,y)$ producing irreducible invariant algebraic curves has been presented in Theorem 3.3. The Puiseux series of Theorem 3.3 take the following form

$$\begin{aligned} y^{(1)}(x) &= x^3 - x^2 + \frac{1}{4}(3-\beta-\nu)x + \frac{1}{6}(\nu+3\beta-3-2e) + \sum_{l=4}^{\infty} c_l^{(1)} x^{3-l}; \\ y^{(2)}(x) &= -x^3 - x^2 + \frac{1}{4}(\nu-\beta-3)x + \frac{1}{6}(\nu-3\beta-3+2e) + \sum_{l=4}^{\infty} c_l^{(2)} x^{3-l}. \end{aligned} \tag{4.3}$$

These Puiseux series have arbitrary coefficients $c_6^{(1,2)}$ and exist whenever the following conditions are satisfied

$$\begin{aligned} y^{(1)}(x): \quad \delta &= \frac{3}{160}\beta^3 + \left(\frac{1}{80}\nu - \frac{15}{16}\right)\beta^2 + \left(\frac{123}{32} - \frac{21}{80}\nu - \frac{1}{160}\nu^2\right)\beta \\ &\quad + \left(2 - \frac{1}{10}\beta\right)\sigma + \left(\frac{7}{20}\beta - \frac{7}{4} - \frac{1}{12}\nu\right)e + \frac{1}{6}(\nu-3)(\nu+3); \\ y^{(2)}(x): \quad \delta &= \frac{3}{160}\beta^3 + \left(\frac{15}{16} - \frac{1}{80}\nu\right)\beta^2 + \left(\frac{123}{32} - \frac{21}{80}\nu - \frac{1}{160}\nu^2\right)\beta \\ &\quad - \left(2 + \frac{1}{10}\beta\right)\sigma - \left(\frac{7}{4} + \frac{1}{12}\nu + \frac{7}{20}\beta\right)e - \frac{1}{6}(\nu-3)(\nu+3) \end{aligned} \tag{4.4}$$

Further, we suppose that the series $y^{(1)}(x)$ enters the factorization of the polynomial $F(x,y)$ N_1 times with pairwise distinct values of $c_{6,j}^{(1)}$. Analogously, we suppose that the series $y^{(2)}(x)$ enters the factorization of the polynomial $F(x,y)$ N_2 times with pairwise distinct values of $c_{6,j}^{(2)}$. If $N_2 = 0$, then it follows from Theorem 3.3 that $N_1 = 1$. The resulting irreducible invariant algebraic curve exists whenever the series $y^{(1)}(x)$ terminates at the zero term. This gives the restriction

$$e = \sigma - \frac{3}{16}\beta^2 + \frac{1}{8}(15-\nu)\beta + \frac{1}{16}(\nu+5)(\nu-3). \tag{4.5}$$

Further, we do the same for the case $N_1 = 0$. In such a way we construct irreducible invariant algebraic curves of the first degree with respect to y.

Now let us suppose that $N_1 > 0$ and $N_2 > 0$. We introduce the following variables

$$C_k^{(1)} = \sum_{j=1}^{N_1} \left(c_{6,j}^{(1)}\right)^k; \quad C_k^{(2)} = \sum_{j=1}^{N_1} \left(c_{6,j}^{(2)}\right)^k. \tag{4.6}$$

According to the results of Theorem 3.2 we need to consider the algebraic system

$$\sum_{j=1}^{N_1} c_{l,j}^{(1)} + \sum_{j=1}^{N_2} c_{l,j}^{(2)} = 0, \quad l \geq 4. \tag{4.7}$$

We take the first eleven equations from this system. In addition, the compatibility conditions for both series to exist should be considered. Solving the algebraic sub-system, we obtain three possibilities: $N_1 = N_2$, $N_1 = 2N_2$, and $N_2 = 2N_1$. If the first possibility takes place, then we find

$$C_1^{(1)} = \varrho_1 N_1, \quad C_2^{(1)} = \varrho_1^2 N_1, \qquad C_1^{(2)} = \varrho_2 N_2, \quad C_2^{(2)} = \varrho_2^2 N_2 \tag{4.8}$$

and restrictions on the parameters as given in items 3 and 4. There exist two families of irreducible invariant algebraic curves $F(x,y) = 0$ with $N_1 = 1$ and $N_2 = 1$. The irreducible invariant algebraic curves are presented in items 3 and 4. This fact proves the validity of the following conditions

$$C_k^{(1)} = \varrho_1^k N_1, \qquad C_k^{(2)} = \varrho_2^k N_2, \quad k \in \mathbb{N}. \tag{4.9}$$

According to Theorem 2.5, we see that the algebraic system in question has no other solutions.

If $N_1 = 2N_2$, then we get $C_1^{(2)} = \varrho_2 N_2$ and $C_2^{(2)} = \varrho_2^2 N_2$. Arguing as above, we find $N_2 = 1$. Analogously the case $N_2 = 2N_1$ can be studied. In expressions (4.8) and (4.9) the values ϱ_1 and ϱ_2 either are constants or depend on the parameters of the original differential systems, but not on N_1 and N_2.

The cofactors can be obtained with the help of expression (3.9). □

Note that it is a difficult computational problem to find invariant algebraic curves of items 5 and 6 using the method of undetermined coefficients, the method of extactic polynomial or an algorithm of decomposing the vector field related to the original system into weight-homogenous components. It seems that the classification of irreducible invariant algebraic curves for quintic Liénard differential systems with a linear damping function is presented here for the first time.

5 Conclusion

In this article we have derived necessary and sufficient conditions enabling a planar polynomial differential system (1.1) to have invariant algebraic curves. Our conditions give rise to an algorithm, which is able to perform a classification of irreducible invariant algebraic curves for a given differential system. The algorithm can be easily implemented with the help of computer systems of symbolic computations.

We have presented the general structure in the ring $\mathbb{C}_\infty\{x\}[y]$ for the bivariate polynomials generating irreducible invariant algebraic curves of Liénard differential systems (1.4) with $\deg g \geq 2 \deg f + 1$. Their cofactors have been calculated in an explicit form. Let us emphasize that the method of Puiseux series is also applicable in the case of systems (1.1) with the parameters affecting degrees of the polynomials $P(x, y)$ and $Q(x, y)$. Some examples are given in articles [9, 11]. In addition, the method enables one to find algebraic first-order ordinary differential equations compatible with a higher-order autonomous ordinary differential equation [8]. Moreover, the method of Puiseux series admits a non-autonomous generalization, for more details see [7]. We conclude that the method presented in works [5,6,10] and developed in this article is a powerful tool of finding invariants for ordinary differential equations and systems of ordinary differential equations.

Another way to derive an algebraic system similar to that presented in expression (2.7) is to require that the non-polynomial part in the expression of the cofactor $\lambda(x, y)$ in (2.1) vanishes. This algebraic system coincides with that arising from Theorems 2.3 and 3.2 in the case of Liénard differential systems. For other polynomial differential systems this approach may lead to finding generalized (non-polynomial in x) invariant curves possessing polynomial cofactors. It seems that this topic is also worth studying. Some results concerning non-algebraic invariant curves with polynomial cofactors were obtained in article [18].

6 Acknowledgments

The author would like to thank the reviewers for their valuable comments, which contributed to the improvement of the article. This research was supported by Russian Science Foundation grant 19–71–10003.

7 Appendix

Let us describe a method, which can be used to perform the classification of Puiseux series satisfying an algebraic first-order ordinary differential equation $E(x, y, y_x) = 0$. The left-hand side of this expression can be regarded as a sum of differential monomials given by

$$M[y(x), x] = Cx^l y^{j_0} \left\{\frac{dy}{dx}\right\}^{j_1}, \quad C \in \mathbb{C} \setminus \{0\}, \quad l, j_0, j_1 \in \mathbb{N}_0. \tag{7.1}$$

The set of all the differential monomials of the form (7.1) will be denoted as $\mathbb{M}$. In order to simplify notation the expression $W[x, y(x)]$ will stand for a polynomial in x, $y(x)$, and $y_x(x)$ with coefficients from the field $\mathbb{C}$.

Let us define the map $q : \mathbb{M} \to \mathbb{R}^2$ by the following rules

$$Cx^{q_1} y^{q_2} \mapsto q = (q_1, q_2), \qquad \frac{d^k y}{dx^k} \mapsto q = (-k, 1), \quad q(M_1 M_2) = q(M_1) + q(M_2),$$

where $C \in \mathbb{C} \setminus \{0\}$ is a constant, M_1 and M_2 are differential monomials. We denote the set of all points $q \in \mathbb{R}^2$ corresponding to the differential monomials of equation $E(x, y, y_x) = 0$ as $S(E)$. The convex hull of $S(E)$ is known as *the Newton polygon* of the equation under consideration.

The boundary of the Newton polygon consists of vertices and edges. Selecting all the differential monomials of the original equation that generate the vertices and the edges of the Newton polygon, we obtain a number of balances. The balance for a vertex is defined as the sum of those differential monomials in $E(x, y, y_x)$ that are mapped into the vertex. The balance for an edge is defined as the sum of differential monomials in $E(x, y, y_x)$ whose images belong to the edge. If solutions of the equation $E(x, y, y_x) = 0$ possess an asymptotics of the form $y(x) = c_0 x^r$ with $x \to 0$ or $x \to \infty$, then there exists a balance $W[x, y(x)]$ such that the function $y(x) = c_0 x^r$ satisfies the equation $W[x, y(x)] = 0$. Conversely, the function $y(x) = c_0 x^r$ solving equation $W[x, y(x)] = 0$, where $W[x, y(x)]$ is a balance, is an asymptotics at $x \to 0$ (or $x \to \infty$) for solutions of equation (1.2) whenever for all the differential monomials $M[x, y(x)]$ of the original equation not involved into $W[x, y(x)]$ we have $\operatorname{Re} \varkappa > \operatorname{Re} \varkappa_0$ (or $\operatorname{Re} \varkappa < \operatorname{Re} \varkappa_0$), where $M[x, c_0 x^r] = Bx^{\varkappa}$ and $M_0[x, c_0 x^r] = B_0 x^{\varkappa_0}$ with $M_0[x, y(x)]$ being a differential monomial of the balance $W[x, y(x)]$.

Thus, having found all the power solutions $y(x) = c_0 x^r$ for all the balances, one needs to select those that give asymptotics at $x \to 0$ or $x \to \infty$. Using power asymptotics it is possible to derive asymptotic series possessing these asymptotics as leading-order terms [1,2]. In this article we are interested in Puiseux series near $x = \infty$ that satisfy equation (1.2), therefore we shall focus at the case $r \in \mathbb{Q}$ and $x \to \infty$. Let us suppose that a balance $W[y(x), x]$ of the equation $E(x, y, y_x) = 0$ has a solution $y(x) = c_0 x^r$, which is an asymptotics at $x \to \infty$ and $r \in \mathbb{Q}$.

In order to obtain the structure of the corresponding series one should find the Gâteaux derivative of the balance $W[y(x), x]$ at the solution $y(x) = c_0x^r$:

$$\frac{\delta W}{\delta y}[c_0x^r] = \lim_{s\to 0} \frac{W[c_0x^r + sx^{r-p}, x] - W[c_0x^r, x]}{s} = V(p)x^{\tilde{r}}, \quad \tilde{r} \in \mathbb{Q}.$$

In this expression $V(p)$ is a first-degree polynomial with respect to p. The coefficients of this polynomial depend on c_0 and on the parameters (if any) of the original equation involved into the balance $W[y(x), x]$. The zero p_0 of $V(p)$ is called *the Fuchs index* (or *the resonance*) of the balance $W[y(x), x]$ and its power solution $y(x) = c_0x^r$. Let $\mathrm{lcm}(n, m)$ be the lowest common multiple of two integer numbers n and m. If the Fuchs index p_0 is not a positive rational number, then the number n_0 in expression (1.3) is given by $n_0 = r_2$ where r_2 is defined as $r = r_1/r_2$ with r_1 and r_2 being coprime numbers, $r_1 \in \mathbb{Z}$ and $r_2 \in \mathbb{N}$. Otherwise we obtain $n_0 = \mathrm{lcm}(g_2, r_2)$, where r_2 was defined previously and g_2 is given by $p_0 = g_1/g_2$ with coprime natural numbers g_1 and g_2.

Finally, it is important to verify the existence of the Puiseux series of the form (1.3) with $l_0 = rn_0$. If the balance $W[y(x), x]$ corresponds to a vertex of the Newton polygon, then the Puiseux series always exists and possesses an arbitrary coefficient c_0. In this case the Fuchs index is equal to zero. Now let us suppose that the balance $W[y(x), x]$ corresponds to an edge of the Newton polygon. Substituting series (1.3) into the equation $E(x, y, y_x) = 0$ one can find the recurrence relation for its coefficients. This relation takes the form

$$V\left(\frac{k}{n_0}\right) c_k = U_k(c_0, \ldots, c_{k-1}), \quad k \in \mathbb{N},$$

where U_k is a polynomial of its arguments. Note that U_k can also depend on the parameters (if any) of the original equation. The equation $U_{n_0p_0} = 0$ is called *the compatibility condition*. If the compatibility condition is not satisfied, then the Puiseux series under consideration does not exist. Otherwise the corresponding Puiseux series exists and possesses an arbitrary coefficient $c_{n_0p_0}$. Consequently, we conclude that the Puiseux series in question has uniquely determined coefficients provided that there are no non-negative rational Fuchs indices.

We note that if one wishes to find all the Puiseux series of the form (1.3) that satisfy the original equation, then it is necessary to implement the procedure described above for all the dominant balances and for all their power solutions $y(x) = c_0x^r$ with $r \in \mathbb{Q}$ and $x \to \infty$.

Asymptotic Puiseux series near the point $x_0 \in \mathbb{C}$ can be found introducing the change of variables $w(s) = y(s + x_0)$, $s = x - x_0$ and considering the case $s \to 0$ in the resulting ordinary differential equations.

We also observe that there may exist balances and their power solutions such that the following condition $V(p) \equiv 0$ is valid. If $V(p)$ is identically zero, then one should make the substitution $y(x) = c_0x^r + w(x)$ in equation $E(x, y, y_x) = 0$ and find all the Puiseux series $w(x) = c_1x^{r_1} + \ldots$ of the latter such that $r_1 < r$, $r_1 \in \mathbb{Q}$ and $x \to \infty$. More details and some generalizations can be found in the works by A. D. Bruno [1,2].

References

[1] A. D. Bruno, Asymptotic behaviour and expansions of solutions of an ordinary differential equation, *Russ. Math. Surv.* **59**(2004), No. 3, 429–481. https://doi.org/10.1070/RM2004v059n03ABEH000736; MR2116535; Zbl 1068.34054

[2] A. D. Bruno, *Power geometry in algebraic and differential equations*, Elsevier Science (North-Holland), 2000. MR1773512

[3] J. Chavarriga, H. Giacomini, M. Grau, Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems, *Bull. Sci. Math.* **129**(2005), No. 1, 99–126. https://doi.org/10.1016/j.bulsci.2004.09.002; MR2123262; Zbl 1091.34017

[4] C. Christopher, J. Llibre, J. V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, *Pacific J. Math.* **229**(2007), No. 1, 63–117. https://doi.org/10.2140/pjm.2007.229.63; MR2276503; Zbl 1160.34003

[5] M. V. Demina, Novel algebraic aspects of Liouvillian integrability for two-dimensional polynomial dynamical systems, *Phys. Lett. A* **382**(2018), No. 20, 1353–1360. https://doi.org/10.1016/j.physleta.2018.03.037; MR3782570; Zbl 1398.34050

[6] M. V. Demina, Invariant algebraic curves for Liénard dynamical systems revisited, *Appl. Math. Lett.* **84**(2018), 42–48. https://doi.org/10.1016/j.aml.2018.04.013; MR3808495; Zbl 1391.37041

[7] M. V. Demina, Invariant surfaces and Darboux integrability for non-autonomous dynamical systems in the plane, *J. Phys. A* **51**(2018), 505202. https://doi.org/10.1088/1751-8121/aaecca; MR3895572; Zbl 1411.70025

[8] M. V. Demina, Classifying algebraic invariants and algebraically invariant solutions, *Chaos Solitons Fractals* **140**(2020), 110219. https://doi.org/10.1016/j.chaos.2020.110219; MR4138303

[9] M. V. Demina, Liouvillian integrability of the generalized Duffing oscillators, *Anal. Math. Phys.* **11**(2021), No. 1, Paper No. 25. https://doi.org/10.1007/s13324-020-00459-z; MR4195118; Zbl 07301487

[10] M. V. Demina, The method of Puiseux series and invariant algebraic curves, *Commun. Contemp. Math.* (2021), 2150007. https://doi.org/10.1142/S0219199721500073

[11] M. V. Demina, N. S. Kuznetsov, Liouvillian integrability and the Poincaré problem for nonlinear oscillators with quadratic damping and polynomial forces, *J. Dyn. Control Syst.* **27**(2021), No. 2, 403–415. https://doi.org/10.1007/s10883-020-09513-2; MR4231678; Zbl 07329777

[12] M. V. Demina, D. I. Sinelshchikov, Integrability properties of cubic Liénard oscillators with linear damping, *Symmetry* **11**(2019), 1378. https://doi.org/10.3390/sym11111378

[13] M. V. Demina, D. I. Sinelshchikov, Darboux first integrals and linearizability of quadratic–quintic Duffing–van der Pol oscillators, *J. Geom. Phys.* **165**(2021), 104215. https://doi.org/10.1016/j.geomphys.2021.104215; MR4236454

[14] M. V. Demina, C. Valls, On the Poincaré problem and Liouvillian integrability of quadratic Liénard differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **150**(2020), No. 6, 3231–3251 . https://doi.org/10.1017/prm.2019.63; MR4190110; Zbl 07316379

[15] A. Ferragut, H. Giacomini, A new algorithm for finding rational first integrals of polynomial vector fields, *Qual. Theory Dyn. Syst.* **9**(2010), 89–99. https://doi.org/10.1007/s12346-010-0021-x; MR2737363; Zbl 1216.34001

[16] I. A. García, H. Giacomini, J. Giné, Generalized nonlinear superposition principles for polynomial planar vector fields, *J. Lie Theory* **15**(2005), No. 1, 89–104. MR2115230; Zbl 1073.34003

[17] H. Giacomini, J. Giné, M. Grau, The role of algebraic solutions in planar polynomial differential systems, *Math. Proc. Cambridge Philos. Soc.* **143**(2007), No. 2, 487–508. https://doi.org/10.1017/S0305004107000497; MR2364665; Zbl 1136.34036

[18] I. A. García, J. Giné, Non–algebraic invariant curves for polynomial planar vector fields, *Discrete Contin. Dyn. Syst.* **10**(2004), No. 3, 755–768. https://doi.org/10.3934/dcds.2004.10.755; MR2018878; Zbl 1050.34033

[19] A. Goriely, *Integrability and nonintegrability of dynamical systems*, World Scientific, 2001. https://doi.org/10.1142/9789812811943; MR1857742

[20] Yu. Ilyashenko, S. Yakovenko, *Lectures on analytic differential equations*, Graduate Studies in Mathematics, Vol. 86, American Mathematical Society, 2008. https://doi.org/10.1090/gsm/086; MR2363178

[21] M. N. Lagutinski, On some polynomials and their application for algebraic integration of ordinary differential algebraic equations (in Russian), *Communications of the Kharkov Mathematical Society. The second series* **13**(1912), 200–224.

[22] J. Llibre, C. Valls, Liouvillian first integrals for generalized Liénard polynomial differential systems, *Adv. Nonlinear Stud.* **13**(2013), 819–829. https://doi.org/10.1515/ans-2013-0404; MR3115140; Zbl 1369.34003

[23] K. Odani, The limit cycle of the van der Pol equation is not algebraic, *J. Differential Equations* **115**(1995), No. 1, 146–152. https://doi.org/10.1006/jdeq.1995.1008; MR1308609; Zbl 0816.34023

[24] R. J. Walker, *Algebraic curves*, Springer-Verlag, New York, 1978. MR513824

[25] H. Żołądek, Algebraic invariant curves for the Liénard equation, *Trans. Amer. Math. Soc.* **350**(1998), No. 4, 1681–1701. https://doi.org/10.1090/S0002-9947-98-02002-9; MR1433130; Zbl 0895.34026