PRODUCTION FUNCTIONS HAVING THE CES PROPERTY

LÁSZLÓ LOSONCZI

Abstract. To what measure does the CES (constant elasticity of substitution) property determine production functions? We show that it is not possible to find explicitely all two variable production functions $f(x, y)$ having the CES property. This slightly generalizes the result of R. Sato [16]. We show that if a production function is a quasi-sum then the CES property determines only the functional forms of the inner functions, the outer functions being arbitrary (satisfying some regularity properties). If in addition to CES property homogeneity (of some degree) is required then the (two-variable) production function is either CD or ACMS production function. This generalizes the result of [4] and also makes their proof more transparent (in the special case of degree 1 homogeneity).

1. Introduction

In economics, a production function is a function that specifies the maximal possible output of a firm, an industry, or an entire economy for all combinations of inputs. In general, a production function can be given as $y = f(x_1, x_2, \dots, x_n)$ where y is the quantity of output, $x_1, x_2, \dots, x_n$ are the production factor inputs (such as capital, labour, land or raw materials). We do not allow joint production, i.e. productions process, which has multiple co-products or outputs. Of course both the inputs and output should be positive. Concerning the history of production functions see the working paper of S. K. Mishra [14]. Several aspects of production functions are dealt with in the monograph of R. W. Shephard [17].

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of reals and positive reals respectively.

Definition 1. A function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is called a **production function**.

In the sequel we assume that production functions are twice continuously differentiable. The elasticity of substitution was originally introduced by J. R. Hicks (1932) [10] (in case of two inputs) for the purpose of analyzing changes in the income shares of labor and capital. R. G. D. Allen and J. R. Hicks (1934) [3] suggested two generalizations of Hicks' original two variable elasticity concept. The first concept which we call Hicks' elasticity of substitution is defined as follows.

Date: July 4, 2009.

2000 *Mathematics Subject Classification.* Primary .

Key words and phrases. production function, elasticity.

This research has been supported by the Hungarian Scientific Research Fund (OKTA) Grant NK-68040.

Definition 2. Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be a production function with non-vanishing first partial derivatives. The function

$$H_{ij}(\mathbf{x}) = -\frac{\frac{1}{x_i f_i} + \frac{1}{x_j f_j}}{\frac{f_{ii}}{(f_i)^2} - \frac{2f_{ij}}{f_i f_j} + \frac{f_{jj}}{(f_j)^2}} \quad (\mathbf{x} \in \mathbb{R}_+^n,\ i, j = 1, \dots, n,\ i \neq j) \tag{1}$$

(where the subscripts of f denote partial derivatives i.e. $f_i = \dfrac{\partial f}{\partial x_i}, f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$, all partial derivatives are taken at the point $\mathbf{x}$ and the denominator is assumed to be different from zero) is called the **Hicks' elasticity of substitution of the ith production variable (factor) with respect to the jth production variable (factor)**.

The other concept (thoroughly investigated by R. G. D. Allen [2], and H. Uzawa [20] is more complicated.

Definition 3. Let $f : \mathbb{R}_+^n \to \mathbb{R}_+$ be a production function. The function

$$A_{ij}(\mathbf{x}) = -\frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{x_i x_j} \frac{F_{ij}}{F} \quad (\mathbf{x} \in \mathbb{R}_+^n,\ i, j = 1, \dots, n,\ i \neq j) \tag{2}$$

where F is the determinant of the bordered matrix

$$M = \begin{pmatrix} 0 & f_1 & \dots & f_n \\ f_1 & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \dots & \vdots \\ f_n & f_{n1} & \dots & f_{nn} \end{pmatrix} \tag{3}$$

and F_{ij} is the co-factor of the element f_{ij} in the determinant F ($F \neq 0$ is assumed and all derivatives are taken at the point $\mathbf{x}$) is called the **Allen's elasticity of substitution of the ith production variable (factor) with respect to the jth production variable (factor)**.

It is a simple calculation to show that in case of two variables Hicks' elasticity of substitution coincides with Allen's elasticity of substitution.

Definition 4. A twice differentiable production function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is said to satisfy the **CES (constant elasticity of substitution)-property** if there is a constant $\sigma \in \mathbb{R}, \sigma \neq 0$ such that

$$H_{ij}(\mathbf{x}) = \sigma \qquad (\mathbf{x} \in \mathbb{R}_+^n,\ i, j = 1, \dots, n,\ i \neq j). \tag{4}$$

In the sequel we discuss that to what measure does the CES property (4) determine the production function.

2. Cobb-Douglas and Arrow-Chenery-Minhas-Solow type production functions

C. W. Cobb and P. H. Douglas [6] studied how the distribution of the national income can be described by help of production functions. The outcome of their study was the production function

$$f(\mathbf{x}) = Cx_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n} \quad (\mathbf{x} \in \mathbb{R}_+^n)$$

where $C > 0, \alpha_i \neq 0\,(i = 1, \dots, n)$ are constants satisfying $\alpha := \sum_{i=1}^{n} \alpha_i \neq 0$. We call this **Cobb-Douglas (or CD) production function.**

In 1961 K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [4] introduced a new production function

$$f(\mathbf{x}) = (\beta_1 x_1^{\frac{m}{\beta}} + \dots + \beta_n x_n^{\frac{m}{\beta}})^{\beta} \quad (\mathbf{x} \in \mathbb{R}_+^n)$$

where $\beta_i > 0\ (i = 1, \dots, n), m \neq 0, \beta \neq 0$ are real constants. We shall refer to this function as **Arrow-Chenery-Minhas-Solow (or ACMS) production function.**

The CD and ACMS production functions have the CES property, namely as it is easy to check $H_{ij}(\mathbf{x}) = 1$ for the CD functions and $H_{ij}(\mathbf{x}) = \dfrac{1}{1 - \frac{m}{\beta}}$ for the ACMS production functions if $\frac{m}{\beta} \neq 1$, for $\frac{m}{\beta} = 1$ the denominator of $H_{i,j}$ is zero, hence it is not defined.

3. Homogeneous, sub- and superhomogeneous functions

Definition 5. A function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is called is said to be *homogeneous of degree* $m \in \mathbb{R}$ if

$$F(t\mathbf{x}) = t^m F(\mathbf{x})$$

holds for all $\mathbf{x} \in \mathbb{R}_+^n, t > 0$.

Definition 6. A function $F : \mathbb{R}_+^n \to \mathbb{R}_+$ is called is said to be *subhomogeneous of degree* $m \in \mathbb{R}$ if

$$F(t\mathbf{x}) \leq t^m F(\mathbf{x})$$

holds for all $\mathbf{x} \in \mathbb{R}_+^n$ and for all $t > 1$. The function F is called *superhomogeneous of degree* $m \in \mathbb{R}$ if the reverse inequality holds.

Homogeneous (sub and superhomogeneous) functions of degree 1 will simply be called homogeneous (sub and superhomogeneous) functions.

If F is a production function, then in economy also the terms *constant return to scale, decreasing and increasing return to scale* are used to designate homogeneous, subhomogeneous and superhomogeneous (production) functions respectively.

It is well-known that differentiable homogeneous functions F of degree m can be characterized by *Euler's PDE*

$$x_1F_{x_1}(\mathbf{x})+\cdots+x_nF_{x_n}(\mathbf{x})=mF(\mathbf{x})\quad(\mathbf{x}\in\mathbb{R}_+^n).$$

It is not so much known, that similar characterizations hold for sub- and superhomogeneous function (compare with L. Losonczi [11]).

Theorem 7. *Suppose that $F:\mathbb{R}_+^n\to\mathbb{R}_+$ is a differentiable function on its domain. F is subhomogeneous of degree m, i.e.*

$$F(t\mathbf{x})\le t^mF(\mathbf{x}) \tag{5}$$

holds for all $\mathbf{x}\in\mathbb{R}_+^n$ and for all $t>1$ if and only if

$$x_1F_{x_1}(\mathbf{x})+\cdots+x_nF_{x_n}(\mathbf{x})\le mF(\mathbf{x})\quad(\mathbf{x}\in\mathbb{R}_+^n) \tag{6}$$

F is superhomogeneous of degree m, i.e. the reverse inequality of (5) *holds if and only if the reverse of* (6) *is satisfied.*
If strict inequality holds in (6) *or in its reverse then also* (5) *or its reverse is satisfied with strict inequality.*

Remark 1. (5) *(or its reverse) holds for $\mathbf{x}\in\mathbb{R}_+^n, t\in]0,1[$ if and only if the reverse of* (6) *(or* (6)*) is satisfied.*

Proof. We prove the statement only for subhomogeneous functions, the superhomogeneous case is analogous.
Necessity. Deducting F from (5), dividing by $t-1>0$ and taking the limit $t\to1+0$ we obtain (6).

Sufficiency. Replace in (6) $\mathbf{x}$ by $t\mathbf{x}$ and rearrange it as

$$\frac{tx_1F_{x_1}(t\mathbf{x})+\cdots+tx_nF_{x_n}(t\mathbf{x})}{F(t\mathbf{x})}\le m$$

where $t>1$. This equation can be rewritten as

$$t\frac{d}{dt}(\ln F(t\mathbf{x}))\le m,\quad\text{or}\quad\frac{d}{dt}(\ln F(t\mathbf{x}))\le\frac{m}{t}.$$

Integrating the latter inequality from $t=1$ to $t>1$ and omitting the ln sign we obtain (5), completing the proof of sufficiency.
The statement concerning strict inequalities is obvious. □

4. The most general two variable CES function

Suppose that $f:\mathbb{R}_+^2\to\mathbb{R}_+$ is a two variable CES production function, then

$$-\frac{\frac{1}{x_1f_1}+\frac{1}{x_2f_2}}{\frac{f_{11}}{(f_1)^2}-\frac{2f_{12}}{f_1f_2}+\frac{f_{22}}{(f_2)^2}}=\sigma\quad(x_1,x_2\in\mathbb{R}_+) \tag{7}$$

where $\sigma \in \mathbb{R}, \sigma \neq 0$ is a constant. (7) is partial differential equation (PDE) of second order which can be reduced to two first order equations. We shall find the general solution of the first equation. We partially follow R. Sato [16] who found the solution of a special Cauchy problem for the said equation. The left hand side of (7) can be written as

$$-\frac{\frac{1}{x_1 f_1}+\frac{1}{x_2 f_2}}{\frac{f_{11}}{(f_1)^2}-\frac{2f_{12}}{f_1 f_2}+\frac{f_{22}}{(f_2)^2}}=\frac{x_1 f_1+x_2 f_2}{x_1 x_2\left(-\frac{f_{11}f_2}{f_1}+2f_{12}-\frac{f_{22}f_1}{f_2}\right)}=\frac{x_1+x_2 u}{x_1 x_2\left(\frac{\partial u}{\partial x_1}-\frac{1}{u}\frac{\partial u}{\partial x_2}\right)}$$

where

$$u(x_1,x_2):=\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} \quad (x_1,x_2 \in \mathbb{R}_+).$$

By (7) the new unknown function u satisfies the first order PDE

$$\frac{\partial u}{\partial x_1}-\frac{1}{u}\frac{\partial u}{\partial x_2}=\frac{u}{\sigma x_1}+\frac{1}{\sigma x_2}.$$

This PDE is simplified if we introduce the function $v=\ln u$ provided that $u(x_1,x_2)>0$ (otherwise, if $u(x_1,x_2)<0$, we use the substitution $v=\ln(-u)$). Restricting ourselves to the first case, the transformed equation reads

$$e^v\frac{\partial v}{\partial x_1}-\frac{\partial v}{\partial x_2}=\frac{e^v}{\sigma x_1}+\frac{1}{\sigma x_2},$$

or

$$e^v\frac{\partial}{\partial x_1}\left(v-\ln x_1^{\frac{1}{\sigma}}\right)=\frac{\partial}{\partial x_2}\left(v+\ln x_2^{\frac{1}{\sigma}}\right).$$

This equation is further simplified if we use the new unknown function

$$w(x_1,x_2):=v(x_1,x_2)-\ln x_1^{\frac{1}{\sigma}}+\ln x_2^{\frac{1}{\sigma}}.$$

Then

$$e^v=e^w\left(\frac{x_1}{x_2}\right)^{\frac{1}{\sigma}},\quad \frac{\partial}{\partial x_1}\left(v-\ln x_1^{\frac{1}{\sigma}}\right)=\frac{\partial w}{\partial x_1},\quad \frac{\partial}{\partial x_2}\left(v+\ln x_2^{\frac{1}{\sigma}}\right)=\frac{\partial w}{\partial x_2}$$

hence

$$e^w\left(\frac{x_1}{x_2}\right)^{\frac{1}{\sigma}}\frac{\partial w}{\partial x_1}-\frac{\partial w}{\partial x_2}=0. \tag{8}$$

(8) is a first order homogeneous quasi-linear partial differential equation in two variables. Taking its general solution in implicit form $\Phi(x_1,x_2,w)=0$ it is known (see [19], pp. 279-283) that for Φ the linear homogeneous PDE

$$e^w\left(\frac{x_1}{x_2}\right)^{\frac{1}{\sigma}}\frac{\partial \Phi}{\partial x_1}-\frac{\partial \Phi}{\partial x_2}+0\frac{\partial \Phi}{\partial w}=0$$

holds. Its characteristic system is

$$\frac{dx_1}{e^w\left(\frac{x_1}{x_2}\right)^{\frac{1}{\sigma}}}=\frac{dx_2}{-1}=\frac{dw}{0}$$

or

$$\frac{dw}{dx_2}=0,\quad \frac{dx_1}{dx_2}=-e^w\left(\frac{x_1}{x_2}\right)^{\frac{1}{\sigma}}.$$

First we find two independent first integrals of this system of ordinary differential equations. From the first equation we get $w=C_0$ (C_0 is an arbitrary constant) then with $e^{C_0}=C_1>0$ separating the variables in the second equation we obtain

$$\frac{dx_1}{x_1^{\frac{1}{\sigma}}}=-C_1\frac{dx_2}{x_2^{\frac{1}{\sigma}}}.$$

Integrating we get

$$\begin{aligned} \ln x_1 &= -C_1\ln x_2+C_2 && \text{if } \sigma=1\\ x_1^{1-\frac{1}{\sigma}} &= -C_1x_2^{1-\frac{1}{\sigma}}+C_2 && \text{if } \sigma\neq 1. \end{aligned} \tag{9}$$

The first integrals are the solutions for C_1,C_2 of the system consisting of (9) and $e^w=C_1$. These are $C_1=e^w, C_2=\ln x_1+e^w\ln x_2$ if $\sigma=1$ and $C_1=e^w, C_2=x_1^{1-\frac{1}{\sigma}}+e^wx_2^{1-\frac{1}{\sigma}}$ if $\sigma\neq 1$. Finally the general solution of (8)

$$\Phi(e^w,\ln x_1+e^w\ln x_2)=0,\qquad \text{if } \sigma=1,$$

$$\Phi(e^w,x_1^{1-\frac{1}{\sigma}}+e^wx_2^{1-\frac{1}{\sigma}})=0,\qquad \text{if } \sigma\neq 1,$$

where Φ is an arbitrary differentiable function. Going back to the original variables we obtain

$$\begin{aligned} \Phi\left(\frac{f_1}{f_2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}},\ \ln x_1+\frac{f_1}{f_2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}}\ln x_2\right) &= 0, && \text{if } \sigma=1\\ \Phi\left(\frac{f_1}{f_2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}},\ x_1^{1-\frac{1}{\sigma}}+\frac{f_1}{f_2}\left(\frac{x_2}{x_1}\right)^{\frac{1}{\sigma}}x_2^{1-\frac{1}{\sigma}}\right) &= 0, && \text{if } \sigma\neq 1 \end{aligned} \tag{10}$$

The next step in finding the production function f would be to solve (10) for the ratio $\frac{f_1}{f_2}$ as a function of x_1,x_2 i.e. find a function G such that $\frac{f_1}{f_2}=G(x_1,x_2)$. Then solving the second linear PDE

$$\frac{\partial f}{\partial x_1}-G(x_1,x_2)\frac{\partial f}{\partial x_2}=0$$

we obtain the the most general CES functions f.

Unfortunately we cannot find all solutions $\frac{f_1}{f_2}$ from (10), as this ratio appears in both variables of Φ.. We can however find several families of Φ for which the solution can be found.

For CES functions of more than two variables the situation is even more complicated.

5. Quasi-sum form CES production functions

Definition 8. A function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is called a **quasi-sum**, if there exist continuous strict monotone functions $g_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, \dots, n)$ and there exist an interval $I \subseteq \mathbb{R}$ of positive length and a continuous strict monotone function $g : I \to \mathbb{R}_+$ such that for every $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ we have $g_1(x_1) + \dots + g_n(x_n) \in I$ and

$$f(\mathbf{x}) = g\left(g_1(x_1) + \dots + g_n(x_n)\right). \tag{11}$$

The justification for studying production functions of quasi-sum form is that these functions appear as solutions of the general bisymmetry equation and they are related to the problem of consistent aggregation, see J. Aczél and Gy. Maksa [1], Gy. Maksa [13].

Our first observation is that *if a production function is of quasi-sum form* (11) *then its Hicks' elasticity of substitution* of the ith production variable with respect to the jth production variable *does not depend on the function* g.

Write $h(\mathbf{x}) = g_1(x_1) + \dots + g_n(x_n)$ then

$$f(\mathbf{x}) = g\left(h(\mathbf{x})\right) = g\left(g_1(x_1) + \dots + g_n(x_n)\right) \qquad (x \in \mathbb{R}_+^n).$$

A simple calculation shows that

$$\begin{aligned} f_{x_i}(\mathbf{x}) &= g'\left(h(\mathbf{x})\right) g_i'(x_i) \\ f_{x_i x_i}(\mathbf{x}) &= g''\left(h(\mathbf{x})\right) \left(g_i'(x_i)\right)^2 + g'\left(h(\mathbf{x})\right) g_i''(x_i) \\ f_{x_i x_j}(\mathbf{x}) &= g''\left(h(\mathbf{x})\right) g_i'(x_i) g_j'(x_j) \end{aligned}$$

thus

$$H_{ij}(\mathbf{x}) = \frac{-\frac{1}{x_i g'(h) g_i'} - \frac{1}{x_j g'(h) g_j'}}{\frac{g''(h)(g_i')^2 + g'(h) g_i''}{(g'(h) g_i')^2} - \frac{2g''(h) g_i' g_j'}{(g'(h))^2 g_i' g_j'} + \frac{g''(h)(g_j')^2 + g'(h) g_j''}{(g'(h) g_j')^2}} = \frac{-\frac{1}{x_i g_i'} - \frac{1}{x_j g_j'}}{\frac{g_i''}{(g_i')^2} + \frac{g_j''}{(g_j')^2}} \tag{12}$$

where the derivatives of g_i $(i = 1, \dots, n)$ are taken at the point x_i and h is taken at $\mathbf{x}$. This proves our claim.

For quasi sums however the CES property determines the functional forms of the inner functions g_i.

Theorem 9. *Suppose that the production function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is of quasi-sum form* (11) *where the functions $g, g_i\,(i = 1, \dots, n)$ are twice differentiable and have non-vanishing first derivatives. If f satisfies the CES-property, then the functions $g_i\,(i = 1, \dots, n)$ have the following forms*

$$g_i(x) = \begin{cases} \dfrac{\sigma x^{1-\frac{1}{\sigma}}}{C_i(\sigma - 1)} + D_i, & \text{if } \sigma \neq 1, \\ \\ \dfrac{\ln x}{C_i} + D_i, & \text{if } \sigma = 1, \end{cases} \tag{13}$$

where C_i, D_i are arbitrary nonzero constants.
If $n = 2, \sigma \neq 1$ then, in addition to the functions (13)*, g_1, g_2 may have the form*

$$g_1(x) = \frac{\ln\left|\frac{\sigma d_1 x^{1-\frac{1}{\sigma}}}{\sigma-1} + C_1\right|}{d_1} + D_1,\ \ g_2(x) = \frac{\ln\left|\frac{-\sigma d_1 x^{1-\frac{1}{\sigma}}}{\sigma-1} + C_2\right|}{-d_1} + D_2, \tag{14}$$

where $d_1 \neq 0, D_1, D_2$ are arbitrary constants, C_1, C_2 are constants satisfying the conditions

$$\operatorname{sign} C_1 = \operatorname{sign}\frac{(\sigma - 1)}{\sigma d_1}, \quad \textit{and} \quad \operatorname{sign} C_2 = -\operatorname{sign}\frac{(\sigma - 1)}{\sigma d_1}. \tag{15}$$

Conversely, if g_i have the forms (13), (14) *(with* (15) *satisfied) then* (4) *holds.*

Proof. By the identity

$$\frac{g''(x)}{(g'(x))^2} = -\frac{d}{dx}\left(\frac{1}{g'(x)}\right)$$

we can rewrite (12) as

$$H_{ij}(\mathbf{x}) = \frac{-\left(\dfrac{1}{x_i}\dfrac{1}{g_i'} + \dfrac{1}{x_j}\dfrac{1}{g_j'}\right)}{\left(\dfrac{1}{g_i'}\right)' + \left(\dfrac{1}{g_j'}\right)'}.$$

This shows that the substitutions $k_i(x_i) := \dfrac{1}{g_i'(x_i)}$ will simplify our formulae. Indeed, by the help of k_i the equation (4) goes over into

$$\sigma k_i'(x_i) - \frac{1}{x_i}k_i(x_i) = -\left(\sigma k_j'(x_j) - \frac{1}{x_j}k_j(x_j)\right).$$

Here the right hand side depends only on x_j while the left hand side depends only on x_i, hence both sides must be constant (depending only the subscript i). Thus we conclude that

$$k_i'(x_i) - \frac{1}{\sigma x_i} k_i(x_i) = d_i \quad (i = 1, \ldots, n). \tag{16}$$

For the constants d_i we have $d_i + d_j = 0$ if $i, j \in \{1, \ldots, n\}, i \neq j$.

If $n = 2$ then we have only one equation: $d_1 + d_2 = 0$, hence $d_2 = -d_1$, with arbitrary $d_1 \in \mathbb{R}$.

If $n \geq 3$ then all d_i's must be zero, as $d_1 + d_2 = d_1 + d_3 = \cdots = d_1 + d_n = 0$, hence $d_2 = d_3 = \cdots = d_n = -d_1$. From $d_2 + d_3 = 0$ we get $d_1 = 0$, thus $d_2 = \cdots = d_n = 0$.

Thus we proved that (4) *holds if and only if*

$$g_i(x) = \int \frac{dx}{k_i(x)}, \quad (x \in \mathbb{R}_+, i = 1, \ldots, n)$$

where k_i *satisfy* (16), *with* $d_1 \in \mathbb{R}, d_2 = -d_1$, *if* $n = 2$, *and* $d_1 = \cdots = d_n = 0$, *if* $n \geq 3$.

It is a simple exercise to show that *the general solution of the linear inhomogeneous first order differential equation*

$$k'(x) - \frac{1}{\sigma x} k(x) = d \qquad (x \in I \subseteq \mathbb{R}_+)$$

is

$$k(x) = \begin{cases} \dfrac{\sigma d x}{\sigma - 1} + C x^{\frac{1}{\sigma}}, & \text{if } \sigma \neq 1, \\ \\ d x \ln x + C x, & \text{if } \sigma = 1, \end{cases}$$

where $C\in\mathbb{R}$ is an arbitrary constant. Further, for $d\neq 0$ using the substitutions $u=\dfrac{\sigma d x^{1-\frac{1}{\sigma}}}{\sigma-1}+C$ resp. $u=d\ln x+C$ in the integrations we have

$$\int\frac{dx}{k(x)}=\begin{cases}\dfrac{\sigma x^{1-\frac{1}{\sigma}}}{C(\sigma-1)}+D, & \text{if } d=0, C\neq 0, \sigma\neq 1,\\[2ex] \dfrac{\ln x}{C}+D, & \text{if } d=0, C\neq 0, \sigma=1,\\[2ex] \dfrac{\ln\left|\frac{\sigma d x^{1-\frac{1}{\sigma}}}{\sigma-1}+C\right|}{d}+D, & \text{if } d\neq 0, \sigma\neq 1,\\[2ex] \dfrac{\ln|d\ln x+C|}{d}+D, & \text{if } d\neq 0, \sigma=1,\end{cases}\tag{17}$$

where $D\in\mathbb{R}$ is an arbitrary constant.

If $n=2$ then, in agreement with the previous calculations, we get g_1, g_2 from (11) by putting into it $C=C_1, C_2$; $D=D_1, D_2$; $d=d_1, -d_1$ respectively. Thus, assuming $d_1\neq 0$ we obtain that

$$g_1(x)=\frac{\ln\left|\frac{\sigma d_1 x^{1-\frac{1}{\sigma}}}{\sigma-1}+C_1\right|}{d_1}+D_1,\quad g_2(x)=\frac{\ln\left|\frac{-\sigma d_1 x^{1-\frac{1}{\sigma}}}{\sigma-1}+C_2\right|}{-d_1}+D_2,\qquad \text{if } \sigma\neq 1,$$

$$g_1(x)=\frac{\ln|d_1\ln x+C_1|}{d_1}+D_1,\quad g_2(x)=\frac{\ln|-d_1\ln x+C_2|}{-d_1}+D_2,\qquad \text{if } \sigma=1.$$

These functions should be defined for all positive numbers. This requirement excludes the solutions g_1, g_2 for $\sigma=1$, as in this case the function $x\to d_1\ln x+C_1$ always has a positive zero $x_0=e^{-C_1/d_1}$ thus g_1 is not defined at x_0. For $\sigma\neq 1$ the situation is different. In this case g_1, g_2 are defined for all positive numbers if and only if the functions $x\to\frac{\sigma d_1 x^{1-\frac{1}{\sigma}}}{\sigma-1}+C_1$, $x\to\frac{-\sigma d_1 x^{1-\frac{1}{\sigma}}}{\sigma-1}+C_2$ do not have positive zeros, i.e if $-\frac{C_1(\sigma-1)}{\sigma d_1}<0$, and $\frac{C_2(\sigma-1)}{\sigma d_1}<0$, or if

$$\operatorname{sign}C_1=\operatorname{sign}\frac{(\sigma-1)}{\sigma d_1},\quad\text{and}\quad\operatorname{sign}C_2=-\operatorname{sign}\frac{(\sigma-1)}{\sigma d_1}$$

hold, which is exactly (8). □

6. Homogeneous CES production functions

Here we show that CES property and homogeneity (of some degree) explicitely determine the production functions, moreover they are either CD or ACMS production functions. This generalizes and somewhat clarifies analogous result of [4].

Theorem 10. *Suppose that $P : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a twice differentiable two-variable production function, homogenous of degree $m \neq 0$ and satisfying* (7)*. Then*

$$P(x,y) = \begin{cases} Cx^{\alpha}y^{m-\alpha}, & \text{if } \sigma = 1 \\ \\ \left(\beta_1 x^{\frac{m}{\beta}} + \beta_2 y^{\frac{m}{\beta}}\right)^{\beta}, & \text{if } \sigma \neq 1. \end{cases} \tag{18}$$

where $\alpha \neq 0$ is arbitrary nonzero constant such that $m - \alpha \neq 0$ holds, C, β_1, β_2 are arbitrary positive constants, $\beta = \dfrac{m\sigma}{\sigma - 1} \neq 0$ (and due to this $\dfrac{m}{\beta} \neq 1$).

Remark 2. *If P is homogeneous of degree zero then by the homogeneity equation $xP_x(x,y) + yP_y(x,y) = 0$. Hence $\frac{1}{xP_x} + \frac{1}{yP_y} = 0$ which makes the function H_{ij} indeterminate. Thus the assumption $m \neq 0$ in Theorem 10 is natural.*

Remark 3. (18) *shows that for $\sigma = 1$ the function P is a CD function while for $\sigma \neq 1$ our production function P is an ACMS function.*

Proof. For the sake of simplicity we shall denote the variables of P by x, y. Then (7) has the form

$$\sigma = -\frac{\frac{1}{xP_x(x,y)} + \frac{1}{yP_y(x,y)}}{\frac{P_{xx}(x,y)}{(P_x(x,y))^2} - \frac{2P_{xy}(x,y)}{P_x(x,y)P_y(x,y)} + \frac{P_{yy}(x,y)}{(P_y(x,y))^2}}. \tag{19}$$

As P is homogeneous of degree m it satisfies the partial differential equation

$$xP_x(x,y) + yP_y(x,y) = mP(x,y). \tag{20}$$

Differentiating (20) with respect to x we get

$$P_x + xP_{xx} + yP_{yx} = mP_x$$

where here and in the following P and its derivatives are taken at the point (x,y). Hence

$$P_{xx} = -\frac{y}{x}P_{yx} + \frac{m-1}{x}P_x \quad \text{and similarly} \quad P_{yy} = -\frac{x}{y}P_{xy} + \frac{m-1}{y}P_y$$

Substituting these into (19) we obtain that

$$\sigma = \frac{-\left(\frac{1}{xP_x} + \frac{1}{yP_y}\right)}{-xyP_{xy}\left(\frac{1}{xP_x} + \frac{1}{yP_y}\right)^2 + (m-1)\left(\frac{1}{xP_x} + \frac{1}{yP_y}\right)}.$$

Simplifying by the numerator we get that

$$xyP_{xy}\left(\frac{1}{xP_x}+\frac{1}{yP_y}\right)=m-1+\frac{1}{\sigma}.$$

Using again the homogeneity equation we have $\frac{1}{xP_x}+\frac{1}{yP_y}=\frac{mP}{xP_xyP_y}$ thus finally

$$\frac{PP_{xy}}{P_xP_y}=1-\frac{1}{m}+\frac{1}{\sigma m} \tag{21}$$

Case 1: $\sigma=1$. Now we can rewrite (21) in the form

$$\frac{PP_{xy}-P_xP_y}{P^2}=0, \quad \text{or} \quad (\ln P)_{xy}=0,$$

hence by integration we conclude that there exist differentiable functions g,h such that

$$\ln P(x,y)=g(x)+h(y), \quad P(x,y)=e^{g(x)+h(y)}.$$

Substituting P into the homogeneity equation (20) we obtain

$$xg'(x)e^{g(x)+h(y)}+yh'(y)e^{g(x)+h(y)}=me^{g(x)+h(y)},$$

or

$$xg'(x)=m-yh'(y).$$

Here the right hand side depends only on x, while the left one only on y, thus both sides must be a constant α and g,h have to satisfy the equations

$$g'(x)=\frac{\alpha}{x}, \qquad h'(y)=\frac{m-\alpha}{y}.$$

These equations imply that $\alpha\neq 0, m-\alpha\neq 0$ otherwise the partial derivatives P_x,P_y would be zero, making the function H_{ij} indeterminate.

Integrating we obtain $g(x)=\alpha\ln x+D_1, h(y)=(m-\alpha)\ln y+D_2$ where $D_1,D_2\in\mathbb{R}$ are arbitrary constants, and

$$P(x,y)=e^{g(x)+h(y)}=e^{\alpha\ln x+D_1+(m-\alpha)\ln y+D_2}=e^{D_1+D_2}x^\alpha y^{m-\alpha}=Cx^\alpha y^{m-\alpha}$$

where $C:=e^{D_1+D_2}$ is an arbitrary positive constant. This proves (18) in the case $\sigma=1$.

Case 2: $\sigma\neq 1$. Let H be defined by $P(x,y)=H(x,y)^\beta$, where β is a constant to be determined later. Substituting the derivatives

$$P_x=\beta H^{\beta-1}H_x,\ P_y=\beta H^{\beta-1}H_y,\ P_{xy}=\beta(\beta-1)H^{\beta-2}H_xH_y+\beta H^{\beta-1}H_{xy}$$

of P into (21) we get after some simplifications that

$$1-\frac{1}{\beta}+\frac{1}{\beta}\frac{HH_{xy}}{H_xH_y}=1-\frac{1}{m}+\frac{1}{\sigma m}. \tag{22}$$

Let $\beta=\dfrac{m\sigma}{\sigma-1}$ then $\beta\neq 0$ as $m\neq 0,\sigma\neq 0$ further $\dfrac{m}{\beta}\neq 1$ otherwise $1=\dfrac{\sigma}{\sigma-1}$ which is impossible. (22) simplifies to $H_{xy}(x,y)=0$. Thus there exist differentiable functions g,h such that

$$H(x,y)=g(x)+h(y),\quad \text{hence}\quad P(x,y)=(g(x)+h(y))^{\beta}.$$

substituting P into the homogeneity equation (20) we obtain after some simplifications that

$$\beta xg'(x)-mg(x)=mh(y)-\beta yh'(y).$$

Here, again, the right hand side depends only on x, while the left one only on y, thus both sides must be a constant α and g,h have to satisfy the equations

$$g'(x)-\frac{m}{\beta x}g(x)=\frac{\alpha}{\beta x},\qquad h'(y)-\frac{m}{\beta y}h(y)=-\frac{\alpha}{\beta y}.$$

The general solutions of these linear differential equations are

$$g(x)=\frac{-\alpha}{m}+\beta_1x^{\frac{m}{\beta}}\qquad h(y)=\frac{\alpha}{m}+\beta_2y^{\frac{m}{\beta}},$$

where $\beta_1,\beta_2\in\mathbb{R}$ are arbitrary constants, and

$$P(x,y)=(g(x)+h(y))^{\beta}=\left(\beta_1x^{\frac{m}{\beta}}+\beta_2y^{\frac{m}{\beta}}\right)^{\beta}.$$

Here β_1,β_2 must be positive, otherwise P would not be defined for all positive x,y. □

7. Closing remarks

For production functions of $n>2$ variables the approach in section 6 does not work, as the CES property involves partial derivatives with respect to two variables while Euler's PDE characterizing homogeneous functions involves all partial derivatives. There were several attempts to extend the two variable result to more variables, see e.g. D. McFadden [8], H. Uzawa [20]. CD and ACMS production functions (of several variables) have been characterized by the homogeneity (of some degree) and quasi-sum (or quasi-linear) form, see W. Eichorn [7], B. Nyul [12], F. Stehling [18]. The Hick's elasticity of substitution has been generalized into several directions, see among others R. Färe and L. Jansson [9], C. Blackorby and R. R. Russell [5], N. S. Revankar [15].

References

[1] J. Aczél, Gy. Maksa, *Solution of the rectangular $m \times n$ generalized bisymmetry equation and of the problem of consistent aggregation*, J. Math. Anal. Appl. Vol 203, (1996), 104-126.
[2] R. G. D. Allen, Mathematical Analysis for Economists, London, Macmillan, 1938.
[3] R. G. D. Allen, J. R. Hicks, *A Reconsideration of the Theory of Value, Pt. II*, Economica, February-May 1934, 1, N.S. 196-219.
[4] K. J. Arrow, H. B. Chenery, B. S. Minhas, R. M. Solow, *Capital-Labor Substitution and Economic Efficiency*, The Review of Economics and Statistics, Vol. 43, No. 3 (Aug., 1961), 225-250.
[5] C. Blackorby and R. R. Russell, *Will the Real Elasticity of Substitution Please Stand Up? (A comparison of the Allen/Uzawa and Morishima Elasticities)*, The American Economic Review, Vol. 79, No. 4 (1989), 882-888.
[6] C. W. Cobb, P. H. Douglas, A theory of production. The American Economic Reviewv Vol. 18 (1928), 139-165.
[7] W. Eichorn, *Characterization of CES production functions by quasilinearity*, in Production Theory (W. Eichorn, R. Henn, O. Opitz and R. W. Shephard eds.), Springer-Verlag, 1974, pp. 21-33.
[8] D. McFadden, *Constant Elasticity of Substitution Production Functions*, The Review of Economic Studies, Vol. 30, No. 2 (Jun., 1963), 73-83.
[9] R. Färe, L. Jansson, *On VES and WDI production functions*, Internat. Econ. Review, vol. 16, no. 3, (1975), 745-750.
[10] J. R. Hicks, Theory of Wages, London, Macmillan, 1932.
[11] L. Losonczi, *Subhomogene Mittelwerte*, Acta Math. Acad. Sci. Hung. Vol. 22, No. 1-2, (1973), 187-195.
[12] B. Nyul, *Production functions and their characterizations*, (in Hungarian), Alk. Mat. Lapok (to appear).
[13] Gy. Maksa, *Solution of the generalized bisymmetry type equations without surjectivity assumptions*, Aeq. Math. Vol. 57 (1999), 50-74.
[14] S. K. Mishra, *A Brief History of Production Functions*, Working Paper Series Social Science Research Network (SSRN), http://ssrn.com/abstract=1020577.
[15] N. S. Revankar, *A Class of Variable Elasticity of Substitution Production Functions*, Econometrica, Vol. 39, No. 1, (1971), 61-71.
[16] R. Sato, *The Most General Class of CES Functions*, Econometrica, Vol. 43, No. 5/6 (Sep. - Nov., 1975), 999-1003.
[17] R. W. Shephard, Cost and Production Functions, Princeton, New Jersey, Princeton University Press, 1970.
[18] F. Stehling, *Eine neue Charakterisierung der CD- und ACMS-Produktiosfunktionen*, Operations Research-Verfahren 21 (1975), 222-238.
[19] K. Sydsæter, P. Hammond, A. Seierstad, A. Strøm, Further Mathematics for Economic Analysis, Prentice Hall (2005).
[20] H. Uzawa, *Production Functions with Constant Elasticities of Substitution*, The Review of Economic Studies, Vol. 29, No. 4 (Oct., 1962), 291-299.

Faculty of Economics, University of Debrecen, H-4028 Debrecen, Kassai út 26, Hungary
E-mail address: losi@math.klte.hu