On estimation of generalized logarithmic series distribution*

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In this paper we have studied the estimation of generalized logarithmic series distribution (GLSD) by the method of weighted discrepancies between observed and expected frequencies. The maximum likelihood, minimum chi-square and the discrimination information methods are special cases of the weighted discrepancies method. A new weighted technique, the empirical weighted rates of change (EWRC) for estimating the GLSD parameters has been obtained. We have fitted the GLSD to several zero-truncated biological data by different methods and observed that in most of the cases the GLSD provided a better fit than the usual logarithmic series distribution (LSD) by using EWRC method.

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Probability distributions. Estimations.

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The generalized logarithmic series distribution (GLSD) characterized by two parameters α and β was defined by *Jain* and *Gupta* [1973]. The probability function of the GLSD model is given by

$$P(X=x)=\begin{cases}\dfrac{\theta\ \Gamma(\beta x)\alpha^{x}(1-\alpha)^{\beta x-x}}{x!\ \Gamma(\beta x-x+1)} & ;\ x=1,\ 2\ \ldots,\\ 0 & \text{otherwise}\end{cases} \qquad /1/$$

where $\beta \geq 1$, $0<\alpha<\beta^{-1}$ and $\theta=-\dfrac{1}{\log(1-\alpha)}$.

The GLSD model /1/ is also a limiting form of the zero-truncated generalized negative binomial distribution of *Jain* and *Consul* [1971]. *Patel* [1981] also defined GLSD and obtained the estimates of the parameters by the method of moments. The model /1/ reduces to the simple logarithmic series distribution (LSD) when $\beta=1$. *Patil* [1962] studied the estimation of LSD. The GLSD model is a member of *Gupta* [1974] modified power series distribution and also can be found in Lagrangian probability distributions of *Consul* and *Shenton* [1972]. *Famoye* [1987] showed that the GLSD is unimodal and the mode is at the point $x=1$. Some methods of sampling from GLSD /1/ are provided by *Famoye* [1997]. *Mishra* [1979], and *Mishra–Tiwary* [1985] showed that the GLSD provides a very close fit to the observations coming from various fields such as medicine, engineering etc. *Tripathi-Gupta* [1988] gave the another generalization of the logarithmic series and geometric distributions.

Jani [1977] obtained the minimum variance estimators, *Famoye* [1995] acquired the moment estimators, *Mishra* [1979] and *Jani–Shah* [1979] discussed the use of maximum likelihood and moment method of estimation for the two parameter GLSD /1/. *Mishra–Tiwary* [1985] suggested an alternative method of estimation based on the first three moments. *Mishra–Hassan* [1996], [1997] recommended a quick and simple method for the estimation and they also obtained the Bayesian estimate of GLSD.

In this paper we study the estimation of the parameters of GLSD /1/ using the maximum likelihood (ML), minimum chi-square (MC), weighted discrepancy (WD) and empirical weighted rate of change (EWRC) methods in the same line as has been performed by *Famoye–Lee* [1992] in case of generalized Poisson distribution (GPD).

1. Maximum likelihood method

Let a random sample of size N be taken from the GLSD /1/ and let the observed frequencies be f_x; $x = 1, 2 \ldots k$ so that $\sum_{x=1}^{k} f_x = N$, where k is the largest of the observed values having non–zero frequencies. The likelihood equation of the GLSD /1/ can be written as

$$L = \frac{\theta^N \alpha^{N\bar{x}} \prod_{\chi=1}^{K} \prod_{j=1}^{x-1} (\beta x - j)^{f_x} (1-\alpha)^{(\beta-1)N\bar{x}}}{\prod_{i=1}^{K} (X_i !)^{f_x}} . \qquad /2/$$

The log likelihood function is given as

$$\begin{aligned} \log L = N \log\theta + N \bar{x} \log\alpha + \sum_{x=1}^{k} \sum_{j=1}^{x-1} f_x \log(\beta x - j) + \\ + (\beta - 1) N \bar{x} \log(1-\alpha) - \sum f_i \log X_2 . \end{aligned} \qquad /3/$$

The two likelihood equations can be obtained as

$$\frac{\partial \log L}{\partial \alpha} = -\frac{N\theta}{(1-\alpha)} + \frac{N\bar{x}}{\alpha} - \frac{(\beta-1)N\bar{x}}{(1-\alpha)} = 0 , \qquad /4/$$

$$\frac{\partial \log \mathrm{L}}{\partial \beta} = \frac{-N\bar{x}}{\theta} + \sum_{x=1}^{k} \sum_{j=1}^{x-1} \frac{x f_x}{\beta x - j} = 0 , \qquad /5/$$

where $\bar{x}$ is the sample mean. From equation /4/, we get

$$\beta = \frac{1}{\alpha} - \frac{\theta}{\bar{x}} . \qquad /6/$$

Putting this in equation /5/, we get

$$\varphi(\alpha) = -\frac{N\bar{x}}{\theta} + \sum_{x=1}^{k} \sum_{j=1}^{x-1} \frac{x f_x}{\left(\frac{1}{\alpha} - \frac{\theta}{\bar{x}}\right) x - j} = 0 . \qquad /7/$$

The previous equation does not seem to be directly solvable and hence some iteration method can be used to solve it. For this we find second derivatives of log L as

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{N\overline{x}}{\alpha^2} - \frac{(\beta-1)\ N\overline{x}}{(1-\alpha)^2} - \frac{N\ \theta}{(1-\alpha)^2} = \frac{N\overline{x}}{\alpha^2} - \frac{N\left[\theta + (\beta-1)\overline{x}\right]}{\left(1-\alpha^2\right)}, \qquad /8/$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\sum_{x-1}^{k} \sum_{j=1}^{x-1} \frac{x^2 f_x}{(\beta x - j)^2}\ , \qquad /9/$$

$$\frac{\partial^2 \log L}{\partial \beta\ \partial \alpha} = -\frac{N\ \overline{x}}{1-\alpha}\ , \qquad /10/$$

$$\frac{\partial^2 \log L}{\partial \alpha\ \partial \beta} = -\frac{N\ \overline{x}}{1-\alpha}\ . \qquad /11/$$

The values of these second derivatives can be put in the following equation in the matrix form as

$$\begin{bmatrix} \dfrac{\partial^2 \log L}{\partial \alpha^2} & \dfrac{\partial^2 \log L}{\partial \beta \partial \alpha} \\ \dfrac{\partial^2 \log L}{\partial \alpha \partial \beta} & \dfrac{\partial^2 \log L}{\partial \beta^2} \end{bmatrix}_{\alpha_0, \beta 0} \begin{bmatrix} \hat{\alpha} - \alpha_0 \\ \\ \hat{\beta} - \beta_0 \end{bmatrix} = \begin{bmatrix} \dfrac{-\partial \log L}{\partial\ \alpha} \\ \dfrac{-\partial \log L}{\partial \beta} \end{bmatrix}_{\alpha_0\ , \beta 0}, \qquad /12/$$

where $\hat{\alpha}$, and $\hat{\beta}$ are the ML estimators of α and β respectively and α_0, β_0 are the initial values of the parameters. For initial values the moment estimators can be used or it can be obtained by equating the first three observed relative frequencies to the corresponding theoretical probabilities. The system of two equations may be used repeatedly till a good approximation of α and β are obtained.

2. Weighted discrepancies (WD) method

Let f_x denote the observed frequencies $x = 0,\ 1,\ 2,\ ....\ k$. Obviously, k is the largest of the observations. Let $N = \sum_{k=1}^{k} f_x$.

The corresponding relative frequencies are given by

$$n_x = \frac{f_x}{N}; \qquad x = 0,\ 1,\ ....\ k\ . \qquad /13/$$

The log likelihood function can be written as

$$\log L = \sum_{x=1}^{k} N n_x \log p_x\left(\alpha,\beta\right) \ . \qquad /14/$$

The likelihood equations are

$$\sum_{x=1}^{k} n_x \frac{\partial}{\partial\alpha} \log p_x = 0\,, \qquad /15/$$

$$\sum_{x=1}^{k} n_x \frac{\partial}{\partial\beta} \log p_x = 0 \ . \qquad /16/$$

Again as $\sum_{x=1}^{k} p_x = 1$, we have

$$\sum_{x=1}^{k} p_x \frac{\partial}{\partial\alpha} \log p_x = 0 \text{ and} \qquad /17/$$

$$\sum_{x=1}^{k} p_x \frac{\partial}{\partial\beta} \log p_x = 0\ . \qquad /18/$$

Subtracting /17/ from /15/ and /18/ from /16/, we get

$$\sum_{x=1}^{k} \left(n_x - p_x\right) \frac{\partial}{\partial\alpha} \log\ p_x = 0\,, \qquad /19/$$

$$\sum_{x=1}^{k} \left(n_x - p_x\right) \frac{\partial}{\partial\beta} \log p_x = 0\ . \qquad /20/$$

Substituting the corresponding expressions of the derivatives to /19/ and /20/, we get

$$\sum_{x=1}^{k} \left(n_x - p_x\right) \left[\frac{N}{\left(1-\alpha\right)\log\left(1-\alpha\right)} + \frac{N\bar{x}}{\alpha} - \frac{\left(\beta-1\right)N\bar{x}}{\left(1-\alpha\right)} \right] = 0 \text{ and} \qquad /21/$$

$$\sum_{x=1}^{k}\left(n_x - p_x\right)\left[N\bar{x}\log\left(1-\alpha\right)+\sum_{x=1}^{k}\sum_{J=1}^{x-1}\frac{x\,f_x}{\left(\beta x - j\right)}\right]=0\ , \qquad /22/$$

which has referred to *Kemp* [1986] as an equation from minimum discrimination information and ML estimation and called as weighted discrepancies estimation method.

3. Minimum chi-square (MC) method

We know that

$$\chi^2 = \sum_{X=1}^{K}\frac{\left(n_x - p_x\right)^2}{p_x} \qquad /23/$$

is approximately distributed as chi-square. Differentiating it with respect to α and β, we obtain

$$\sum_{x=1}^{k}\left(n_x - p_x\right)\left(1+\frac{n_x}{p_x}\right)\frac{\partial}{\partial\alpha}\log p_x = 0\ , \qquad /24/$$

$$\sum_{x=1}^{k}\left(n_x - p_x\right)\left(1+\frac{n_x}{p_x}\right)\frac{\partial}{\partial\beta}\log p_x = 0\ . \qquad /25/$$

Substituting the corresponding expressions of the derivatives to /24/ and /25/ we get

$$\sum_{x=1}^{k}\left(n_x - p_x\right)\left(1+\frac{n_x}{p_x}\right)\left[\frac{N}{\left(1-\alpha\right)\log\left(1-\alpha\right)}+\frac{N\bar{x}}{\alpha}-\frac{\left(\beta-1\right)N\bar{x}}{\left(1-\alpha\right)}\right]=0 \text{ and} \qquad /26/$$

$$\sum_{x=1}^{k}\left(n_x - p_x\right)\left(1+\frac{n_x}{p_x}\right)\left[N\bar{x}\log\left(1-\alpha\right)+\sum_{x=1}^{k}\sum_{j=1}^{x-1}\frac{x\,n_x}{\left(\beta x - j\right)}\right]=0\ . \qquad /27/$$

By a similar argument in section 3 the resulting equations /26/ and /27/ are known as minimum chi-square equations.

The weights in equations /15/ and /16/ for ML method depend only on the observed frequencies while the weighted discrepancies method equations /21/ and /22/ including the minimum chi-square method equations /26/ and /27/, both have weights depending on the parameters as well as observed frequencies.

4. Empirical weight rates of change (EWRC) method

The expression

$$\frac{\partial}{\partial \theta_j} \log p_x \; ; \quad j = 1,2 \, , \qquad /28/$$

where $\theta_1 = \alpha, \; \theta_2 = \beta$, is common to /15/, /16/ and /19/, /20/ and /26/, /27/ which are, the ML equations, the weighted discrepancies equations and minimum chi-square equations respectively. The common term /28/ can be seen as the relative rates of change in the probabilities on the parameters as α and β change. We refer to /28/ as the score function and it is being weighted by the relative frequencies in case of ML estimation method as in equations /15/ and /16/ and weighted by the discrepancy between observed relative frequency and estimated probability in case of WD estimation method as in equations /26/ and /27/. In order to obtain an estimation which is closer to the actual parameter value, it is quite natural to consider the combination of these two methods of estimation. Thus, we will use a weighting factor which is the product of the weights of ML and WD methods. This leads to equations

$$\sum_{x=1}^{k} n_x (n_x - p_x) \frac{\partial}{\partial \theta_j} \log p_x = 0, \qquad j = 1,2 \, , \qquad /29/$$

where $\theta_1 = \alpha, \; \theta_2 = \beta$.

Estimators obtained from /29/ will be referred as empirical weighted rates of change estimators (EWRC). This method weights the scoring function by $n_x \left(n_x - p_x \right)$ which weights the discrepancy by factor n_x. If large discrepancies occur on the extreme x values, then small weights are applied. Meanwhile, if large discrepancies occur on the more frequent x values, large weights are applied. Therefore this method can be viewed as a generalization of WD method.

5. Fitting to the GLSD

In this section the method will be presented on a biological example. In our paper we have fitted the logarithmic series distribution (LSD) and GLSD /1/ to the same zero truncated biological data which was used by *Jani* and *Shah* [1979] though they only used the method of moment estimation. Here we have operated with the different methods of estimation like ML, WD, MC and EWRC to find the estimators for fitting the LSD and GLSD.

Data provided in Tables 1 and 2 are the zero-truncated data of *P. Garman (Jani–Shah* [1979]) on counts of the number of European red mites on apple leaves.

Table 1

Estimation of LSD model

Number of mites per leaf	Leaves observed	Expected frequency of LSD methods of estimation				
		ML	Moments	MC	WD	EWRC
1	38	44.05	43.46	42.53	40.58	41.26
2	17	15.67	6.24	0.09	18.09	17.07
3	10	9.03	8.09	9.01	9.06	9.03
4	9	3.97	4.53	4.25	3.56	4.56
5	3	2.49	2.71	2.79	2.72	3.89
6	2	1.34	1.69	1.57	1.34	1.56
7	1	1.56	1.08	1.79	2.09	2.01
≥8	0	1.89	2.20	2.97	2.56	0.62
Total	80	80.00	80.00	80.00	80.00	80.00
Mean	2.1500	.	.	.	.	.
S.D	1.4504	.	.	.	.	.
χ^2	.	2.30	1.81	1.038	0.935	0.80508
D.f	.	2	2	2	2	2
$P(\chi^2)$	.	0.3166	0.40	0.595	0.6265	0.6686
	Estimates					
α		0.7578	0.7473	0.7398	0.7296	0.7216

Table 2

Estimation of GLSD model

Number of mites per leaf	Leaves observed	Expected frequency of LSD methods of estimation				
		ML	Moments	MC	WD	EWRC
1	38	40.56	39.10	38.46	38.16	38.89
2	17	18.56	17.40	16.03	15.98	16.05
3	10	9.34	9.73	10.63	10.21	9.81
4	9	4.79	5.83	5.26	5.81	5.34
5	3	2.59	3.55	3.29	3.75	3.89
6	2	2.06	2.17	1.38	1.56	1.34

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Number of mites per leaf	Leaves observed	Expected frequency of LSD				
		methods of estimation				
		ML	Moments	MC	WD	EWRC
7	1	0.89	1.27	2.97	3.01	2.79
≥8	0	1.21	0.95	1.98	1.52	1.89
Total	80	80.00	80.00	80.00	80.00	80.00
Mean	2.1500	.	.	.	.	.
S.D	1.4504	.	.	.	.	.
χ^2	.	1.38	0.16	0.102	0.097	0.084
D.f	.	1	1	1	1	1
$P(\chi^2)$	.	0.2401	0.69	0.75	0.76	0.772
	Estimates					
α		0.8904	0.8898	0.8878	0.8823	0.8789
β		0.9526	0.9129	0.9103	0.91001	0.9001

It is evident from Tables 1 and 2 that in all the cases GLSD /1/ provides a better fit than the usual logarithmic series distribution. We notice that all the estimation techniques, the ML, the WD, the moment and the MC method do not perform well in comparison to the EWRC method in estimating the LSD and GLSD parameters. Also, we observed that EWRC method seems to be better than any of the other methods for fitting the data either to LSD or GLSD.

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