ON THE VECTORIAL REPRESENTATION OF BASIC COLOUR-PERCEPTION AND ITS USE IN COLOUR-MEASUREMENT

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1. Introduction

It is usual to distinguish between two levels of colorimetry, a basic one, where Grassmann's laws are valid and an advanced, perceptual one. In this paper we will stay within the boundaries of basic colorimetry, where linear metrics can be applied. Within these limits it is usual to define three primary colours and use the laws of additive colour mixture to express the problems of tristimulus colorimetry.

In the following an alternative interpretation of this system, based on vector calculus, will be given, and applied to discuss tristimulus colorimetry.

2. Colour transformations

The colour $\underline{S}$[xx] of an unknown spectral distribution can be expressed as the additive sum of three reference stimuli ($\underline{R}$, $\underline{G}$, $\underline{B}$):

$$\underline{S} = R\underline{R} + G\underline{G} + B\underline{B} \qquad 1$$

where R, G and B are the tristimulus values.

Using this equation for all the monochromatic radiations finaly the spectral tristimulus values (STV) $\bar{r}(\lambda)$, $\bar{g}(\lambda)$, $\bar{b}(\lambda)$ are reached, by the help of which the tristimulus values of a colour can be calculated if the spectral distribution

[x] Now with Tungsram Inc. Lamp Works, Budapest.

[xx] Colours are represented by underlined capital letters.

$\underline{S}_\lambda$ of the radiation representing the colour is known:

$$R=\int_{380}^{780} S_\lambda \bar{r}(\lambda)\,d\lambda,\quad G=\int_{380}^{780} S_\lambda \bar{g}(\lambda)\,d\lambda,\quad B=\int_{380}^{780} S_\lambda \bar{b}(\lambda)\,d\lambda \qquad 2$$

The STV-s are characteristic to the $\underline{R}$, $\underline{G}$, $\underline{B}$ reference stimuli, but not to their spectral distribution, as the same colour can be produced by different spectral distributions (metamerism).

It is well known that the same colour matching equation as seen in Equ.1 can be written for other reference stimuli, thus e.g.:

$$\underline{S} = F\underline{F} + H\underline{H} + N\underline{N} \qquad 3$$

Here again equations of the form of Equ.2 hold, only the STV functions ($\bar{f}(\lambda)$, $\bar{h}(\lambda)$, $\bar{n}(\lambda)$) have to be determined.

Equations coupling the two systems of reference stimuli are

$$\begin{aligned} \underline{F} &= R_f\,\underline{R} + G_f\,\underline{G} + B_f\,\underline{B} \\ \underline{H} &= R_h\,\underline{R} + G_h\,\underline{G} + B_h\,\underline{B} \\ \underline{N} &= R_n\,\underline{R} + G_n\,\underline{G} + B_n\,\underline{B} \end{aligned} \qquad 4$$

The tristimulus values expressing the colour attributes of colour $\underline{S}$ in the two systems (Equ.1 and 3.) are coupled by similar equations. Writing the tristimulus values as column vectors it can be shown that equation

$$\begin{bmatrix} F \\ H \\ N \end{bmatrix} = \begin{bmatrix} R_f & R_h & R_n \\ G_f & G_h & G_n \\ B_f & B_h & B_n \end{bmatrix}^{-1} \begin{bmatrix} R \\ G \\ B \end{bmatrix} \qquad 5$$

holds. By denoting the inverse of the matrix in this equation by $\underline{\underline{L}}$ this can be written in a simpler form:

$$\begin{bmatrix} F \\ H \\ N \end{bmatrix} = \underline{\underline{L}} \begin{bmatrix} R \\ G \\ B \end{bmatrix} \qquad 6$$

Using the form of Equ.2. this can be rewritten as

$$\begin{bmatrix} \int_{380}^{780} S_\lambda \, \bar{f}(\lambda)\, dx \\ \int_{380}^{780} S_\lambda \, \bar{h}(\lambda)\, d\lambda \\ \int_{380}^{780} S_\lambda \, \bar{n}(\lambda)\, d\lambda \end{bmatrix} = \underline{\underline{L}} \begin{bmatrix} \int_{380}^{780} S_\lambda \, \bar{r}(\lambda)\, d\lambda \\ \int_{380}^{780} S_\lambda \, \bar{g}(\lambda)\, d\lambda \\ \int_{380}^{780} S_\lambda \, \bar{b}(\lambda)\, d\lambda \end{bmatrix} \qquad 7$$

This equation has to hold for every S_λ distribution, thus

$$\begin{bmatrix} \bar{f}(\lambda) \\ \bar{h}(\lambda) \\ \bar{n}(\lambda) \end{bmatrix} = \underline{\underline{L}} \cdot \begin{bmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{bmatrix} \qquad 8$$

gives the transformation equations for the STV-s. It is very important to notice that $\underline{\underline{L}}$ is independent of λ, and the equations have to hold for every wavelength.

It is well known that the CIE fixed in 1933 the STV-s of the so called average observer, and fixed also a linear transformation of these, a transformation where all the STV-s are non-negative numbers. It is usual to denote this system of STV-s by $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ (CIE-STV-s).

A tristimulus colorimeter should show sensitivity distributions in accordance to these CIE-STV-s. If, however, the instrumental STV-s are a linear transformation of the CIE ones, i.e.

$$\left[\bar{a}_i\,(\lambda)\right] = \underline{\underline{L}}\,\left[\bar{x}_j\,(\lambda)\right], \text{ (with } i=1\ldots4,\ j=1\ldots3) \qquad 9$$

where $\bar{a}_i\,(\lambda)$ represents the instrumental STV-s (it is usual to use four detectors, thus $i = 1\ldots4$), and $\bar{x}_1\,(\lambda) = \bar{x}(\lambda)$, $\bar{x}_2\,(\lambda) = \bar{y}\,(\lambda)$ and $\bar{x}_3\,(\lambda) = \bar{z}\,(\lambda)$, an exact colour measurement is still possible.

In practice it is usual to set $\underline{\underline{L}}$ in Equ.9. as a unity matrix (using the $\overline{a}_i(\lambda)$ curves instead of the $\overline{x}_i(\lambda)$ ones), but even in the more general case of an arbitrary matrix $\underline{\underline{L}}$ Equ.9. does not hold for every wavelength in question (380 nm ≤ λ ≤ 780 nm) exactly. This leads to measurement errors. Budinszky [1] showed a method of finding an optimal $\underline{\underline{L}}$ matrix based on a minimum calculus.

In the following we shall discuss a vector representation of the spectral power distributions and show how vector calculus can be applied to describe colorimetry. By the help of this we shall get an answer on the problem of optimal transformation as well.

3. Vectorial representation of colour measurement

As well known the human eye can distinguish between spectral distributions only in the form of integrals. All the different P_λ spectral distributions that yield the same

$$X=\int_{380}^{780} P_\lambda \overline{x}(\lambda)\,d\lambda, \quad Y=\int_{380}^{780} P_\lambda \overline{y}(\lambda)\,d\lambda, \quad Z=\int_{380}^{780} P_\lambda \overline{z}(\lambda)\,d\lambda \qquad 10$$

tristimulus values are metameric, we cannot distinguish them.

In Equ.5. we used already the column vector representation of the tristimulus values. Using the system of the reference stimuli as basic vectors the manyfold of colours can be represented as the elements of this tridimensional space, where the tristimulus values give the coordinates of the colour-vector.

It is straight-forward to regard the spectral power distributions as elements of a multidimensional vector space [2-4]. It can be shown that in such a representation all the laws of vector calculus hold. The only exception would be the multiplication by a negative number, as this operation would lead out of the set of spectral distributions. If, however, the system of spectral distributions is completed with these "virtual" distributions, in a similar way as the concept of virtual colours is used, this problem is solved without any difficulty. Thus all the continuous or quadraticly integrable functions are elements of this vector space. The number of dimensions of this space can be reached by finding a basis. Such a basis could be the system of the sine or cosine functions used in the Fourier-series of the functions. From this it can be seen that our vector space is countably infinite.

It can be shown that inner products between the elements of this linear space can be defined in the following form:

f (x) function is represented by vector $\underline{f}$

g (x) function is represented by vector $\underline{g}$

and

$$\int f(x)\, g(x)\, dx = c \text{ (real)} \longrightarrow \underline{f} \cdot \underline{g} = c \qquad 11$$

As both the P_λ power distributions and the $\bar{x}_i$ ($\bar{x}_1 = \bar{x}(\lambda)$, $\bar{x}_2 = \bar{y}(\lambda)$, $\bar{x}_3 = \bar{z}(\lambda)$) STV functions can be represented in this vector space by vectors ($\bar{P}$ and $\bar{x}_i$), the colour measurement is, according to Equ.11., just the inner product of the $\bar{P}$ vector and the $\bar{x}_i$ spectral tristimulus vector.

4. Vectorial picture of colour vision

The spectral power distribution of a given colour gives a vector $\underset{\sim}{P}^x$ in the space of power distributions. In the same space the STV-s of the CIE-system ($\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$) are represented by three vectors: $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$. These vectors form a threedimensional subspace; in the following we will refere to this subspace as to the colour measuring subspace.

The visual perception of a spectral power distribution is the orthogonal projection of this power distribution vector into the colour-measuring subspace. The colour perception is influenced only by this projection and not by the "arbitrarily" chosen triplet of STV-s, forming a basis of the subspace. The projection can be expressed as the linear combination of the reference stimuli vectors $\underline{x}_i$, ($i = 1...3$), but just as well in an orthonormal system of basic vectors (see e.g. 3), reached as linear transformation of the reference stimuli vectors $\underline{x}_i$ ($i = 1...3$). Budinszky [1] used in his calculation an orthonormal system $\underline{k}_i$ ($i = 1...3$) where

$$k_i\, k_j = \delta_{ij} \begin{cases} = 0, & \text{if } i \neq j \\ = 1, & \text{if } i = j \end{cases} \qquad i, j = 1 \ldots 3 \qquad 12$$

(We will come back to this problem in Chapter 6.)

Let us try to find that particular basis, where the tristimulus values are coordinates of the projection.

It can be shown that every spectral power distribution vector has one and only one projection in the colour-measuring subspace. The projection of $\underset{\sim}{P}^x$ will be marked by $\underline{P}_o$. The tristimulus values of the spectral power distribution $\underset{\sim}{P}^x$ are the $\underset{\sim}{P}^x \underline{x}_1$, $\underset{\sim}{P}^x \underline{x}_2$, $\underset{\sim}{P}^x \underline{x}_3$ inner products. We will show now that the $P^x \underline{x}_i$ $(i = 1, 2, 3)$ tristimulus values are the coordinates in the inverted system of $\underline{x}_i$ $(i = 1 \ldots 3)$: $\underline{y}_i$ $(i = 1 \ldots 3)$, where $\underline{y}_i\, \underline{x}_j = \delta_{ij}$ see Equ. 12., i.e.:

$$\sum_{i=1}^{3} \left(\underset{\sim}{P}^x \underline{x}_i \right) \underline{y}_i = \underline{P}_o, \qquad \text{where } \underline{y}_i\, \underline{x}_j = \delta_{ij} \qquad 13$$

Let us describe $\underline{P}_o$ in the system of $\underline{y}_i$ $(i = 1 \ldots 3)$ in the following way

$$\underline{P}_o = r_1 \underline{y}_1 + r_2 \underline{y}_2 + r_3 \underline{y}_3 \qquad 14$$

As $\underline{P}_o$ is the projection of $\underset{\sim}{P}^x$,

$$\underset{\sim}{P}^x \underline{x}_i = \underline{P}_o\, \underline{x}_i, \quad i = 1 \ldots 3 \qquad 15$$

and thus

$$\sum_{i=1}^{3} (\underset{\sim}{P}^x \underline{x}_i)\, \underline{y}_i = \sum_{i=1}^{3} (\underline{P}_o\, \underline{x}_i)\, \underline{y}_i = \sum_{i=1}^{3} (r_1 \underline{y}_1 \underline{x}_i + r_2 \underline{y}_2 \underline{x}_i + r_3 \underline{y}_3 \underline{x}_i)\, \underline{y}_i$$

and this, by using 13 and 14, gives:

$$\sum_{i=1}^{3} (\underset{\sim}{P}^x \underline{x}_i)\, \underline{y}_i = \sum_{i=1}^{3} r_i\, \underline{y}_i = \underline{P}_o \qquad 16$$

Thus we reached the very simple result that for all those spectral power distributions for which P_o is the projection (i.e. for all the colours metameric with P_o), the STV-s represent the projection as coordinates in the invert system of basic vectors.

By the help of these results we get a somewhat better insight into the colour vision and measurement mechanism. Postulating the spectral power distribution space, and the colour measuring subspace, the Grassmann laws are reached very easily:

According to the first Grassmann law, in case of additive colour mixture, the final colour depends only on the colour and not on the spectral distribution of the primeries. As the spectral power distribution and colour measuring spaces are linear, and vector addition corresponds to additive colour mixture, just the projections of the spectral distributions into the colour measuring subspace show these characteristics. Grassmann's second law (the necessity of three independent data to describe a colour) shows that the colour measuring subspace is three-dimensional.

Grassmann's third law that the colour mixing series are continuous is a consequence of the linearity of the vector spaces.

Most important among these is the first Grassmann law, describing metamerism. In vector representation this means that an (infinite) number of spectral distributions can have the same projection in the colour measuring subspace. Fig. 1. visualizes this in a simplified form: $\vec{x_1}$ and $\vec{x_2}$ represent the basis of the colour measuring subspace (let us take it now as only two-dimensional) and $\vec{e}$ the other n-2 coordinates ($n \rightarrow \infty$) of the spectral power distribution space. $\vec{P^x}$ is a vector in this space, and $\vec{P_o}$ is its projection in the subspace. At the same time $\vec{P^x_{met}}$ is an other spectral distribution yielding $\vec{P_o}$ as projection, i.e. the same colour, thus $\vec{P^x}$ and $\vec{P^x_{met}}$ represent metameric colours.

This gives a - theoretical - tool for finding a basis of the colour measuring subspace. It is necessary to determine a high number of pairs of spectral

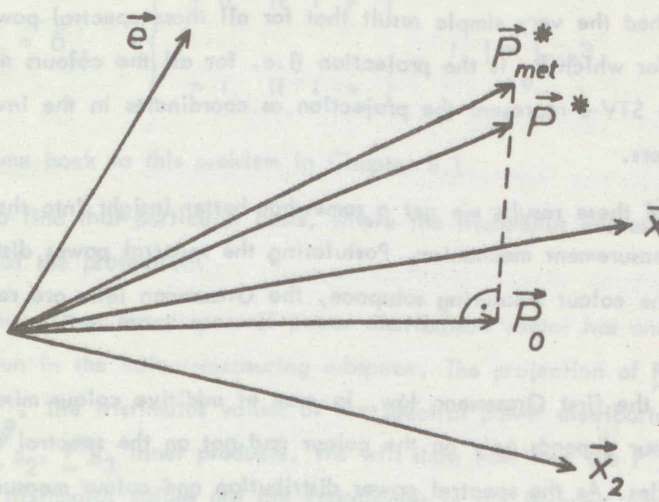

Fig. 1. Schematic representation of the spectral power distribution and colour measuring spaces, and the colour measurement of two metameric colour stimuli ($\overrightarrow{P^x}$ and $\overrightarrow{P^x_{met}}$).

distributions being metameric. If one calculates the difference of every two distributions forming a pair this difference vectors form a subspace, and the colour measuring subspace contains the vectors perpendicular to these difference vectors.

By the help of this experiment it is thus possible to determine the colour measuring subspace. The $\underline{x}_i$ STV-s (i.e. the $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ functions) are a basis of this space, determined by the help of some further requirements: One of the basic vectors should coincide with the $V(\lambda)$ luminosity curve, all the coordinates of the vectors forming the basis should be non-negative, and the vector, representing the equienergetic spectrum should be of equal distance from the basis. These demands determine the CIE STV-s unambiguously, and by this we can reach these distributions - theoretically - without using the concept of primary colours.

5. The transformation of an instrumental system

Tristimulus values of colours can be determined either by measuring the spectral power distribution P_λ and using the equations of Equ.10., or by duplicating the STV functions by detector - filter combinations and thus performing the integrations of Equ.10. in the detectors themselves. In practice the filter-detector combinations never match the STV functions perfectly, and thus a transformation of the instrumental results into the CIE tristimulus values might increase the measuring accuracy.

From the discussions in Chapter 3 and 4 it is obvious that an exact transformation is possible only if the instrumental STV "vectors" lie within the subspace spanned by the CIE STV vectors $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$.

The instrumental subspace is not necessarily threedimensional (this depends on the number of detectors used, in usual tristimulus colorimeters it is four dimensional, due to the fact that the $\overline{x}(\lambda)$ curve shows two maxima), in the following we will take it as five dimensional giving a further degree of freedom for the construction. Increasing the number of detectors enables a more complete transformation, but the practical circuitry becomes increasingly complicated, five channels seem to be a reasonable compromise. Let us mark these instrumental STV vectors by $\underline{a}_i$ (i = 1 ...5).

Measuring $\underset{\sim}{P}^x$ by this instrument yields the $\underset{\sim}{P}^x\ \underline{a}_i$ values as results, i.e. the measurement gives us the projection of $\underset{\sim}{P}^x$ in the subspace spanned by the vectors $\underline{a}_1$, $\underline{a}_2$, ... $\underline{a}_5$.

Let us denote the coordinates of this projection by p_i (i = 1 ... 5), and the projection of $\underset{\sim}{P}^x$ in this subspace by $\underset{\sim}{P}$. Thus

$$\begin{aligned} \underset{\sim}{P}^x\underline{a}_1 &= \underset{\sim}{P}\underline{a}_1 = (p_1\underline{a}_1 + p_2\underline{a}_2 + p_3\underline{a}_3 + p_4\underline{a}_4 + p_5\underline{a}_5)\ \underline{a}_1 \\ \underset{\sim}{P}^x\underline{a}_2 &= \underset{\sim}{P}\underline{a}_2 = (p_1\underline{a}_1 + p_2\underline{a}_2 + p_3\underline{a}_3 + p_4\underline{a}_4 + p_5\underline{a}_5)\ \underline{a}_2 \qquad 17 \\ \underset{\sim}{P}^x\underline{a}_5 &= \underset{\sim}{P}\underline{a}_5 = (p_1\underline{a}_1 + p_2\underline{a}_2 + p_3\underline{a}_3 + p_4\underline{a}_4 + p_5\underline{a}_5)\ \underline{a}_5 \end{aligned}$$

Performing the multiplication and using a column vector notation for p_i ($i = 1 \ldots 5$): $\vec{p}$, writing also the five results of the measurement ($\underset{\sim}{P}\underline{a}_i$ ($i = 1 \ldots 5$)) in this form: $\vec{P}$, and realizing that the $\underline{a}_i\underline{a}_k$ products form the a_{ik} elements of a matrix $\underline{\underline{A}}$, Equ.17. can be written in a simpler form:

$$\vec{P} = \underline{\underline{A}}\ \vec{p} \qquad 18$$

and thus the coordinates are given by

$$\vec{p} = \underline{\underline{A}}^{-1}\ \vec{P} \qquad 19$$

$\vec{P}$ is the "instrumental value", the tristimulus values can be calculated from this in the following way:

The instrument produces the $\underset{\sim}{P} = \sum_{i=1}^{5} p_i\underline{a}_i$ vector, and the tristimulus values are the

$$\underset{\sim}{P}\,\underline{x}_1,\ \underset{\sim}{P}\,\underline{x}_2,\ \underset{\sim}{P}\,\underline{x}_3 \quad \text{products.}$$

Thus e.g.

$$\underset{\sim}{P}\underline{x}_1 = (p_1\underline{a}_1 + p_2\underline{a}_2 + p_3\underline{a}_3 + p_4\underline{a}_4 + p_5\underline{a}_5)\ \underline{x}_i =$$

$$= p_1\underline{a}_1\underline{x}_1 + p_2\underline{a}_2\underline{x}_2 + \ldots + p_5\underline{a}_5\underline{x}_5 \qquad 20$$

Using again the column vector and for the $\underline{x}_i\ \underline{a}_k$ product the m_{ik} matrix element notation, the tristimulus values ($\vec{X}$) are:

$$\vec{X} = \underline{\underline{M}}\ \vec{p} \qquad 21$$

where m_{ik} is an element of matrix $\underline{\underline{M}}$.

By the help of Equ.19. this gives

$$\vec{X} = \underline{\underline{M}}\ \underline{\underline{A}}^{-1}\ \vec{P} \qquad 22$$

Thus matrix $\underline{\underline{M}}\ \underline{\underline{A}}^{-1}$ gives the final transformation. To visualize this, Fig. 2 shows an oversimplified version of the measurement: $\vec{P}^x$ is the spectral power distribution in the total spectral power distribution space, consisting of the

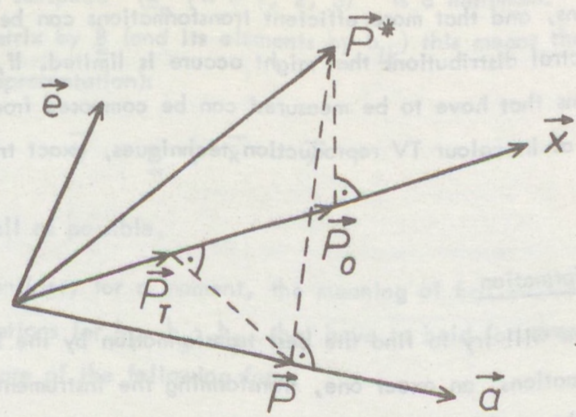

Fig.2. Schematic representation of the spectral power distribution, colour measuring and instrumental subspaces, showing an ideal and an instrumental measurement together with a transformation ($\vec{P}^x \rightarrow \vec{P}_o$ and $\vec{P}^x \rightarrow \vec{P} \rightarrow \vec{P}_T$)

independent instrumental subspace ($\vec{a}$), the colour measuring subspace ($\vec{x}$) and the $N - N_a - N_x$ dimensional space (here N_a and N_x represents the number of dimensions of the instrumental and colour measuring subspace) visualized by vector $\vec{e}$.

A correct colour measurement projects $\vec{P}^x$ onto the $\vec{x}$ axis and yields thus the colour $\vec{P}_o$. The instrument performs instead of this the projection onto axis $\vec{a}$, resulting in the "instrumental value" $\vec{P}$, the transformation according to Equ.21. performs the projection onto the $\vec{x}$ axis giving the final value $\vec{P}_T$.

The aim of the transformation technique is to decrease the average $\left|\underline{P}_T - \underline{P}_o\right|$ value. It will be shown in the next Chapter that the method outlined above gives the "best" transformation for all the monochromatic colours. As normal colours are composed by monochromatic radiations the transformation will give better tristimulus values also for most of the practical colours. It is, however, quite obvious that it might deteriorate the measurement results of some special spectral distributions, and that more efficient transformations can be found, if the number of spectral distributions that might occure is limited. If, e.g. the spectral distributions that have to be measured can be composed from three basic distributions as in colour TV reproduction techniques, exact transformations can be found.

6. Optimal transformation

In the following we will try to find the best transformation by the help of two successive transformations, an exact one, transforming the instrumental STV vectors into an orthogonal system, and a second one giving the smallest mean error in transforming all the monochromatic radiations.

Let us denote the instrumental basis by $\underline{a}_i$ (i = 1 ...5) and the orthogonal ones by $\underline{k}_i$ (i = 1...5). Due to orthogonality

$$\underline{k}_i \, \underline{k}_j = \delta_{ij} \qquad 23$$

where δ_{ij} is the Kronecker-symbol (see Equ.12.).

Using the column vector representation the transforming matrix ($\underline{\underline{O}}$) between $\underline{a}_i$ and $\underline{k}_m$ (i, m = 1 ...5) is found from the following equation:

$$\underline{\underline{O}} \cdot \vec{a} = \vec{k} \qquad 24$$

Thus, if the elements of matrix $\underline{\underline{O}}$ are denoted by σ_{ik}, it can be written that

$$\underline{k}_i = \sum_{k=1}^{5} \sigma_{ik} \, \underline{a}_k \quad (i = 1 \ldots 5) \qquad 25$$

Putting Equ.25. into Equ.23.,and taking into consideration that multiplication is commutative, 15 equations can be found for the 25 elements of the matrix. The other elements can be chosen arbitrarily. Thus it is always possible to use instead of the instrumental functions an orthogonal transformation of these.

The problem is now to find such a transformation matrix that the difference between the transformed orthogonal instrumental vectors and the basis of the colour measuring subspace (x_n (n = 1, 2, 3) is a minimum.

Denoting this matrix by $\underline{\underline{B}}$ (and its elements by b_{ik}) this means that (by using column vector representation):

$$\vec{h} = \underline{\underline{B}} \cdot \vec{k} - \vec{x} \qquad 26$$

should be as small as possible.

(Let us now reconsider, for a moment, the meaning of Equ. 26.: It covers really three equations for $\underline{h}_1$, $\underline{h}_2$, $\underline{h}_3$, that have to hold for every (wavelength) dimension, thus are of the following form:

$$h_1(\lambda) = \left[b_{11}\ k_1(\lambda) + b_{12}\ k_2(\lambda) + .. + b_{15}\ k_5(\lambda)\right] - \overline{x}(\lambda) \qquad 27$$

and similarly for $\overline{y}(\lambda)$ and $\overline{z}(\lambda)$ with $h_2(\lambda)$, $h_3(\lambda)$ and b_{21} ... b_{35}.)

Instead of finding the minimum of Equ. 26. it is reasonable to look for the minimum of

$$H_i = \left|\underline{h}_1\right|^2 \qquad (i = 1, 2, 3) \qquad 28$$

the norm of vector $\underline{h}_i$, this is, due to Equ. 11. an other description of

$$H_i = \int_{380}^{780} \left[h_i(\lambda)\right]^2 d\lambda \quad (i = 1, 2, 3) \qquad 29$$

The minimum requirements are of the following form:

$$\frac{\partial H_1 (b_{11}, b_{12}, \cdots, b_{15})}{\partial b_{12}} = 0 \qquad 30$$

Writing Equ. 27. and 29. into Equ. 30., and by using the orthogonality criterium of Equ. 23. we get that the matrix elements of $\underline{\underline{B}}$ have to be of the following form

$$b_{ik} = \underline{x}_i\ \underline{k}_k \qquad 31$$

Comparing Equ.31. with Equ.21. and 22. it is seen that matrix $\underline{\underline{M}}$ is equal to matrix $\underline{\underline{B}}$, if the instrumental system is orthogonal; and thus in the vector transformation technique of Chapter 5 matrix $\underline{\underline{A}}$ constructed from the instrumental STV vectors has the same meaning as the transformation of the instrumental STV vectors into an orthogonal basic vector system.

Thus the transformation equation of Equ.22. yields the "best" transformation of the instrumental subspace into the colour measuring one. (Matrix $\underline{\underline{M}}$ performs a rotation within the instrumental subspace giving the "best" threedimensional subspace.)

7. Application of the matrix transformation technique

The above described technique has been used to correct some crude photocell-filter combinations. In each combination a red sensitive Si-photoelement and two colour filters have been used. We calculated the difference between the uncorrected instrumental STV functions and the CIE STV functions, as well as the same values using the transformed STV-s, always correcting the absolute values of the curves to give the same area under the curve, and calculated the mean deviations.

Table I and II show some representative results of this calculation. As seen from Table II although filter combination No.2 is better than combination No.3, the transformed results of combination 3 are the best ones. This means that it is possible to find a combined optimum for filter thickness and transformation.

Table I.

Filter combinations used

	combination 1	combination 2	combination 3
$\underline{a}_1$:	BG 18 + OG 1	BG 18 + GG 20	BG 18 + GG 20
$\underline{a}_2$:	BG 18 + BG 3	BG 18 + BG 1	BG 18 + BG 3
$\underline{a}_3$:	BG 18 + VG 4	BG 18 + VG 4	BG 18 + GG 10
$\underline{a}_4$:	BG 18 + BG 24	BG 18 + BG 24	BG 18 + BG 24

Table II.

Filter combination	Instrumental-CIE STV-s			Transformed-CIE STV-s			Mean deviation from CIE values	
	x	y	z	x	y	z	instum.	transf.
1	1,427	0,244	0,731	1,294	0,175	0,565	0,541	0,474
2	0,944	0,244	0,731	0,887	0,103	0,706	0,399	0,382
3	0,968	0,463	0,731	0,925	0,268	0,561	0,439	0,372

the values in column 2-7 are calculated as $\sqrt{\sum_{i=1}^{n}\left[\bar{a}_{jq}(\lambda_i) - \bar{x}_j(\lambda_i)\right]^2}$

where $\bar{a}_{jq}$ refers to the instrumental STV-s, $\bar{x}_j$ to the CIE ones (j: x, y, z; q = 0 for the instrumental and is t for the transformed one).

Table III gives, as an example, the values of the transformation matrix for combination 3.

Table III.

Transformation matrix of filter combination No.3

0,966	0,834	0,092	0,029
-0,193	0,308	1,134	-0,043
0,0374	-2,103	-0,042	1,393

Fig. 3 shows the spectral distribution of the instrumental, the transformed and the CIE $\bar{y}(\lambda)$ curve. It is seen that the transformation improves the curve form considerably.

Our aim is to use the matrix transformation technique in a tristimulus light source colorimeter, thus we calculated the chromaticity coordinates for several light sources as well Fig. 4. and 5. shows the spectral distribution of some sources used in this calculation, Table IV-VI presents the chromaticity co-ordinates of these sources for the instrumental, the transformed and the CIE STV-s (column A, B and C) as well as differences between the instrumental and CIE, as well as the transformed and CIE chromaticity coordinates (column D

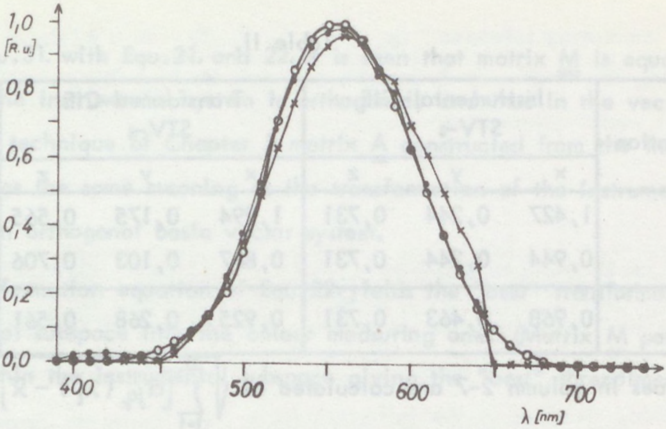

Fig. 3. A realized $\bar{y}(\lambda)$ curve: —×—×— The transformed $\vec{y}_1(\lambda)$ curve: —•—•— and the CIE one: —o—o—

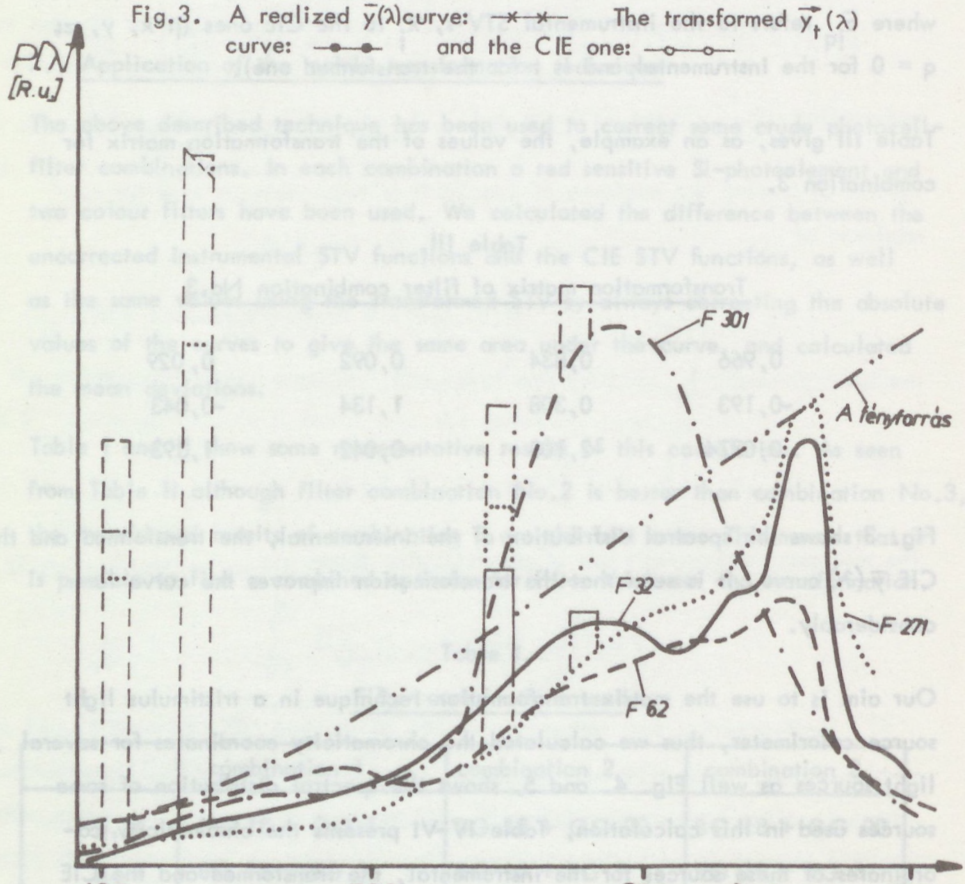

Fig. 4. Some spectral power distributions used in the calculations

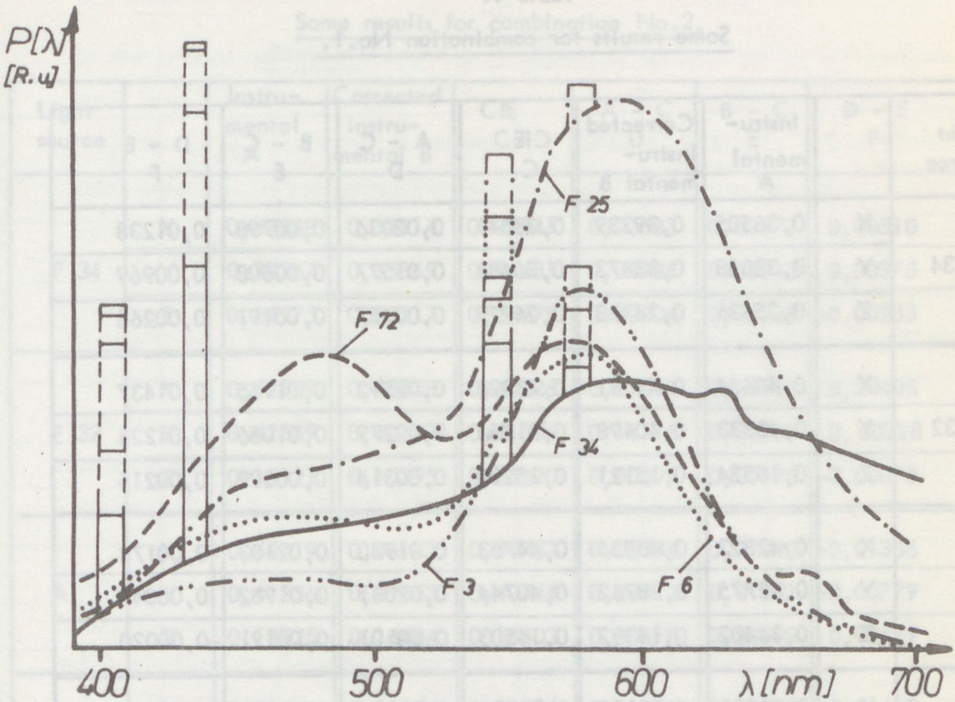

Fig. 5. Some spectral power distributions used in the calculations

and E). The difference of column D and E visulaizes the amount of amelioration (column F). For every filter combination the mean relative amelioration (F_{mean}/D_{mean}) has been calculated as well. It is interesting to note that for the relatively bad combination No.1 a strong amelioration was achieved, the relatively good combination No.2 is not much better after transformation as combination No.1, on the other hand combination No.3 shows a mean amelioration comparable to that of combination No.1, and gives at the same time relatively good results.

The instrumental realization of these results is under way, together with the development of a new computer program enabling the determination of the optimal filter thickness together with the best transformation matrix.

Table IV

Some results for combination No.1.

Light source		Instrumental A	Corrected instrumental B	CIE C	A - C D	B - C E	D - E F
F 34	X	0,36505	0,39339	0,38541	0,02036	0,00798	0,01238
	Y	0,38058	0,35873	0,36481	0,01577	0,00608	0,00969
	Z	0,25436	0,24788	0,24979	0,00457	0,00191	0,00266
F 32	X	0,40634	0,44381	0,43226	0,02592	0,01155	0,01437
	Y	0,43833	0,40498	0,41554	0,02279	0,01056	0,01223
	Z	0,15534	0,15121	0,15220	0,00314	0,00099	0,00215
A	X	0,42823	0,46856	0,44753	0,01930	0,02103	-0,00173
	Y	0,42775	0,38762	0,40744	0,02031	0,01982	0,00049
	Z	0,14402	0,14382	0,14503	0,00101	0,00121	-0,00020
F 6	X	0,31513	0,33185	0,33436	0,01923	0,00251	0,01672
	Y	0,36326	0,35568	0,35373	0,00953	0,00195	0,00758
	Z	0,32161	0,31247	0,31191	0,00970	0,00056	0,00914
F 25	X	0,37345	0,40352	0,40807	0,03462	0,00455	0,03007
	Y	0,41555	0,39115	0,38820	0,02735	0,00295	0,02440
	Z	0,21100	0,20534	0,20373	0,00727	0,00161	0,00566
Mean:					0,01606	0,00635	0,00971

$$\frac{F_{mean}}{D_{mean}} = 0,60$$

Table V

Some results for combination No.2.

Light source		Instru-mental A	Corrected instru-mental B	CIE C	A - C D	B - C E	D - E F
F 34	X	0,37658	0,38914	0,38541	0,00883	0,00373	0,00510
	Y	0,37367	0,35869	0,36481	0,00986	0,00612	0,00375
	Z	0,24975	0,25217	0,24979	0,00004	0,00238	-0,00233
F 32	X	0,42209	0,43838	0,43226	0,01017	0,00612	0,00405
	Y	0,42669	0,40767	0,41554	0,11115	0,00787	0, 00328
	Z	0,15121	0,15395	0,15220	0,00099	0,00175	-0,00076
A	X	0,44719	0,46143	0,44753	0,00034	0,01390	-0,01356
	Y	0,41356	0,49453	0,40744	0,00512	0,01291	-0,00779
	Z	0,13925	0,14404	0,14503	0,00578	0,00099	0,00479
F 6	X	0,31466	0,32874	0,33436	0,01970	0,00562	0,01408
	Y	0,36351	0,35127	0,35373	0,00978	0,00246	0,00731
	Z	0,32183	0,32000	0,31191	0,00992	0,00808	0,00184
F 25	X	0,41149	0,40572	0,40807	0,00342	0,00235	0,00107
	Y	0,39032	0,38701	0,38820	0,00212	0,00119	0,00094
	Z	0,19819	0,20727	0,20373	0,00554	0,00354	0,00200
Mean:					0,00685	0,00527	0,00158

$$\frac{F_{mean}}{D_{mean}} = 0,23$$

Table VI

Some results for combination No. 3.

Light source		Instrumental A	Corrected instrumental B	CIE C	A - C D	B - C E	D - E F
F 3	X	0,35216	0,37174	0,37838	0,02622	0,00664	0,01958
	Y	0,39146	0,38328	0,37631	0,01516	0,00697	0,00818
	Z	0,25638	0,24498	0,24531	0,01107	0,00033	0,01074
F 62	X	0,35840	0,37580	0,36774	0,00934	0,00806	0,00128
	Y	0,31432	0,30373	0,31043	0,00389	0,00670	-0,00281
	Z	0,32728	0,32048	0,32183	0,00545	0,00135	-0,00410
F 72	X	0,29276	0,30548	0,30288	0,01012	0,00260	0,00752
	Y	0,33212	0,32421	0,32619	0,00593	0,00198	0,00395
	Z	0,37512	0,37031	0,37092	0,00420	0,00061	0,00359
F 271	X	0,42077	0,44352	0,43085	0,01008	0,01267	-0,00259
	Y	0,35955	0,34057	0,35368	0,00587	0,01311	-0,00724
	Z	0,21968	0,21591	0,21547	0,00421	0,00044	0,00377
F 301	X	0,39476	0,41572	0,41997	0,02521	0,00425	0,02096
	Y	0,40150	0,38548	0,38150	0,02000	0,00398	0,01602
	Z	0,20374	0,19880	0,19853	0,00521	0,00027	0,00494
Mean:					0,01080	0,00466	0,00613

$$\frac{F_{mean}}{D_{mean}} = 0,57$$

8. Summary

In the present work the vector representation of spectral power distributions have been used. By the help of this it was possible to give a visual interpretation of such concepts as metamerism, spectral tristimulus values of an instrument and its transformation into the colour measuring subspace, a space corresponding to the human colour vision mechanism.

Transformation techniques for transforming the spectral sensitivity distribution functions of a tristimulus colorimeter as close as possible to the CIE spectral tristimulus values have been discussed. Possibilities and limitations of this technique are shown theoretically as well as on practical examples. It is possible to achieve a twofold mean amelioration of the experimental results by this technique.

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