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Dimension of the space of unitary equivariant translation invariant tensor valuations[☆]

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ABSTRACT

Following the work of Semyon Alesker in the scalar valued case and of Thomas Wannerer in the vector valued case, the dimensions of the spaces of continuous translation invariant and unitary equivariant tensor valuations are computed. In addition, a basis in the vector valued case is presented.

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1. Introduction

For a real vector space \mathbb{V} of finite dimension and an abelian semigroup $(\mathcal{A}, +)$, we write $\mathcal{K}(\mathbb{V})$ to denote the space of convex bodies in \mathbb{V} (i.e., compact convex sets) equipped with the Hausdorff metric. We call an operator $Z : \mathcal{K}(\mathbb{V}) \rightarrow \mathcal{A}$ a *valuation* if

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$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L)$$

holds for any $K, L \in \mathcal{K}(\mathbb{V})$ satisfying that $K \cup L \in \mathcal{K}(\mathbb{V})$. Typical choices for the semigroup \mathcal{A} are the field of real numbers \mathbb{R} , or the vector space \mathbb{V} itself, and more generally the space $\mathbb{S}^d(\mathbb{V})$ of symmetric rank d tensors of \mathbb{V} . Also $\mathcal{A} = \mathcal{K}(\mathbb{V})$ equipped with Minkowski addition leads to many interesting valuations.

One of the principal goals in the theory of valuations is to obtain characterizations of known operators as the only valuations satisfying certain simple geometric and topological properties. The fundamental result in this direction goes back to 1952, when Hadwiger proved that, for $\mathbb{V} = \mathbb{R}^n$, the linear combinations of intrinsic volumes are the only continuous real-valued valuations being invariant under rigid motions of \mathbb{R}^n (see [22]).

Hadwiger's result can be generalized in different directions. One of them is to change the group acting on $\mathcal{K}(\mathbb{V})$ and classify the continuous real-valued translation invariant valuations that are invariant under the linear action of some group G . The space of such valuations is finite-dimensional precisely when G acts transitively on the unit sphere [3]. For the first nontrivial case $G = \mathrm{U}(m)$, this was achieved by Alesker [5] and refined by Bernig and Fu [10]. After this breakthrough, several other groups have been successfully studied. We refer the reader to [1,8,9,11,13,17,34] and references therein for some results in this direction.

Another important generalization of Hadwiger's theorem consists of changing the target space \mathcal{A} . The case $\mathcal{A} = \mathbb{S}^d(\mathbb{V})$ of tensor-valued valuations is of particular interest and has been thoroughly studied, specially under equivariance assumptions with respect to orthogonal and special linear groups (see e.g. [2,12,20,21,24]). The space of $\mathrm{U}(m)$ -equivariant valuations is much less understood, and is the object of the present paper.

Other current research directions in valuation theory include the following. Real-valued and tensor-valued valuations defined on lattice polytopes have been studied in [15,31]. A very active area is the study of valuations taking values in the space of convex bodies (see e.g., [27,28] and the references in [29]). Also, important results on valuations defined in several function spaces have been recently obtained (cf. e.g., [6,15,16,30]).

In this paper we begin the study of unitary-equivariant valuations on complex vector spaces. To state our results precisely, let us introduce some notation. We denote the space of continuous translation invariant \mathbb{R} -valued valuations on $\mathcal{K}(\mathbb{V})$ by $\mathrm{Val} = \mathrm{Val}(\mathbb{V})$. The subspace of k -homogeneous valuations (i.e. such that $Z(\lambda K) = \lambda^k Z(K)$ for any convex body K and $\lambda > 0$) is denoted by Val_k . Given a linear action of a closed subgroup $G \subset \mathrm{GL}(\mathbb{V})$ on a finite dimensional \mathbb{R} -vector space \mathbb{W} , we say that a valuation $Z : \mathcal{K}(\mathbb{V}) \rightarrow \mathbb{W}$ is G -equivariant if $Z(\varphi(K)) = \varphi Z(K)$ holds for any $\varphi \in G$ and $K \in \mathcal{K}(\mathbb{V})$. The space of \mathbb{W} -valued continuous translation invariant G -equivariant valuations is naturally identified with the subspace $(\mathrm{Val} \otimes \mathbb{W})^G$ of G -invariants in $\mathrm{Val} \otimes \mathbb{W}$ (the symbol \otimes is used in this paper for tensor products over \mathbb{R}).

We will focus on continuous translation invariant and $\mathrm{U}(m)$ -equivariant tensor-valued valuations on $\mathcal{K}(\mathbb{R}^{2m})$ for $m \geq 2$. So in our case, \mathbb{V} will be \mathbb{C}^m , viewed as the real vector

space \mathbb{R}^{2m} . The group G will be the unitary group $U(m)$ with its defining action on $\mathbb{V} = \mathbb{C}^m$, and $\mathbb{W} = S^d(\mathbb{R}^{2m})$ will be the d th symmetric tensor power over \mathbb{R} of \mathbb{V} . For the homogeneity degree $k = 0, \dots, 2m$ of a valuation, we set $\ell = \min\{k, 2m - k\}$, and write $\lfloor \frac{\ell}{2} \rfloor$ for the lower integer part of $\frac{\ell}{2}$. In the scalar valued case, Alesker [5] proved that $\dim_{\mathbb{R}} \text{Val}_k^{U(m)} = 1 + \lfloor \frac{\ell}{2} \rfloor$, and he provided two different sets of bases for $\text{Val}_k^{U(m)}$. In the vector valued case, Wannerer [35] obtained $\dim_{\mathbb{R}} (\text{Val}_k \otimes \mathbb{R}^{2m})^{U(m)} = 2 \lfloor \frac{\ell}{2} \rfloor$ (see also [33, Theorem 6.14]).

Our main result is the determination of the dimension of the space of $U(m)$ -equivariant tensor valued valuations of all ranks:

Theorem 1.1. *For $m \geq 2$, $k = 0, \dots, 2m$ and $d \geq 0$, using the notation $f := \lfloor \frac{d}{2} \rfloor$ and $\ell := \min\{k, 2m - k\}$, the dimension of $(\text{Val}_k \otimes S^d(\mathbb{R}^{2m}))^{U(m)}$ is as follows:*

	$\dim_{\mathbb{R}} ((\text{Val}_k \otimes S^d(\mathbb{R}^{2m}))^{U(m)})$
$d = 0$	$1 + \lfloor \frac{\ell}{2} \rfloor$
$d = 2f > 0, \quad \ell = 0$	1
$d = 2f > 0, \quad 1 \leq \ell < m$	$3\ell f^2 + 2 \lfloor \frac{\ell}{2} \rfloor - 2f^2 + 2f + 1$
$d = 2f > 0, \quad \ell = m$	$3mf^2 + 2 \lfloor \frac{m}{2} \rfloor - 3f^2 + 2f + 1$
$d = 2f + 1, \quad \ell = 0$	0
$d = 2f + 1, \quad 1 \leq \ell < m$	$3\ell f^2 + 3\ell f + 2 \lfloor \frac{\ell}{2} \rfloor - 2f^2$
$d = 2f + 1, \quad \ell = m$	$3mf^2 + 3mf + 2 \lfloor \frac{m}{2} \rfloor - 3f^2 - f$

To compute these dimensions, we first obtain the multiplicity in $\mathbb{C} \otimes \text{Val}_k$ of each irreducible $U(m)$ -module direct summand of $\mathbb{C} \otimes S^d(\mathbb{R}^{2m})$ (see Theorem 3.5). This intermediate result may also be of independent interest.

Once its dimension is known, a natural goal is to determine a basis for $(\text{Val}_k \otimes S^d(\mathbb{R}^{2m}))^{U(m)}$. So far this has been only known for rank $d = 0$ by Alesker [5]. Here we construct such a basis in the vector valued case (i.e. for $d = 1$). To this end, we use the area measures introduced by Wannerer [35] (see Section 4 for definitions). We write $S(\mathbb{V})$ to denote the unit sphere in a euclidean vector space \mathbb{V} , and $\text{Area}(\mathbb{V})$ to denote the space of smooth area measures, which is a certain class of translation invariant valuations taking values in the space of signed measures of $S(\mathbb{V})$. To each $\Psi \in \text{Area}(\mathbb{V})$, one can assign the smooth vector valued valuation $C(\Psi)$ defined by

$$C(\Psi)(K) = \int_{S(\mathbb{V})} u d\Psi(K, du)$$

for any convex body K .

For $\mathbb{V} = \mathbb{C}^m$, Wannerer [35] gave a complete description of the space $\text{Area}^{U(m)}$ of $U(m)$ -equivariant area measures. In particular, he introduced a certain family $\Delta_{k,q} \in \text{Area}^{U(m)}$ with specially nice properties (see Section 4).

Theorem 1.2. Consider the \mathbb{C} -vector space structure on $\text{Val}_k \otimes_{\mathbb{R}} \mathbb{C}^m$ given by $\alpha(Z \otimes u) = Z \otimes (\alpha u)$ for $\alpha \in \mathbb{C}$, $u \in \mathbb{C}^m$, and $Z \in \text{Val}_k$. Then, for $m \geq 2$, a \mathbb{C} -vector space basis of $(\text{Val}_k \otimes_{\mathbb{R}} \mathbb{C}^m)^{\text{U}(m)}$ is given by the family $C(\Delta_{k,q})$ where $0 \leq k < 2m$ and $\max(0, k - m) < q \leq \frac{k}{2}$.

One obtains the following immediate corollary:

Corollary 1.3. For $m \geq 2$, an \mathbb{R} -vector space basis of $(\text{Val}_k \otimes_{\mathbb{R}} \mathbb{R}^{2m})^{\text{U}(m)}$ is

$$\{C(\Delta_{k,q}), \sqrt{-1} \cdot C(\Delta_{k,q}) \mid 0, k - m < q \leq \frac{k}{2}\}.$$

2. Branching rules

Let us recall some background material from representation theory. For technical reasons we shall work with complex representations. So given a compact Lie group G , by a G -module we shall mean a finite dimensional complex vector space V endowed with an action of G via \mathbb{C} -linear transformations, such that the corresponding group homomorphism $G \rightarrow \text{GL}(V)$ is smooth. Write \overline{V} for the isomorphism class of V . The set of isomorphism classes of G -modules is a commutative monoid with addition given by $\overline{V} + \overline{W} = \overline{V \oplus W}$. The *Grothendieck group* of this monoid consists of formal differences of isomorphism classes of G -modules. It is a free abelian group R_G freely generated by τ_G , the set of isomorphism classes of irreducible G -modules. In fact R_G is a ring, called the *representation ring of G* , with multiplication given by $\overline{V} \cdot \overline{W} = \overline{V \otimes W}$. For ease of language or notation we shall frequently identify the isomorphism class $\overline{V} \in R_G$ with a G -module V representing the isomorphism class \overline{V} . We shall denote by $V \downarrow_H^G$ the restriction of the G -module V to a subgroup H of G .

Let us briefly sketch the strategy in the proof of Theorem 1.1. We will consider the complexifications

$$\begin{aligned} \text{Val}_{k,\mathbb{C}} &:= \mathbb{C} \otimes \text{Val}_k \\ \mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}) &:= \mathbb{C} \otimes \mathbb{S}^d(\mathbb{R}^{2m}). \end{aligned}$$

Note that while \mathbb{R}^{2m} is an irreducible real representation of $\text{U}(m)$, its complexification $\mathbb{S}_{\mathbb{C}}^1(\mathbb{R}^{2m}) = \mathbb{C} \otimes \mathbb{R}^{2m}$ is the direct sum of the natural $\text{U}(m)$ -module \mathbb{C}^m and its dual (over \mathbb{C}). It follows that $\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m})$ is a self-dual $\text{U}(m)$ -representation.

Clearly

$$\dim_{\mathbb{R}}(\text{Val}_k \otimes \mathbb{S}^d(\mathbb{R}^{2m}))^{\text{U}(m)} = \dim_{\mathbb{C}}(\text{Val}_{k,\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}))^{\text{U}(m)}. \quad (1)$$

Using that $\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m})$ is a self-dual $\text{U}(m)$ -representation, we obtain the standard isomorphism

$$(\mathrm{Val}_{k,\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}))^{U(m)} \cong \mathrm{Hom}_{U(m)}(\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}), \mathrm{Val}_{k,\mathbb{C}}). \quad (2)$$

Starting from known decompositions into irreducible $\mathrm{SO}(2m)$ -modules, and restricting those to $U(m)$, we will determine the decomposition into irreducible summands of the $U(m)$ -modules $\mathrm{Val}_{k,\mathbb{C}}$ and $\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m})$. Combining (1) and (2), and using Schur's Lemma will then yield the dimension of $(\mathrm{Val}_k \otimes \mathbb{S}^d(\mathbb{R}^{2m}))^{U(m)}$.

Next we turn to a parametrization of $\tau_{O(2m)}$ based on partitions, where $O(2m)$ is the full orthogonal group. This material can be found for example in [32], [19], [23] (the notation in these sources is different, and they mainly work in the context of complex linear algebraic groups and not with compact Lie groups). Set

$$\Pi_m = \{\lambda = (\lambda_1, \dots, \lambda_m) \mid \lambda_1 \geq \dots \geq \lambda_m \geq 0, \quad \lambda_i \in \mathbb{Z}\}.$$

For $p \leq m$ we shall treat Π_p as a subset of Π_m , by identifying $(\lambda_1, \dots, \lambda_p) \in \Pi_p$ with $(\lambda_1, \dots, \lambda_p, 0, \dots, 0)$. Set

$$\Pi_m^+ = \{\lambda \in \Pi_m \mid \lambda_m > 0\}.$$

Now we have

$$\tau_{O(2m)} = \{[\lambda], [\lambda]^\circ, [\mu] \mid \mu \in \Pi_m^+, \lambda \in \Pi_m \setminus \Pi_m^+\}.$$

Here $[\lambda]^\circ = [\lambda] \cdot \varepsilon$, where ε is the 1-dimensional $O(2m)$ -module given by the determinant. Note that [32, p. 418, Theorem 2] labels the elements of $\tau_{O(2m)}$ by partitions such that the sum of the lengths of the first two columns of their Young diagram is at most $2m$. Denoting by λ'_1 the length of the first column of the Young diagram of λ , our $[\lambda]^\circ$ corresponds to the partition whose Young diagram is obtained by replacing the first column of the Young diagram of λ by a column of length $2m - \lambda'_1$. For example, for the partition $\lambda = (1^k) = (1, \dots, 1)$ (with k components 1 for some $0 \leq k \leq m$), we have that $[\lambda]$ is the k th exterior power of the natural $O(2m)$ -module \mathbb{C}^{2m} , and $[\lambda]^\circ$ is the $(n - k)$ th exterior power of the natural $O(2m)$ -module \mathbb{C}^{2m} . We mention that the irreducible $O(2m)$ -modules are all defined over \mathbb{R} , that is, they are complexifications of irreducible real $O(2m)$ -modules.

Next we turn to the unitary group $U(m) = O(2m) \cap \mathrm{GL}(\mathbb{C}^m)$, where $\mathrm{GL}(\mathbb{C}^m)$ is the complex general linear group. We have

$$\tau_{U(m)} = \{\{\mu; \lambda\} \mid \lambda \in \Pi_p^+, \mu \in \Pi_q^+, p + q \leq m\}.$$

Here $\{\mu; \lambda\}$ is the irreducible $U(m)$ -module with highest weight

$$(\lambda_1, \dots, \lambda_p, 0, \dots, 0, -\mu_q, \dots, -\mu_1).$$

This is the restriction to the maximal compact subgroup $U(m)$ of the general linear group $\mathrm{GL}(\mathbb{C}^m)$ of its representation obtained by tensoring by the $(-\mu_1)$ th power

of the determinant representation the Schur module associated with the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_1, \dots) \in \Pi_m$ (cf. [32, p. 278, (8.1.3)]). For example, the natural $U(m)$ -module \mathbb{C}^m is $\{0; 1\}$, where in order to simplify the notation we write $\{j; i\} := \{(j, 0, \dots, 0); (i, 0, \dots, 0)\}$. The dual of the natural $U(m)$ -module \mathbb{C}^m is $\{1; 0\}$, and the k th exterior power of \mathbb{C}^m is $\{(0); (1^k)\}$.

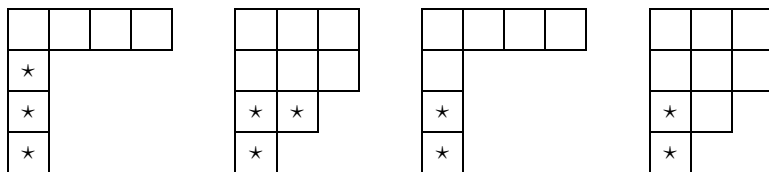
In certain formulae below elements of $R_{U(m)}$ denoted by $\{\mu; \lambda\}$ where $\lambda \in \Pi_p^+$, $\mu \in \Pi_q^+$, and $p + q > m$ will also occur. They can be expressed as an integral linear combination of elements in $\tau_{U(m)}$ by a repeated application of the following modification rule given by King [25, p. 433], see also [14, Section 3].

Set $h = p + q - m - 1$. In case it is possible to remove a boundary strip of h boxes from the Young diagram of λ , starting at the foot of the first column, and we obtain a Young diagram of a partition, then we denote this partition by $\lambda - h$. Otherwise we say that $\lambda - h$ does not exist. Similarly we define $\mu - h$. Now

$$\{\mu; \lambda\} = \begin{cases} 0 \in R_{U(m)} & \text{if any of } \lambda - h, \mu - h \text{ does not exist;} \\ (-1)^{x+y-1} \{\mu - h; \lambda - h\} & \text{if both } \lambda - h, \mu - h \text{ exist} \end{cases} \quad (3)$$

where the boundary h -strip removed from the Young diagram of λ ends in the x th column, and the boundary h -strip removed from the Young diagram of μ ends in the y th column. Note that in the special case $p + q = m + 1$, i.e. when $h = 0$, the outcome of the above rule is $\{\mu; \lambda\} = -\{\mu; \lambda\}$, which implies $\{\mu; \lambda\} = 0$ whenever $p + q = m + 1$.

Example 2.1. We draw the Young diagram of the partitions $(4, 1, 1, 1)$ and $(3, 3, 2, 1)$ with a boundary 3-strip (formed by the boxes denoted by \star), and the Young diagram of these partitions with a boundary 2-strip:



Thus we have $\{(4, 1, 1, 1); (3, 3, 2, 1)\} = \{(4); (3, 3)\} \in R_{U(4)}$

whereas $\{(4, 1, 1, 1); (3, 3, 2, 1)\} = 0 \in R_{U(5)}$.

For a triple of partitions $\lambda, \mu, \nu \in \Pi_m$ denote by $c_{\mu, \nu}^\lambda$ the associated Littlewood-Richardson coefficient (cf. e.g. [18, A.8]). It is determined by the equality

$$\{0; \mu\} \cdot \{0; \nu\} = \sum_{\lambda \in \Pi_m} c_{\mu, \nu}^\lambda \{0; \lambda\} \in R_{U(m)}.$$

The equality $\{0; \mu\} \cdot \{0; \nu\} = \{0; \nu\} \cdot \{0; \mu\}$ implies $c_{\mu, \nu}^\lambda = c_{\nu, \mu}^\lambda$.

Theorem 2.2 (King [25]). For any $[\lambda] \in \tau_{O(2m)}$ we have

$$[\lambda] \downarrow_{U(m)}^{O(2m)} = \sum_{\nu, \mu, \xi, (2\beta)' \in \Pi_m} c_{\mu, \nu}^\lambda c_{\xi, (2\beta)'}^\mu \{\xi; \nu\} \in R_{U(m)} \quad (4)$$

where $(2\beta)'$ stands for a partition such that all columns of its Young diagram have even length (i.e. the transpose of a partition with even parts).

The above statement can be found in [25, p. 440, (4.21)]; the notation B used there is explained at [25, p. 435-436]. For fixed $\xi, \nu, \beta \in \Pi_m$ we have

$$\begin{aligned} \sum_{\lambda, \mu \in \Pi_m} c_{\xi, \nu}^\mu c_{\mu, \beta}^\lambda \{0; \lambda\} &= (\{0; \xi\} \cdot \{0; \nu\}) \cdot \{0; \beta\} \\ &= (\{0; \xi\} \cdot \{0; \beta\}) \cdot \{0; \nu\} = \sum_{\lambda, \mu \in \Pi_m} c_{\xi, \beta}^\mu c_{\mu, \nu}^\lambda \{0; \lambda\}. \end{aligned}$$

It follows that for each $\lambda, \xi, \nu, \beta \in \Pi_m$ we have

$$\sum_{\mu \in \Pi_m} c_{\xi, \nu}^\mu c_{\mu, \beta}^\lambda = \sum_{\mu \in \Pi_m} c_{\xi, \beta}^\mu c_{\mu, \nu}^\lambda.$$

So (4) can be rewritten as

$$[\lambda] \downarrow_{U(m)}^{O(2m)} = \sum_{\nu, \mu, \xi, (2\beta)' \in \Pi_m} c_{\xi, \nu}^\mu c_{\mu, (2\beta)'}^\lambda \{\xi; \nu\} \in R_{U(m)}. \quad (5)$$

We note that in the special case when $\lambda, \xi, \nu \in \Pi_{\lfloor \frac{m}{2} \rfloor}$, the same formula for the multiplicity of $\{\xi; \nu\}$ as a summand in $[\lambda] \downarrow_{U(m)}^{O(2m)}$ appears also in [23, Section 2.3.1].

For the complexification of the symmetric tensor power $\mathbb{S}^d(\mathbb{R}^{2m})$ of the defining $O(2m)$ -module \mathbb{R}^{2m} we have

$$\overline{\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m})} = [d] + [d-2] + [d-4] + \cdots, \quad (6)$$

where $d \in \mathbb{N}_0$ is identified with $(d, 0, \dots, 0) \in \Pi_m$ (see e.g. [19, Section 5.2.3]).

Lemma 2.3. For $m \geq 2$ we have

$$[d] \downarrow_{U(m)}^{O(2m)} = \sum_{i+j=d} \{j; i\}$$

where in order to simplify the notation we write

$$\{j; i\} := \{(j, 0, \dots, 0); (i, 0, \dots, 0)\}.$$

Proof. By the Littlewood-Richardson Rule (see for example [32, p. 498, Theorem] or [18, p.456, A.8]) we have that if $c_{\alpha,\beta}^\lambda \neq 0$ for some $\lambda, \alpha, \beta \in \Pi_m$, then $\alpha_1 \leq \lambda_1, \dots, \alpha_m \leq \lambda_m$, and similarly $\beta_1 \leq \lambda_1, \dots, \beta_m \leq \lambda_m$. Moreover, $\sum_{i=1}^m \lambda_i = \sum_{i=1}^m \alpha_i + \sum_{i=1}^m \beta_i$. Apply (5) for the special case $\lambda = (d)$. By the above remark, $c_{\mu,(2\beta)}^{(d)} \neq 0$ holds only if $\beta = (0)$ and $\mu = (d)$, and in this case $c_{\mu,(2\beta)}^{(d)} = 1$. So (5) reduces to $[d] \downarrow_{U(m)}^{O(2m)} = \sum_{\xi, \nu \in \Pi_m} c_{\xi, \nu}^{(d)}$. Again by the above remark, $c_{\xi, \nu}^{(d)} \neq 0$ holds if and only if $\xi = (j)$ and $\nu = (d-j)$ for some $j \in \{0, \dots, d\}$, and $c_{(j);(d-j)}^{(d)} = 1$ by Pieri's rule (see e.g. [18, p. 455, A.7]). \square

Lemma 2.4. Fix non-negative integers i, j , and assume that $m \geq 2$.

- (i) Suppose $\xi - h = (j, 0^{m-1})$ and $\nu - h = (i, 0^{m-1})$, where $\xi \in \Pi_p^+$, $\nu \in \Pi_q^+$, $p, q \leq m$, and $h = p + q - m - 1$. Then $j > 0$, $i > 0$,

$$\xi = (j, 1^{m-1}), \quad \nu = (i, 1^{m-1}),$$

and in this case $\{\xi - h; \nu - h\} = -\{j; i\}$.

- (ii) There is no $\xi \in \Pi_m$ and positive integer h such that $\xi - h = (j, 1^{m-1})$.

Proof. (i) If $\xi - h = (j, 0^{m-1})$, then all boxes of the Young diagram of ξ below the first row must belong to the boundary h -strip that we remove to get the Young diagram of $(j, 0^{m-1})$. The inequality

$$p - 1 \leq h = p + q - m - 1$$

follows, implying that $q \geq m$. Thus we have $q = m$ and $p - 1 = h$. Similarly, $p = m$ and $h = m - 1$, so the boundary h -strip removed from the Young diagram of ξ (respectively ν) consists of the boxes in the first column beginning from the second row. Moreover, if $\xi = (j, 1^{m-1})$ and $\nu = (i, 1^{m-1})$, then applying (3), we have $x = y = 1$ in the formula, hence the modification rule (3) says $\{\xi - h; \nu - h\} = -\{j; i\}$.

(ii) Suppose $\xi - h = (j, 1^{m-1})$ for some $\xi \in \Pi_m$. Unless $h = 0$, to get the Young diagram of $\xi - h$ we must remove the bottom box in the first column of the Young diagram of ξ . Thus $\xi - h$ has at most $m - 1$ non-zero parts, whereas $(j, 1^{m-1})$ has m non-zero parts. \square

3. The space of valuations as a $U(m)$ -module

Throughout this section we assume $m \geq 2$. Introduce the notation

$$\langle \lambda \rangle := [\lambda] \downarrow_{SO(2m)}^{O(2m)}.$$

For $\lambda \in \Pi_m$ with $\lambda_m = 0$ the restriction $[\lambda] \downarrow_{SO(2m)}^{O(2m)}$ is an irreducible $SO(2m)$ -module with highest weight λ , whereas for $\mu \in \Pi_m$ with $\mu_m > 0$ the restriction $[\mu] \downarrow_{SO(2m)}^{O(2m)}$ is

Table 1
The pairs $([\lambda], \{j; i\})$ in Proposition 3.2.

		$\{0; e\}$ $e = 0$	$\{0; e\}$ $e = 1$	$\{0; e\}$ $e \geq 2$	$\{1; e - 1\}$ $e = 2$	$\{1; e - 1\}$ $e \geq 3$	$\{j; e - j\}$ $e - j \geq j \geq 2$
$[e]$	$e \neq 1$	1	—	1	1	1	1
$[(e, 2^h)]$	$2 \mid h > 0$	—	—	1	1	2	3
$[(e, 2^h)]$	$2 \nmid h$	—	—	0	1	1	1
$[(e + 2, 2^h)]$	$2 \nmid h$	1	1	1	1	1	1
$[(e - 2, 2^h)]$	$2 \nmid h$	—	—	0	—	0	1

the direct sum of two non-isomorphic irreducible $\mathrm{SO}(2m)$ -modules with highest weight μ and $(\mu_1, \dots, \mu_{m-1}, -\mu_m)$ (cf. [32, p. 422, Theorem]). Furthermore, for $k \in \{0, 1, \dots, m\}$ set

$$\ell(k) := \min\{k, 2m - k\}.$$

Theorem 3.1. (Alesker, Bernig, Schuster [7, Theorem 1]) The $\mathrm{SO}(2m)$ -module $\mathrm{Val}_{k, \mathbb{C}}$ admits the decomposition

$$\overline{\mathrm{Val}_{k, \mathbb{C}}} = \sum_{(g, 2^h) \in \Pi_{\ell(k)}, g \neq 1} \langle (g, 2^h) \rangle$$

where $(g, 2^h)$ stands for the partition $(g, 2, \dots, 2, 2)$, and where the notation introduced in Section 2 is extended in the obvious way to locally finite dimensional $\mathrm{SO}(2m)$ -representations in which each irreducible $\mathrm{SO}(2m)$ -module has finite multiplicity.

Note that for any $\lambda \in \Pi_m$ we have

$$\langle \lambda \rangle \downarrow_{\mathrm{U}(m)}^{\mathrm{SO}(2m)} = [\lambda] \downarrow_{\mathrm{U}(m)}^{\mathrm{O}(2m)},$$

whence

$$\overline{\mathrm{Val}_{k, \mathbb{C}}} \downarrow_{\mathrm{U}(m)}^{\mathrm{O}(2m)} = \sum_{(g, 2^h) \in \Pi_{\ell(k)}, g \neq 1} [(g, 2^h)] \downarrow_{\mathrm{U}(m)}^{\mathrm{O}(2m)} \in R_{\mathrm{U}(m)}.$$

Proposition 3.2. Table 1 above gives all pairs $([\lambda], \{j; i\})$ where $\lambda \in \Pi_m$ is of the form $\lambda = (g, 2^h)$, $g \neq 1$, $j \leq i$, such that $\{j; i\}$ has non-zero coefficient (given in the table) in the expansion (5) of $[\lambda] \downarrow_{\mathrm{U}(m)}^{\mathrm{O}(2m)}$.

Proof. Recall that by the Littlewood-Richardson Rule (see for example [32, p. 498, Theorem] or [18, p. 456, A.8]), $c_{\alpha, \delta}^\eta \neq 0$ for $\alpha \in \Pi_p$, $\delta \in \Pi_q$ implies that $\alpha_s \leq \eta_s$ and $\delta_s \leq \eta_s$ for each $s = 1, \dots, m$, $\sum_{s=1}^m \eta_s = \sum_{s=1}^m (\alpha_s + \delta_s)$, and $\eta \in \Pi_{p+q}$. Therefore if $c_{(i, 0^{m-1}), (j, 0^{m-1})}^{\mu, (g, 2^h)}$ is non-zero, then $\mu \in \{(i+j), (i+j-1, 1), (i+j-2, 2)\}$, $(2\beta)' = (2^t)$ with $t \in \{h+1, h, h-1\}$ or $(2\beta)' = (2^{h-1}, 1, 1)$, and $i+j \in \{g-2, g, g+2\}$.

Table 2

The non-zero multiplicities in Proposition 3.4 (ii).

		$\{0; e\}$ $e = 0$	$\{0; e\}$ $e = 1$	$\{0; e\}$ $e \geq 2$	$\{1; e - 1\}$ $e = 2$	$\{1; e - 1\}$ $e \geq 3$	$\{j; e - j\}$ $e - j \geq j \geq 2$
$[(e, 2^{m-1})]$	$2 \nmid m$	—	—	1	0	1	2
$[(e, 2^{m-1})]$	$2 \mid m$	—	—	0	0	0	0
$[(e + 2, 2^{m-1})]$	$2 \mid m$	1	1	1	1	1	1
$[(e - 2, 2^{m-1})]$	$2 \mid m$	—	—	0	—	0	1

Next we indicate the calculation of the value $\dim_{\mathbb{C}}(\text{Hom}_{U(m)}(\{j, e - j\}, [(e, 2^h)])) = 3$ for $e - j \geq j \geq 2, 2 \mid h > 0$. By Pieri's rule (see e.g. [18, p. 455, A.7]) we have $c_{(e-j), (j)}^{\mu} = 1$ for $\mu \in \{(e), (e - 1, 1), (e - 2, 2)\}$. It follows easily from the Littlewood-Richardson rule that for $\mu \in \{(e), (e - 1, 1), (e - 2, 2)\}$, we have $c_{\mu, (2\delta)}^{(e, 2^h)} \neq 0$ holds only if $(2\delta)' = (2^h)$, and in this case $c_{\mu, (2\delta)}^{(e, 2^h)} = 1$. We draw the corresponding Littlewood-Richardson tableau:

★	★	1	1	1
★	★			
1	1			

★	★	1	1	1
★	★			
1	2			

★	★	1	1	1
★	★			
2	2			

Or by similar considerations, the values $\dim_{\mathbb{C}}(\text{Hom}_{U(m)}(\{1; e - 1\}, [(e + 2, 2^h)])) = 1$ and $\dim_{\mathbb{C}}(\text{Hom}_{U(m)}(\{1; e - 1\}, [(e, 2^h)])) = 1$ for $2 \nmid h$ correspond to the tableaux

★	★	1	1	1
★	★			
★	★			
★	★			

★	★	1
★	★	
★	1	
★	2	

The remaining entries in Table 1 are calculated similarly. \square

Remark 3.3. The number in Table 1 in the row labeled by $[\lambda]$ and the column labeled by $\{j; i\}$ is the coefficient of $\{j; i\}$ in the expansion (5) of $[\lambda] \downarrow_{U(m)}^{O(2m)}$. Note that for certain $[\lambda]$ and $\{j; i\}$ this number is not equal to the *multiplicity* of the irreducible $U(m)$ -module $\{j; i\}$ as a summand in $[\lambda] \downarrow_{U(m)}^{O(2m)}$. Indeed, the expansion (5) of $[\lambda] \downarrow_{U(m)}^{O(2m)}$ may contain terms $\{\xi; \nu\}$ where $\{\xi; \nu\}$ does not belong to $\tau_{U(m)}$ (namely when $\xi \in \Pi_p^+$, $\nu \in \Pi_q^+$ with $p + q > m$). When such an element $\{\xi; \nu\}$ is expanded in terms of the basis $\tau_{U(m)}$ of $R_{U(m)}$ using King's modification rules (3), basis elements of the form $\{j; i\}$ may appear with non-zero integer coefficient. The necessary modifications in Table 1 in order to get the *multiplicity* of $\{j; i\}$ as a summand in $[\lambda] \downarrow_{U(m)}^{O(2m)}$ are taken into account in Proposition 3.4 below.

Proposition 3.4.

- (i) For $h < m - 1$, Table 1 gives the non-zero multiplicities of summands of the form $\{j; i\}$ with $j \leq i$ in the $U(m)$ -modules $[(g, 2^h)] \downarrow_{U(m)}^{O(2m)}$.
- (ii) The non-zero multiplicities of the summands of the form $\{j; i\}$ with $j \leq i$ in the $U(m)$ -modules $[(g, 2^{m-1})] \downarrow_{U(m)}^{O(2m)}$ are given in Table 2.

Proof. To get the multiplicity of $\{j; i\}$ in $[(g, 2^h)] \downarrow_{U(m)}^{O(2m)}$ we may start with the expansion (5), which expresses $[(g, 2^h)] \downarrow_{U(m)}^{O(2m)}$ as a non-negative integer linear combination of symbols $\{\xi; \nu\}$, where $\xi, \nu \in \Pi_m$. The terms $\{\xi; \nu\}$ not belonging to $\tau_{U(m)}$ need to be rewritten in terms of the basis $\tau_{U(m)}$ by a possibly iterated use of King's modification rules (3). Then we can collect the coefficient of $\{j; i\}$ in $[(g, 2^h)] \downarrow_{U(m)}^{O(2m)}$ with respect to the basis $\tau_{U(m)}$. By (3) and Lemma 2.4 (i), (ii), apart from $\{j; i\}$ the only $\{\xi; \nu\}$ whose expansion with respect to the basis $\tau_{U(m)}$ involves a basis element of the form $\{j; i\}$ is $\{(j, 1^{m-1}); (i, 1^{m-1})\}$, and the coefficient of $\{j; i\}$ in $\{(j, 1^{m-1}); (i, 1^{m-1})\}$ with respect to the basis $\tau_{U(m)}$ is -1 . Now $\{(j, 1^{m-1}); (e-j, 1^{m-1})\}$ occurs with coefficient 1 in the expansion (5) of $[(e, 2^{m-1})] \downarrow_{U(m)}^{O(2m)}$, and does not occur in $[(g, 2^h)] \downarrow_{U(m)}^{O(2m)}$ for $h < m-1$ or if $g \neq e$. Consequently, by Proposition 3.2, the non-zero multiplicities of $\{j; e-j\}$ (with $j \leq e-j$) in the $U(m)$ -modules $[(g, 2^h)] \downarrow_{U(m)}^{O(2m)}$ are given by Table 1 for $h < m-1$ or $g \neq e$, whereas for $h = m-1$, $g = e$ and $j \geq 1$ we need to subtract 1 from the corresponding entry of Table 1, and that is how we obtained Table 2. \square

Now we are in position to give the multiplicities of the irreducible $U(m)$ -modules $\{j; i\}$ as a summand of $\text{Val}_{k, \mathbb{C}}$.

Theorem 3.5. *The table below gives the non-zero multiplicities of summands of the form $\{j; i\}$ with $i \geq j$ in $\text{Val}_{k, \mathbb{C}} \downarrow_{U(m)}^{O(2m)}$ for $0 \leq k \leq 2m$:*

$\text{Val}_{k, \mathbb{C}}$	$\{0, 0\}$	$\{0; 1\}$	$\{0; e\}$ $e \geq 2$	$\{1; 1\}$	$\{1; e-1\}$ $e \geq 3$	$\{j; e-j\}$ $e-j \geq j \geq 2$
$k \in \{0, 2m\}$	1	0	0	0	0	0
$1 \leq \ell(k) < m$	$1 + \lfloor \frac{\ell(k)}{2} \rfloor$	$\lfloor \frac{\ell(k)}{2} \rfloor$	$\ell(k)$	$\ell(k) + \lfloor \frac{\ell(k)}{2} \rfloor$	$2\ell(k) - 1$	$3\ell(k) - 2$
$k = m$	$1 + \lfloor \frac{m}{2} \rfloor$	$\lfloor \frac{m}{2} \rfloor$	$m-1$	$m + \lfloor \frac{m}{2} \rfloor - 1$	$2m-2$	$3m-3$

Proof. By Theorem 3.1 and Proposition 3.4 we have that for $e \geq 3$, and $0 < \ell(k) < m$, the multiplicity of $\{1; e-1\}$ in $\text{Val}_{k, \mathbb{C}} \downarrow_{U(m)}^{O(2m)}$ is

$$1 + 2 \cdot \#\{h: 0 < h \leq \ell(k) - 1, 2 \mid h\} + 2 \cdot \#\{h: h \leq \ell(k) - 1, 2 \nmid h\} = 2\ell(k) - 1,$$

whereas the multiplicity of $\{1; 1\}$ in $\text{Val}_{k, \mathbb{C}} \downarrow_{U(m)}^{O(2m)}$ is

$$1 + \#\{h: 0 < h \leq \ell(k) - 1, 2 \mid h\} + 2 \cdot \#\{h: h \leq \ell(k) - 1, 2 \nmid h\} = \ell(k) + \left\lfloor \frac{\ell(k)}{2} \right\rfloor.$$

Similar considerations yield the rest of the table. \square

Proposition 3.6. *The dimensions of the spaces of $U(m)$ -module homomorphisms from $[e] \downarrow_{U(m)}^{O(2m)}$ to $\text{Val}_{k, \mathbb{C}}$ ($e \geq 0$; $0 \leq k \leq 2m$) are the following (for greater legibility, the notation $\ell := \ell(k)$ is used in some places):*

	$\dim_{\mathbb{C}}(\text{Hom}_{U(m)}([e], \text{Val}_{k, \mathbb{C}}))$
$e = 0, \quad \ell(k) > 0$	$1 + \lfloor \frac{\ell}{2} \rfloor$
$e = 1, \quad \ell(k) > 0$	$2 \lfloor \frac{\ell}{2} \rfloor$
$e = 2, \quad 0 < \ell(k) < m$	$3\ell + \lfloor \frac{\ell}{2} \rfloor$
$e \geq 3, \quad 0 < \ell(k) < m$	$3\ell e - 3\ell - 2e + 4$
$e = 2, \quad k = m$	$3m + \lfloor \frac{m}{2} \rfloor - 1$
$e \geq 3, \quad k = m$	$3me - 3m - 3e + 5$
$e = 0, \quad \ell(k) = 0$	1
$e \neq 0, \quad \ell(k) = 0$	0

Proof. By Lemma 2.3, the $U(m)$ -module $[e] \downarrow_{U(m)}^{O(2m)}$ is multiplicity free, and it is the direct sum of the modules $\{j; e - j\}$. Therefore by Schur's Lemma, $\dim_{\mathbb{C}}(\text{Hom}_{U(m)}([e], \text{Val}_{k, \mathbb{C}}))$ equals the sum of the multiplicities of the irreducible $U(m)$ -module direct summands $\{j; e - j\}$ of $\text{Val}_{k, \mathbb{C}} \downarrow_{U(m)}^{O(2m)}$. These sums of multiplicities can be easily determined using Theorem 3.5, and taking into account that the multiplicity of $\{j; i\}$ in $\text{Val}_k \downarrow_{U(m)}^{O(2m)}$ equals the multiplicity of $\{i; j\}$ by (5) and by $c_{\mu, \nu}^{\lambda} = c_{\nu, \mu}^{\lambda}$. \square

Proposition 3.7. *The dimensions of the spaces $\text{Hom}_{U(m)}(\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}), \text{Val}_{k, \mathbb{C}})$ ($d \geq 0$; $0 \leq k \leq 2m$) are the following (we shall use the notation $f := \lfloor \frac{d}{2} \rfloor$ and $\ell = \ell(k)$):*

	$\dim_{\mathbb{C}}(\text{Hom}_{U(m)}(\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}), \text{Val}_{k, \mathbb{C}}))$
$d = 0$	$1 + \lfloor \frac{\ell}{2} \rfloor$
$d = 2f > 0, \quad \ell = 0$	1
$d = 2f > 0, \quad 1 \leq \ell < m$	$3\ell f^2 + 2 \lfloor \frac{\ell}{2} \rfloor - 2f^2 + 2f + 1$
$d = 2f > 0, \quad \ell = m$	$3mf^2 + 2 \lfloor \frac{m}{2} \rfloor - 3f^2 + 2f + 1$
$d = 2f + 1, \quad \ell = 0$	0
$d = 2f + 1, \quad 1 \leq \ell < m$	$3\ell f^2 + 3\ell f + 2 \lfloor \frac{\ell}{2} \rfloor - 2f^2$
$d = 2f + 1, \quad \ell = m$	$3mf^2 + 3mf + 2 \lfloor \frac{m}{2} \rfloor - 3f^2 - f$

Proof. The $U(m)$ -module $\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m})$ is multiplicity free by (6) and by Lemma 2.3. Therefore by Schur's Lemma and by (6) we have

$$\dim_{\mathbb{C}}(\text{Hom}_{U(m)}(\mathbb{S}_{\mathbb{C}}^d(\mathbb{R}^{2m}), \text{Val}_{k, \mathbb{C}})) = \sum_{p=0}^{\lfloor \frac{d}{2} \rfloor} \dim_{\mathbb{C}}(\text{Hom}_{U(m)}([d - 2p], \text{Val}_{k, \mathbb{C}})).$$

Thus the statement follows easily from Proposition 3.6. \square

Proof of Theorem 1.1. The result is an immediate consequence of Proposition 3.7 by (1) and (2). \square

Remark 3.8.

(i) The case $d = 0$ in Proposition 3.7, i.e. the equality

$$\dim_{\mathbb{C}}((\text{Val}_{k,\mathbb{C}})^{\text{U}(m)}) = 1 + \left\lfloor \frac{\ell(k)}{2} \right\rfloor$$

is due to Alesker [4, Theorem 6.1].

(ii) The special case $d = 1$ in Proposition 3.7, namely that

$$\dim_{\mathbb{C}}((\text{Val}_{k,\mathbb{C}} \otimes \mathbb{S}_{\mathbb{C}}^1(\mathbb{R}^{2m})^{\text{U}(m)}) = 2 \left\lfloor \frac{\ell(k)}{2} \right\rfloor,$$

is due to Wannerer [35], see also [33, Theorem 6.14].

4. A basis for vector valued valuations in terms of area measures

In this section we construct a \mathbb{C} -vector space basis of $(\text{Val} \otimes_{\mathbb{R}} \mathbb{C}^m)^{\text{U}(m)}$, the complex vector space of $\text{U}(m)$ -equivariant translation invariant vector valued valuations on \mathbb{C}^m for $m \geq 2$.

In the scalar valued case, Bernig and Fu [10] constructed a basis of $\text{Val}^{\text{U}(m)}$ consisting of the so-called hermitian intrinsic volumes $\mu_{k,q}$, defined for $0 \leq k \leq 2m$ and $0, k - m \leq q \leq \frac{k}{2}$. These valuations are even and hence characterized by their Klain function [26]. The Klain function of an even valuation $\varphi \in \text{Val}_k(\mathbb{V})$ is a function Kl_{φ} on the k -Grassmannian $\text{Gr}_k(\mathbb{V})$ given by $\varphi(A) = \text{Kl}_{\varphi}(E) \text{vol}_k(A)$ for $A \subset E \in \text{Gr}_k(\mathbb{V})$.

For $k \leq m$, the Klain function of $\mu_{k,q}$ is

$$\text{Kl}_{\mu_{k,q}}(E) = \sum_{i=q}^{\lfloor \frac{k}{2} \rfloor} (-1)^{i+q} \binom{i}{q} \sigma_i(\cos^2 \theta_1, \dots, \cos^2 \theta_{\lfloor k/2 \rfloor}) \quad (7)$$

where θ_i is the i th elementary symmetric function, and $\theta_1, \dots, \theta_{\lfloor k/2 \rfloor}$ are the Kähler angles of the k -dimensional linear subspace E . These angles are characterized as follows. Let ψ_E be the endomorphism of E that maps $u \in E$ to the orthogonal projection of $\sqrt{-1}u$ to E . Then ψ_E has eigenvalues

$$\pm \sqrt{-1} \cos \theta_1, \dots, \pm \sqrt{-1} \cos \theta_{\lfloor k/2 \rfloor},$$

plus a zero eigenvalue if k is odd. For $k > m$,

$$\text{Kl}_{\mu_{k,q}}(E) = \text{Kl}_{\mu_{2m-k, m-k+q}}(E^{\perp}). \quad (8)$$

On the other hand, Wannerer [34] introduced the space $\text{Area}(\mathbb{V})$ of smooth area measures on a euclidean vector space \mathbb{V} . These are certain translation invariant valuations on \mathbb{V} , taking values on the space of signed measures of the unit sphere $S(\mathbb{V})$.

Thus, if $\Phi \in \text{Area}(\mathbb{V})$ and $A \subset \mathbb{V}$ is a convex body, then $\Phi(A, \cdot)$ is a signed measure on $S(\mathbb{V})$. The globalization map $\text{glob}: \text{Area}(\mathbb{V}) \rightarrow \text{Val}(\mathbb{V})$ and the centroid map $C: \text{Area}(\mathbb{V}) \rightarrow \text{Val}(\mathbb{V}) \otimes \mathbb{V}$ are then defined by

$$\text{glob}(\Phi)(A) = \Phi(A, S(\mathbb{V})), \quad C(\Phi)(A) = \int_{S(\mathbb{V})} u \, d\Phi(A, u).$$

Given a linear subspace $E \subset \mathbb{V}$ there exists a restriction map r from $\text{Area}(\mathbb{V})$ to $\text{Area}(E)$ characterized as follows. Given a Borel set $U \subset S(V)$, let $\overline{U} = (U + E^\perp) \cap S(\mathbb{V})$. The restriction of $\Phi \in \text{Area}(\mathbb{V})$ is given by

$$r(\Phi)(A, U) = \Phi(A, \overline{U}), \quad A \in \mathcal{K}(E), U \subset S(E). \quad (9)$$

For $\mathbb{V} = \mathbb{C}^m$, the space $\text{Area}_k^{\text{U}(m)}$ of k -homogeneous $\text{U}(m)$ -invariant smooth area measures was described in [34]. We will need the following.

Proposition 4.1 ([34]). *Given $0 \leq k < 2m$, there exists a family $\Delta_{k,q} \in \text{Area}_k^{\text{U}(m)}$ with $0, k - m \leq q \leq \frac{k}{2}$ such that*

- (i) $\text{glob}(\Delta_{k,q}) = \mu_{k,q}$
- (ii) *for every polytope P , and every Borel set $U \subset S(\mathbb{C}^m)$*

$$\Delta_{k,q}(P, U) = \sum_{F \in \mathcal{F}_k} \text{Kl}_{\mu_{k,q}}(\vec{F}) \frac{\text{vol}_{2m-k-1}(N(P, F) \cap U)}{\text{vol}_{2m-k-1}(S^{2m-k-1})} \text{vol}_k(F) \quad (10)$$

where \mathcal{F}_k is the set of k -dimensional faces, $N(P, F)$ is the set of outer unit normal vectors to P at points of F , and \vec{F} is the k -dimensional linear subspace parallel to F .

- (iii) *The restriction $r: \text{Area}(\mathbb{C}^{m+l}) \rightarrow \text{Area}(\mathbb{C}^m)$ corresponding to the inclusion $\mathbb{C}^m \rightarrow \mathbb{C}^{m+l}$ fulfills $r(\Delta_{k,q}) = \Delta_{k,q}$ if $q \geq k - m$.*

Given a p -dimensional real subspace $E \subset \mathbb{C}^m$ and the corresponding restriction map r , it follows from (10) that

$$C(r(\Delta_{k,q}))(A) = c_{m,p,k} C(\Delta_{k,q})(A), \quad A \in \mathcal{K}(E), \quad (11)$$

for $c_{m,p,k} \neq 0$ depending only on m, p, k .

It was shown in [34] that the family $C(\Delta_{k,q})$ with $0, k - m < q \leq \frac{k}{2}$ is \mathbb{R} -linearly independent. Since we already know that $\dim_{\mathbb{C}}((\text{Val}_k \otimes_{\mathbb{R}} \mathbb{C}^m)^{\text{U}(m)}) = \lfloor \frac{\ell(k)}{2} \rfloor$ (see Remark 3.8 (ii)), to prove Theorem 1.2 we only need to show that the above family is in fact \mathbb{C} -linearly independent.

Proof of Theorem 1.2. Let us first assume $k \geq m$. For $1 \leq r \leq \lfloor \frac{2m-k}{2} \rfloor$ consider the following element of $\text{Area}_k^{\text{U}(m)}$:

$$\Psi_{k,r} = \sum_{i=r}^{\lfloor \frac{2m-k}{2} \rfloor} \binom{i}{r} \Delta_{k,k-m+i}.$$

By (7) and (8) we have

$$\text{Kl}_{\text{glob}(\Psi_{k,r})}(F) = \sigma_r(\cos^2 \theta_1, \dots, \cos^2 \theta_{\lfloor \frac{2m-k}{2} \rfloor}),$$

where the θ_i refer to the Kähler angles of F^\perp .

We show next that the vector valuations $C(\Psi_{k,r})$, and hence the $C(\Delta_{k,q})$ with $q > k - m$, are linearly independent elements of the complex vector space $(\text{Val}_k \otimes_{\mathbb{R}} \mathbb{C}^m)^{\text{U}(m)}$. Since this space has dimension $\lfloor \frac{2m-k}{2} \rfloor$, the statement will follow.

Given $k+1-m \leq q \leq \frac{k+1}{2}$, we may consider

$$E = \mathbb{C}^q \oplus \mathbb{R}^{k-2q+1}$$

which is a subspace of \mathbb{C}^m with $\dim_{\mathbb{R}} E = k+1$. Let T_q be the $(k+1)$ -dimensional simplex in E with vertices

$$0, e_1, \sqrt{-1}e_1, \dots, e_q, \sqrt{-1}e_q, e_{q+1}, \dots, e_{k-q+1}$$

We proceed to compute $C(\Psi_{k,r})(T_q)$.

The linear spaces parallel to the k -faces F of T_q with normal vectors $u \in \{e_1, \sqrt{-1}e_1, \dots, e_q, \sqrt{-1}e_q\}$ belong to the $\text{U}(m)$ -orbit of $\mathbb{C}^{q-1} \oplus \mathbb{R}^{k-2q+2} = (\mathbb{C}^{m-k+q-1} \oplus \mathbb{R}^{k-2q+2})^\perp$. Hence the Kähler angles θ_i of F^\perp are given by

$$\cos \theta_1 = \dots = \cos \theta_{m-k+q-1} = 1, \quad \cos \theta_{m-k+q} = \dots = \cos \theta_{\lfloor \frac{2m-k}{2} \rfloor} = 0$$

which yields

$$\text{Kl}_{\text{glob}(\Psi_{k,r})}(F) = \binom{m-k+q-1}{r}.$$

Similarly, the linear spaces parallel to the k -faces F of T_q with normal vectors $u \in \{e_{q+1}, \dots, e_{k-q+1}\}$ belong to the $\text{U}(m)$ -orbit of $\mathbb{C}^q \oplus \mathbb{R}^{k-2q} = (\mathbb{C}^{m-k+q} \oplus \mathbb{R}^{k-2q})^\perp$. Hence the Kähler angles θ_i of F^\perp are given by

$$\cos \theta_1 = \dots = \cos \theta_{m-k+q} = 1, \quad \cos \theta_{m-k+q+1} = \dots = \cos \theta_{\lfloor \frac{2m-k}{2} \rfloor} = 0$$

which yields

$$\text{Kl}_{\text{glob}}(\Psi_{k,r})(F) = \binom{m-k+q}{r}.$$

It remains to consider the k -face F of T_q opposite to 0. Its outer unit normal vector in E is

$$v = \frac{1}{\sqrt{k+1}}(e_1 + \sqrt{-1}e_1 + \cdots + e_q + \sqrt{-1}e_q + e_{q+1} + \cdots + e_{k-q+1})$$

and its normal space F^\perp in \mathbb{C}^m is spanned by

$$v, \sqrt{-1}e_{q+1}, \dots, \sqrt{-1}e_{k-q+1}, e_{k-q+2}, \sqrt{-1}e_{k-q+2}, \dots, e_m, \sqrt{-1}e_m.$$

The Kähler angles θ_i of F^\perp can be obtained from the eigenvalues of ψ_{F^\perp} . The outcome of this computation is

$$\cos \theta_1 = \cdots = \cos \theta_{m-k+q-1} = 1, \quad \cos \theta_{m-k+q} = \sqrt{\frac{k-2q+1}{k+1}}$$

and $\cos \theta_i = 0$ for $i > m-k+q$, which yields

$$\text{Kl}_{\text{glob}}(\Psi_{k,r})(F) = \binom{m-k+q-1}{r} + \binom{m-k+q-1}{r-1} \frac{k-2q+1}{k+1}.$$

Finally, by (10) and (11)

$$\begin{aligned} c_{m,k+1,k} C(\Psi_{k,r})(T_q) &= \frac{1}{2 \cdot k!} \binom{m-k+q-1}{r} (-e_1 - \sqrt{-1}e_1 - \cdots - e_q - \sqrt{-1}e_q) \\ &\quad + \frac{1}{2 \cdot k!} \binom{m-k+q}{r} (-e_{q+1} - \cdots - e_{k-q+1}) \\ &\quad + \frac{\sqrt{k+1}}{2 \cdot k!} \left[\binom{m-k+q-1}{r} + \binom{m-k+q-1}{r-1} \frac{k-2q+1}{k+1} \right] v \\ &= \binom{m-k+q-1}{r-1} \frac{1}{2(k+1)!} \left[(k-2q+1)(e_1 + \sqrt{-1}e_1 + \cdots + \sqrt{-1}e_q) \right. \\ &\quad \left. - 2q(e_{q+1} + \cdots + e_{k-q+1}) \right] \end{aligned}$$

Suppose now that $\sum_i a_i C(\Psi_{k,i}) = 0$ for $a_1, \dots, a_{\lfloor \frac{2m-k}{2} \rfloor} \in \mathbb{C}$. For each $i = 1, \dots, \lfloor \frac{2m-k}{2} \rfloor$, pick $q = k-m+i$. By the above expression we get

$$C(\Psi_{k,r})(T_q) = 0 \Leftrightarrow r > i.$$

By induction this shows $a_1 = \cdots = a_{\lfloor \frac{2m-k}{2} \rfloor} = 0$.

The case $k < m$ can be deduced from the previous one as follows. Suppose

$$\sum_{i=1}^{\lfloor k/2 \rfloor} a_i C(\Delta_{k,i}) = 0 \quad (12)$$

with $a_i \in \mathbb{C}$. Consider the inclusion $\mathbb{C}^k \subset \mathbb{C}^m$. By item (iii) in Proposition 4.1 and (11), the relation (12) holds in \mathbb{C}^k . Hence, we can apply the previous case to conclude $a_i = 0$ for all i . \square

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