Contemporary Mathematics
Volume **259**, 2000

The adjacency matrix of a directed graph over the Grassmann algebra

Péter Körtesi and Jenő Szigeti

ABSTRACT. The main aim of this note is to give a concise and transparent reformulation of Swan's graph theoretical theorem, equivalent to the classical Amitsur-Levitzki theorem on the maximal PI of matrix algebras. Labeling the edges of a directed graph Γ by the anticommutative generators of a Grassmann algebra, we define its adjacency matrix in a usual way and prove that this matrix is nilpotent of index $2n$, where n denotes the number of vertices.

1. PRELIMINARIES

Given a directed graph Γ on the vertex set $V = \{1, 2, ..., n\}$ (loops and multiple edges are allowed), its adjacency matrix can be defined in a natural way, it is an $n \times n$ matrix $A \in M_n(\mathbb{Z})$ having the number $\alpha(i,j)$ of edges oriented from vertex i to vertex j in the (i,j) slot. As is well known, the powers of this matrix can be described in terms of directed sequences of edges in Γ. Sometimes the use of indeterminates is more convenient in similar matrix constructions starting from directed or undirected graphs; a typical example is the so-called skew symmetric adjacency matrix in Tutte's theorem on the existence of complete matchings (in undirected graphs).
If the set $E = \{x_1, x_2, ..., x_N\}$ of oriented edges of Γ is considered as a subset of the indeterminates generating the commutative polynomial algebra $\mathbb{Q}[x_1, x_2, ...]$ over the field of rational numbers, then it is also common to replace the $\alpha(i,j)$'s in the above definition of A by the sum of the x_r's starting from vertex i and terminating at vertex j. The use of the standard $n \times n$ matrix units E_{ij}, $1 \leq i, j \leq n$ enables us to write

$$(1) \qquad A(X) = \sum_{r=1}^{N} x_r E_{\sigma(r)\tau(r)}$$

for this new adjacency matrix, where $\sigma(r) \in V$ and $\tau(r) \in V$ denote the tail end and the head end of the oriented edge x_r, respectively. Clearly, the powers of $A(X) \in M_n(\mathbb{Q}[x_1, x_2, ...])$ encode more information about the directed sequences of edges in Γ than the powers of A. The structure of such sequences can be completely read off the powers of $A(X)$ if we don't allow the x_r's to commute, i.e. if we consider $A(X)$ as an $n \times n$ matrix over the non-commutative polynomial algebra

1991 *Mathematics Subject Classification.* Primary 05C50, Secondary 15A75.
Key words and phrases. Adjacency matrix of a directed graph, Grassmann algebra.
The second author acknowledges the support of OTKA of Hungary, grant no. T23434.

$\mathbb{Q}\langle x_1, x_2, ...\rangle$. The main idea of this note is to consider $A(X)$ as a matrix over the Grassmann (exterior) algebra

$$G = \mathbb{Q}\langle x_1, x_2, ... \mid x_r x_s + x_s x_r = 0 \text{ for all } 1 \le r, s\rangle .$$

generated by the x_r's. We note that G is an associative algebra (infinite dimensional) over $\mathbb{Q}$ with the anticommutative defining relations $x_s x_r = -x_r x_s$ (including $x_r^2 = 0$) on the generators. As a consequence of these relations, for $\pi \in \mathrm{Sym}(\{1, 2, ..., N\}$ we get that

$$(2) \qquad x_{\pi(1)} x_{\pi(2)} ... x_{\pi(N)} = \mathrm{sgn}(\pi) x_1 x_2 ... x_N \quad \text{in } G.$$

Each element $g \in G$ can be uniquely written in the form

$$g = c_g + \sum_{1 \le i_1 < i_2 < ... < i_k} c_g(i_1, i_2, ..., i_k) x_{i_1} x_{i_2} ... x_{i_k}$$

where $c_g, c_g(i_1, i_2, ..., i_k) \in \mathbb{Q}$. We also note that G is Lie nilpotent of index 2, i.e. that $[[f, g], h] = 0$ for all $f, g, h \in G$ ($hg \neq -gh$ in general).

Such a modification of the algebraic environment under $A(X)$ will result in a dramatic change in the behavior of its powers. Before proceeding to the formulation of our result, some further comments are in order.

The famous Amitsur-Levitzki theorem on the minimal PI's of matrix algebras was published in 1950 ([1]); several essentially different proofs appeared in the literature since then. Here we deal two of them: Swan's proof ([5]) is of purely graph theoretical in nature, while -possibly the shortest- Rosset's ([3]) uses the Grassmann algebra. The idea of considering $A(X)$ in $M_n(G)$ was inspired by the above works. In fact, we take Rosset's starting point and use one of his tools to get a particularly transparent theorem on $A(X)$. Swan's theorem on the numbers of even and odd directed Eulerian paths (which is an equivalent reformulation of the Amitsur-Levitzki theorem) will appear as an easy consequence of this theorem.

2. THE NILPOTENCY OF $A(X)$

The identity $A(X)^{N+1} = 0$ in $M_n(G)$ immediately follows from (1) and the relations satisfied by the generators of G. Our main result gives a lesser trivial bound for the index of nilpotency of $A(X)$.

Theorem. *Let $A(X)$ be the adjacency matrix of a directed graph $\Gamma = (V, E)$ over the Grassmann algebra G. Then we have $A(X)^{2n} = 0$ in $M_n(G)$, where $n = |V|$, $E = \{x_1, x_2, ..., x_N\}$ and $G = \mathbb{Q}\langle x_1, x_2, ... \mid x_r x_s + x_s x_r = 0 \text{ for all } 1 \le r, s\rangle$.*

We shall make use of the following consequence of the Cayley-Hamilton theorem and the Newton formulae (see in [3]).

Lemma. *Let Ω be a commutative algebra (with 1) over a field of characteristic zero and $B \in M_n(\Omega)$ an $n \times n$ matrix over Ω, then $\mathrm{tr}(B) = \mathrm{tr}(B^2) = ... = \mathrm{tr}(B^n) = 0$ implies that $B^n = 0$.*

Proof of the Theorem. The multiplication rule of the standard matrix units ensures that the (i, j) entry of the power $A(X)^k$ is

$$(3) \qquad \sum x_{r_1} x_{r_2} ... x_{r_k} ,$$

where the sum is taken over all sequences $x_{r_1}, x_{r_2}, ..., x_{r_k}$ of distinct edges such that

$$i = \sigma(r_1), \tau(r_1) = \sigma(r_2), \tau(r_2) = \sigma(r_3), ..., \tau(r_{k-1}) = \sigma(r_k), \tau(r_k) = j.$$

These sequences are directed paths in Γ from vertex i to vertex j (summands corresponding to directed sequences involving an edge more than once may appear, but vanish as a consequence of the relations on the x_r's). Set $B = A(X)^2$. Clearly, $B \in M_n(G_0)$ with G_0 being the even part of the Grassmann algebra, generated by the monomials in the x_r's of even length. Since G_0 is a commutative $\mathbb{Q}$-subalgebra of G, the above Lemma can be applied to the matrix B. It is enough to show that $\mathrm{tr}(B^k) = \mathrm{tr}(A(X)^{2k}) = 0$ for all integers $1 \le k \le n$. In view of (3), we have

$$\text{(4)} \qquad \mathrm{tr}(A(X)^{2k}) = \sum x_{r_1} x_{r_2} ... x_{r_{2k}} \ ,$$

where the sum is taken over all sequences $x_{r_1}, x_{r_2}, ..., x_{r_{2k}}$ of distinct edges such that

$$\tau(r_1) = \sigma(r_2), \tau(r_2) = \sigma(r_3), ..., \tau(r_{2k-1}) = \sigma(r_{2k}), \tau(r_{2k}) = \sigma(r_1).$$

If $x_{r_1} x_{r_2} ... x_{r_{2k}}$ is a summand in (4) then $x_{r_2} ... x_{r_{2k}} x_{r_1}$ also occurs in (4), moreover

$$x_{r_1} x_{r_2} ... x_{r_{2k}} + x_{r_2} ... x_{r_{2k}} x_{r_1} = 0 \text{ in } G.$$

Thus each directed circuit in Γ of length $2k$ gives rise to exactly k pairwise disjoint pairs of summands of the form (4). In consequence, we obtain that $\mathrm{tr}(A(X)^{2k}) = 0$.

Corollary. *By* (2), *the* (i, j) *entry of the power* $A(X)^N$ *is*

$$\sum x_{\pi(1)} x_{\pi(2)} ... x_{\pi(N)} = \left(\sum \mathrm{sgn}(\pi)\right) x_1 x_2 ... x_N \ ,$$

where the sum is over all directed Eulerian paths $x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(N)}$ *of* Γ *from* i *to* j. *If* $N \ge 2n$ *then our Theorem gives that* $A(X)^N = 0$, *i.e. that* $\sum \mathrm{sgn}(\pi) = 0$ *for the above sum. This is essentially Swan's theorem.*

Remark. *The graph theoretic analogue of the classical Kostant-Rowen theorem* ([2],[4]) *on the standard identity for skew symmetric matrices allows us to formulate the following statement:* $\left(A(X) - A(X)^{\mathrm{T}}\right)^{2n-2} = 0$ *in* $M_n(G)$ *for all directed graphs on* n *vertices (here* $A(X)^{\mathrm{T}}$ *denotes the transpose of* $A(X)$*).*

3. REFERENCES

1. Amitsur, S.A. and Levitzki, J., *Minimal identities for algebras,* Proc. Amer. Math. Soc. **1** (1950),449-463.
2. Kostant, B., *A theorem of Frobenius, a theorem of Amitsur-Levitzki and cohomology theory,* Indiana J. Math. **7** (1958),237-264.
3. Rosset, S., *A new proof of the Amitsur-Levitzki identity,* Israel J. Math., **23** (1976), 187-188.
4. Rowen, L.H., *Standard polynomials in matrix algebras,* Trans. Amer. Math. Soc. **19** (1974),253-284.
5. Swan, R.G., *An application of graph theory to algebra,* Proc. Amer. Math. Soc. **14** (1963),367-380.

Institute of Mathematics, University of Miskolc, Miskolc, Hungary 3515
E-mail address: matkp@gold.uni-miskolc.hu, matszj@gold.uni-miskolc.hu