

The 2-blocking number and the upper chromatic number of $\text{PG}(2, q)$

Gábor Bacsó[†], Tamás Héger[‡], Tamás Szőnyi^{†‡} *

[†]Computer and Automation Institute
Hungarian Academy of Sciences
H-1111 Budapest, Kende u. 13–17, Hungary

[‡]Eötvös Loránd University
Department of Computer Science
H-1117 Budapest, Pázmány P. sétány 1/C, Hungary

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Abstract

A 2-fold blocking set (double blocking set) in a finite projective plane Π is a set of points, intersecting every line in at least two points. The minimum number of points in a double blocking set of Π is denoted by $\tau_2(\Pi)$. Let $\text{PG}(2, q)$ be the Desarguesian projective plane over $\text{GF}(q)$, the finite field of q elements. We show that if q is odd, not a prime, and r is the order of the largest proper subfield of $\text{GF}(q)$, then $\tau_2(\text{PG}(2, q)) \leq 2(q + (q - 1)/(r - 1))$.

For a finite projective plane Π , let $\bar{\chi}(\Pi)$ denote the maximum number of classes in a partition of the point-set, such that each line has at least two points in some partition class. It can easily be seen that $\bar{\chi}(\Pi) \geq v - \tau_2(\Pi) + 1$ (\star) for every plane Π on v points. Let $q = p^h$, p prime. We prove that for $\Pi = \text{PG}(2, q)$, equality holds in (\star) if q and p are large enough.

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1 Introduction

The almost classical area of finite geometries and a very young area, the coloring theory of mixed hypergraphs are combined in this paper. As for general information on the latter, we refer to [21]. Mostly a special case of mixed hypergraphs will be discussed here. To be clear, we give some definitions.

A \mathcal{C} -hypergraph $\mathcal{H} = (X, \mathcal{C})$ has an underlying vertex set X and a set system \mathcal{C} over X . A *proper vertex coloring* of \mathcal{H} is a mapping φ from X to a set of colors, such that each \mathcal{C} -edge $C \in \mathcal{C}$ has at least two vertices with a *common* color, or equivalently $|\varphi(C)| < |C|$.

In particular, a *strict k -coloring* is a mapping $\varphi : X \rightarrow \{1, \dots, k\}$ that uses each of the k colors on at least one vertex and satisfies the requirements above. A different but equivalent view is to consider *color partitions* $X_1 \cup \dots \cup X_k = X$ with k nonempty classes. The correspondence between these two approaches is established by the rule $X_i = \varphi^{-1}(i)$; that is, $x \in X_i$ if and only if $\varphi(x) = i$, for $i = 1, 2, \dots, k$. The *upper chromatic number* of \mathcal{H} , denoted by $\bar{\chi}(\mathcal{H})$, is the largest k admitting a strict k -coloring.

Let us mention here the tight connection with *rainbow colorings* [10]. The coloring above is just a coloring where rainbow objects are forbidden.

As one can see, the above defined hypergraph coloring problem is a counterpart of the traditional one. The general mixed hypergraph model combines the two. This mixed model is better known but here we do not discuss it. These concepts, providing a powerful model for many types of problems, were introduced by Voloshin [19, 20].

Throughout the paper, Π_q denotes an arbitrary finite projective plane of order q , $\text{GF}(q)$ denotes the finite field of q elements, $\text{PG}(2, q)$ and $\text{AG}(2, q)$ stand for the projective and affine plane over $\text{GF}(q)$, respectively. The point-set and the line-set of Π_q will be denoted by \mathcal{P} and \mathcal{L} , respectively, and let $v = |\mathcal{P}| = q^2 + q + 1$.

A projective plane may be considered as a hypergraph, whose vertices and hyperedges are the points and the lines of the plane, respectively. We consider every line as a \mathcal{C} -edge, and specialize the definition of a proper coloring and the upper chromatic number for this specific case.

Definition 1.1. *We say that a coloring of the points of a finite projective plane Π is proper, if every line contains at least two points of the same color. The upper chromatic number $\bar{\chi}(\Pi)$ is the maximum number of colors one may use in a proper coloring.*

Note that if we merge color classes of a proper coloring (i.e., replace two color classes C_i and C_j by $C_i \cup C_j$), then the coloring obtained is also proper.

In [1] the following general bound is given on the upper chromatic number for any projective plane, as a function of the order, and thus a ten-year-old open problem is solved in the coloring theory of mixed hypergraphs.

Result 1.2 (Bacsó, Tuza [1]). *As $q \rightarrow \infty$, any projective plane Π_q of order q satisfies*

$$\bar{\chi}(\Pi_q) \leq q^2 - q - \sqrt{q}/2 + o(\sqrt{q}).$$

Let us consider now a parameter, frequently investigated in connection with finite planes.

Definition 1.3. *In the plane Π , $B \subset \Pi$ is a t -fold blocking set if every line intersects B in at least t points. For $t = 2$, we call B a double blocking set.*

Definition 1.4. *Let $\tau_t = \tau_t(\Pi)$ denote the size of a smallest t -fold blocking set in Π . In particular, τ_2 denotes the size of a smallest double blocking set in Π . The name of this quantity is the t -blocking number of the plane.*

The estimation of the t -blocking number is a challenging problem and it has a large literature. Lower bounds are much more often considered, mostly in $\Pi = \text{PG}(2, q)$. However, due to the lack of constructions, we have only weak upper bounds in general. It is well-known that $\tau_2(\text{PG}(2, q)) \leq 2(q + \sqrt{q} + 1)$ if q is a square. In Section 2 we study multiple blocking sets. We present a result on unique reducibility, and a construction for a small double blocking set in $\text{PG}(2, q)$, if q is an odd power of an odd prime power.

If B is a double blocking set (i.e., a 2-fold blocking set) in Π_q , coloring the points of B with one color and all points outside B with mutually distinct colors, one gets a proper coloring of Π_q with $v - |B| + 1$ colors. To achieve the best possible out of this idea, one should take B as a smallest double blocking set. We have obtained

Proposition 1.5.

$$\bar{\chi}(\Pi_q) \geq v - \tau_2 + 1.$$

Definition 1.6. *A coloring of Π_q is trivial, if it contains a monochromatic double blocking set of size τ_2 , and every other color class consists of one single point.*

We recall a more general result of [1]. Let $\tau_2(\Pi_q) = 2(q+1) + c(\Pi_q)$. Note that Proposition 1.5 claims $\bar{\chi}(\Pi_q) \geq q^2 - q - c(\Pi_q)$.

Result 1.7 (Bacsó, Tuza [1]). *Let Π_q be an arbitrary finite projective plane of order q . Then*

$$\bar{\chi}(\Pi_q) \leq q^2 - q - \frac{c(\Pi_q)}{2} + o(\sqrt{q}).$$

If $c(\Pi_q)$ is not too small (roughly, $c(\Pi_q) > 24q^{2/3}$), we improve this result combinatorially. Moreover, we show that (under some technical conditions) the lower bound of Proposition 1.5 is sharp in $\text{PG}(2, q)$ if q is not a prime, and it is almost sharp if q is a prime and τ_2 is small enough. In the proof we use algebraic results as well, and we also rely on our new upper bound on τ_2 . The precise results are the following.

Theorem 1.8. *Let $h \geq 3$ odd, $\alpha \geq 1$ an integer, p an odd prime, $r = p^\alpha$, $q = r^h$. Then there exist two disjoint blocking sets of size $q + (q - 1)/(r - 1)$ in $\text{PG}(2, q)$. Consequently, $\tau_2(\text{PG}(2, q)) \leq 2(q + (q - 1)/(r - 1))$.*

Together with $\tau_2(\text{PG}(2, q)) \leq 2(q + \sqrt{q} + 1)$ for square q , we immediately get the following result.

Corollary 1.9. *Let r denote the order of the largest proper subfield of $\text{GF}(q)$, q odd (where q is not a prime). Then $\tau_2(\text{PG}(2, q)) \leq 2(q + (q - 1)/(r - 1))$.*

We have the following results regarding the upper chromatic number.

Theorem 1.10. *Let Π_q be an arbitrary projective plane of order $q \geq 8$, and let $\tau_2(\Pi_q) = 2(q + 1) + c(\Pi_q)$. Then*

$$\bar{\chi}(\Pi_q) < q^2 - q - \frac{2c(\Pi_q)}{3} + 4q^{2/3}.$$

Theorem 1.11. *Let $v = q^2 + q + 1$. Suppose that $\tau_2(\text{PG}(2, q)) \leq c_0q - 8$, $c_0 < 8/3$, and let $q \geq \max\{(6c_0 - 11)/(8 - 3c_0), 15\}$. Then*

$$\bar{\chi}(\text{PG}(2, q)) < v - \tau_2 + \frac{c_0}{3 - c_0}.$$

In particular, $\bar{\chi}(\text{PG}(2, q)) \leq v - \tau_2 + 7$.

Theorem 1.12. *Let $v = q^2 + q + 1$, $q = p^h$, p prime. Suppose that $q > 256$ is a square, or $p \geq 29$ and $h \geq 3$ odd. Then $\bar{\chi}(\text{PG}(2, q)) = v - \tau_2 + 1$, and equality is reached only by trivial colorings.*

2 Results on multiple blocking sets

There is a strong connection between proper colorings and double blocking sets, thus we shall examine double blocking sets in order to determine $\bar{\chi}(\text{PG}(2, q))$.

2.1 On the number of t -secants through an essential point of a t -fold blocking set

Definition 2.1. *Let $B \subset \mathcal{P}$. Lines intersecting a point-set B in exactly t points or in more than t points are called a t -secant or a $(> t)$ -secant to B , respectively.*

Definition 2.2. *Let $B \subset \mathcal{P}$ be a t -fold blocking set. A point $P \in B$ is essential if $B \setminus \{P\}$ is not a t -fold blocking set; equivalently, if there is at least one t -secant of B through P . A t -fold blocking set B is minimal, if no proper subset of B is a t -fold blocking set, that is, every point of B is essential.*

Lemma 2.3. *Let S be a t -fold blocking set in $\text{PG}(2, q)$, $|S| = t(q + 1) + k$. Then there are at least $q + 1 - k - t$ distinct t -secants to S through any essential point of S .*

Proof. Let $P \in S$ be essential, and let l be an arbitrary t -secant of S that is not incident with P . Assume that there are less than $q + 1 - k - t$ distinct t -secants through P . We claim that in this case every t -secant through P intersects l in a point of $l \cap S$. Having proved this, we easily get a contradiction: since P is essential, there exists a line e through P that is a t -secant. Choose a point $Q \in e \setminus S$. If the only t -secant through Q is e , then

$|S| \geq t(q+1) + q$ and the statement of the lemma is trivial. Thus we may assume that there is another t -secant, say l^* , through Q . But by our claim every t -secant through P intersects l^* in a point of $l^* \cap S$, which is a contradiction, since $e \cap l^* = Q \notin S$. So now we have to prove the claim above.

Choose homogeneous coordinates $(X : Y : Z)$ in such a way that the common point of the vertical lines, $(\infty) = (0 : 1 : 0) \in S$ and l is the line at infinity (having equation $Z = 0$).

Suppose that $S \cap l = \{(0 : 1 : 0)\} \cup \{(1 : m_i : 0) \mid i = 1, \dots, t-1\}$. Let the affine plane $\text{PG}(2, q) \setminus l$ be coordinatized by affine coordinates $(x, y) = (x : y : 1)$ and let $S \setminus l = \{(x_i, y_i) \mid i = 1, \dots, tq+k\}$. We may assume that $P = (x_1, y_1)$. Let $H(B, M)$ be the Rédei polynomial of $S \setminus \{(\infty)\}$:

$$H(B, M) = \prod_{i=1}^{t-1} (M - m_i) \cdot \prod_{i=1}^{tq+k} (B + x_i M - y_i).$$

Note that $\deg_{B,M} H(B, M) = t(q+1) + k - 1$. A line with slope m and y -intersection b (defined by the equation $Y = mX + b$) intersects S in exactly as many points as the number of (linear) factors vanishing in $H(b, m)$; that is, the multiplicity of the root b of the one-variable polynomial $H(B, m)$ or the multiplicity of the root m of $H(b, M)$, provided that these one-variable polynomials are not identically zero. This latter phenomenon occurs iff we substitute $M = m_i$ for some $i = 1, \dots, t-1$.

Since S is a t -fold blocking set, every pair (b, m) produces at least t factors vanishing in H ; thus by [6] (or see [7]), $H(B, M)$ can be written in the form

$$(B^q - B)^t F_0(B, M) + (B^q - B)^{t-1} (M^q - M) F_1(B, M) + \dots + (M^q - M)^t F_t(B, M),$$

where $\deg(F_i) \leq k + t - 1$. Since $\prod_{i=1}^{t-1} (M - m_i)$ divides $H(B, M)$ and $(M^q - M) = \prod_{m \in \text{GF}(q)} (M - m)$, it also divides $F_0(B, M)$. Let $F_0^*(B, M) = F_0(B, M) / \prod_{i=1}^{t-1} (M - m_i)$. Fix $m \in \text{GF}(q) \setminus \{m_i : i = 1, \dots, t-1\}$. Then $F_0(B, m)$ and $F_0^*(B, m)$ differ only in a nonzero constant multiplier, $0 \neq H(B, m) = (B^q - B)^t F_0(B, m)$ and $\deg_B F_0(B, m) \leq k$. If a line $Y = mX + b$ intersects S in more than t points, then the multiplicity of the root b of $H(B, m)$ is more than t , thus $(B - b)$ divides $F_0^*(B, m)$. Conversely, if $F_0^*(b, m) = 0$, then the line $Y = mX + b$ intersects S in more than t points.

If there are less than $q + 1 - k - t$ distinct t -secants through $P = (a_1, b_1)$, then there are more than k non-vertical ($> t$)-secants through P with slopes different from m_i , $i = 1, \dots, t-1$. Hence there are more than k pairs (b, m) for which $b + mx_1 - y_1 = 0$ and $F_0^*(b, m) = 0$; in other words, the algebraic curves defined by $B + Mx_1 - y_1 = 0$ and $F_0(B, M) = 0$ have more than k points in common. Since $\deg_{B,M} F_0^*(B, M) \leq k$, this implies that $B + Mx_1 - y_1 \mid F_0^*(B, M)$ (e.g., by Bézout's theorem). Geometrically this means that every line passing through $P = (x_1, y_1)$ not through $B \cap l$ is a ($> t$)-secant of B . \square

Remark 2.4. *Lemma 2.3 is similar to Lemma 2.3 in [7]. The proof given there works for $k + t < (q + 3)/2$ (although it is only stated implicitly before the lemma) and it gives a*

somewhat better result, that there are at least $q - k$ distinct t -secants through every point. Note that in the actual applications of the lemma in [7], this condition is assumed.

Corollary 2.5. *Let B be a t -fold blocking set with $|B| \leq (t + 1)q$ points. Then there is exactly one minimal t -fold blocking set in B , namely the set of essential points.*

Proof. Let B' be a minimal t -fold blocking set of size $t(q + 1) + k'$ inside B , and let $P \in B'$. Then P is an essential point of B' , hence there are at least $q + 1 - k' - t$ distinct t -secants to B' through P . At least $q + 1 - k' - t - (|B| - |B'|) \geq 1$ of these must be a t -secant to B as well, thus P is an essential point of B . On the other hand, all essential points of B must be in B' . \square

Remark 2.6. *The case $t = 1$ of the above corollary was already proved by Szőnyi [17]. Recently, Harrach [12] proved that a sufficiently small weighted t -fold $(n - k)$ -blocking set in the projective space $\text{PG}(n, q)$ contains a unique minimal weighted t -fold $(n - k)$ -blocking set. For non-weighted t -fold blocking sets in $\text{PG}(2, q)$, Harrach's result is the same as Corollary 2.5.*

Harrach pointed out the following.

Remark 2.7. *Corollary 2.5 is equivalent to Lemma 2.3.*

Proof. Suppose that Corollary 2.5 holds. Let S be a minimal t -fold blocking set with $|S| = t(q + 1) + k \leq (t + 1)q$. Should there be a point $P \in S$ with $s < q + 1 - k - t$ distinct t -secants through it, add one new point P_i to S on each of the t -secants through P , $1 \leq i \leq s$. Then the extended set S' is a t -fold blocking set and $|S'| \leq (t + 1)q$. Note that $S' \setminus \{P\}$ is also a t -fold blocking set, hence it contains a minimal t -fold blocking set that is different from S . Thus S' would violate Corollary 2.5, a contradiction. \square

Remarks 2.6 and 2.7 show that Lemma 2.3 follows from the results of Harrach [12]. However, as our proof is self-contained and compact enough, we found it worthy to present it here.

2.2 The construction of two small disjoint blocking sets

Next we construct two disjoint blocking sets of Rédei type. $\text{GF}(q)$ denotes the finite field of $q = r^h$ elements of characteristic p , $r = p^\alpha$.

Let $f: \text{GF}(q) \rightarrow \text{GF}(q)$, and consider its graph $U_f = \{(x : f(x) : 1) : x \in \text{GF}(q)\}$ in the affine plane $\text{AG}(2, q)$. The slopes (or directions) determined by U_f is the set $S_f = \{(f(x) - f(y))/(x - y) : x, y \in \text{GF}(q), x \neq y\}$. It is well-known that $U_f \cup \{(1 : m : 0) : m \in S_f\}$ is a blocking set in $\text{PG}(2, q)$. This blocking set has the property that there is a line such that there are precisely q points in the blocking set that are not on this line. Such blocking sets are called *blocking sets of Rédei type*. For more information about these we refer to [11].

Let γ be a primitive element of $\text{GF}(q)^*$, the multiplicative group of $\text{GF}(q)$. Let $d \mid q - 1$, $m = (q - 1)/d$, and let $D = \{x^d : x \in \text{GF}(q)^*\} = \{\gamma^{kd} : 0 \leq k < m\}$. Let $1 \leq t \leq m - 1$. Then $\gamma^{dt} \neq 1$. On the other hand, $\sum_{c \in D} c^t = \sum_{c \in D} c^t \gamma^{dt}$, therefore $\sum_{c \in D} c^t = 0$ follows.

First we copy the ideas of the proof of the Hermite–Dickson theorem on permutation polynomials from [13] to prove a generalization of it to multiplicative subgroups of $\text{GF}(q)^*$, which we will use to find two small disjoint blocking sets in $\text{PG}(2, q)$.

Lemma 2.8. *Let $\text{GF}(q)$ be a field of characteristic p , $d \mid q - 1$, $m = (q - 1)/d$. Let $D = \{x^d : x \in \text{GF}(q)^*\}$ be the set of nonzero d^{th} powers. Let a_1, \dots, a_m be a sequence of elements of D . Then the following two conditions are equivalent:*

- (i) a_1, \dots, a_m are pairwise distinct;
- (ii) $\sum_{i=1}^m a_i^t = 0$ for all $1 \leq t \leq m - 1$, $p \nmid t$.

Proof. Let γ be a primitive element of $\text{GF}(q)^*$, $a_i = \gamma^{\alpha_i d}$. Let $g_i(x) = \sum_{j=0}^{m-1} a_i^{m-j} x^j$. Choose $b = \gamma^{\beta d} \in D$. Then

$$g_i(b) = \sum_{j=0}^{m-1} \gamma^{\alpha_i(m-j)d} \gamma^{d\beta j} = \sum_{j=0}^{m-1} \gamma^{\alpha_i m d} \gamma^{d(\beta - \alpha_i)j} = \sum_{j=0}^{m-1} (\gamma^{jd})^{\beta - \alpha_i} = \begin{cases} m, & \text{if } \beta - \alpha_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

As $m \mid q - 1$, $m \not\equiv 0 \pmod{p}$. Let $g(x) = \sum_{i=1}^m g_i(x) = \sum_{j=0}^{m-1} (\sum_{i=1}^m a_i^{m-j}) x^j$. Then $\deg g(x) < m$, and $g(b) = |\{i \in \{1, \dots, m\} : a_i = b\}| \cdot m \pmod{p}$. Thus a_1, \dots, a_m are pairwise distinct $\iff g(b) = m$ for all $b \in D \iff g(x) \equiv m \iff \sum_{i=1}^m a_i^t = 0$ for all $1 \leq t \leq m - 1$. As $x \mapsto x^p$ is an automorphism of $\text{GF}(q)$, this yields the statement. \square

Theorem 2.9. *Let $\text{GF}(q)$ be a field of characteristic p , $m \mid q - 1$, and let D be the multiplicative subgroup of $\text{GF}(q)^*$ of m elements. Let $g \in \text{GF}(q)[x]$ be a polynomial such that $g(b) \in D$ for all $b \in D$. Then $g|_D : D \rightarrow D$ is a permutation of D if and only if the constant term of $g(x)^t \pmod{x^m - 1}$ is zero for all $1 \leq t \leq m - 1$, $p \nmid t$.*

Proof. Let $b \in D$. If $b \neq 1$, then $b^{m-1} + b^{m-2} + \dots + b + 1 = (b^m - 1)/(b - 1) = 0$, otherwise it equals m . Let $g^{[t]}(x) = g(x)^t \pmod{x^m - 1}$. As $g(x)^t$ and $g^{[t]}(x)$ take the same values on D and $\deg(g^{[t]}(x)) < |D|$, by interpolation we have $g^{[t]}(x) = \sum_{b \in D} \frac{g^t(b)}{m} \left(\left(\frac{x}{b}\right)^{m-1} + \dots + \frac{x}{b} + 1 \right)$. Thus the constant term of $g^{[t]}(x)$ is $\sum_{b \in D} g^t(b)$, hence Lemma 2.8 yields the stated result. \square

Corollary 2.10. *Let $D \leq \text{GF}(q)^*$ be a multiplicative subgroup of m elements. Suppose that $g \in \text{GF}(q)[x]$ maps a coset $c_1 D$ into another coset $c_2 D$. Then this mapping is injective if and only if the constant term of $g(c_1 x)^t \pmod{x^m - 1}$ is zero for all $1 \leq t \leq m - 1$, $p \nmid t$.*

Proof. Apply Theorem 2.9 to $g^*(x) = c_2^{-1} g(c_1 x)$. \square

Now we are ready to prove Theorem 1.8. Recall that $q = r^h$, $h \geq 3$ odd, $r = p^\alpha$, $\alpha \geq 1$, and p is an odd prime.

Proof of Theorem 1.8. Let D be the set of nonzero $(r-1)^{\text{th}}$ powers, $m = (q-1)/(r-1) = r^{h-1} + \dots + r + 1$. Then m is also odd. Let $f, g: \text{GF}(q) \rightarrow \text{GF}(q)$ be two additive functions. Then the directions determined by f are $\{(f(x) - f(y))/(x - y) : x \neq y \in \text{GF}(q)\} = \{f(x)/x : x \in \text{GF}(q)^*\}$, which correspond to the points $(1 : f(x)/x : 0) = (x : f(x) : 0)$ on the line at infinity. The same holds for g as well. Note that interchanging the second and third coordinates is an automorphism of $\text{PG}(2, q)$. Consider the following blocking sets:

$$B_1 = \underbrace{\{(x : f(x) : 1)\}}_{U_1} \cup \underbrace{\{(x : f(x) : 0)\}_{x \neq 0}}_{D_1},$$

$$B_2 = \underbrace{\{(y : 1 : g(y))\}}_{U_2} \cup \underbrace{\{(y : 0 : g(y))\}_{y \neq 0}}_{D_2}.$$

Besides additivity, suppose that g is an automorphism of $\text{GF}(q)$ and that $f(x) = 0 \iff x = 0$. The latter assumption yields $(0 : 0 : 1) \notin D_2$, so $D_2 \cap B_1$ is empty. If $(x : f(x) : 0) = (y : 1 : g(y)) \in D_1 \cap U_2$, then $g(y) = 0$, hence $y = 0$ and $x = 0$, a contradiction. Thus $D_1 \cap U_2$ is also empty. Now we need $U_1 \cap U_2 = \emptyset$. Suppose that $(y : 1 : g(y)) = (x : f(x) : 1)$. Then $x \neq 0$ (otherwise $y = g(y) = 0 \neq 1$), thus $(y : 1 : g(y)) = (x/f(x) : 1 : 1/f(x))$, so $g(x/f(x)) = 1/f(x)$. As g is multiplicative, this yields $g(x)f(x) = g(f(x))$ (\star). We want this equation to have no solutions in $\text{GF}(q)^*$.

Let $g(x) = x^r$ and $f(x) = \frac{1}{a}(x^r + x)$, $a \in \text{GF}(q)^*$. Then g is an automorphism and f is additive. Moreover, if $x \neq 0$, then $f(x) = \frac{1}{a}x(x^{r-1} + 1)$ is zero iff $x^{r-1} = -1$, consequently $1 = x^{m(r-1)} = (-1)^m = -1$ as m is odd, which is impossible in odd characteristic. Hence $f(x) = 0 \iff x = 0$. It is easy to see that $f(x)/x = f(y)/y$ if and only if $(x/y)^{r-1} = 1$, thus $|D_1| = (q-1)/(r-1)$; similarly, $|D_2| = (q-1)/(r-1)$ as well.

Equality (\star) now says $x^r = (\frac{1}{a}(x^r + x))^{r-1} = \frac{x^{r-1}}{a^{r-1}}(x^{r-1} + 1)^{r-1}$, equivalently

$$a^{r-1} = \frac{(x^{r-1} + 1)^{r-1}}{x} =: \psi^*(x). \quad (1)$$

Recall that we want (1) to have no solutions in $\text{GF}(q)^*$. To this end we need to find an $(r-1)^{\text{th}}$ power (i.e., an element of D) that is not in the range of ψ^* . Note that $\psi^*(b) \in D \iff b \in D$. Let $\psi(x) = (x^{r-1} + 1)^{r-1}x^{q-2}$. Then $\psi^*(x)$ and $\psi(x)$ take the same values on $\text{GF}(q)^*$. Thus we need to show that $\psi|_D$ does not permute D . By Theorem 2.9, it is enough to show that the constant term of $\psi^{r-1}(x) \pmod{x^m - 1}$ is not zero.

Consider

$$\psi^{r-1}(x) = \sum_{k=0}^{(r-1)^2} \binom{(r-1)^2}{k} x^{k(r-1) + (r-1)(q-2)}.$$

Since $k(r-1) + (r-1)(q-2) \equiv (k-1)(r-1) \pmod{m}$, the exponents reduced to zero have $k = 1 + \ell \frac{m}{\gcd(m, r-1)}$. As $\binom{(r-1)^2}{1} \equiv 1 \pmod{p}$, it is enough to show that $\binom{(r-1)^2}{k} \equiv 0 \pmod{p}$ for the other possible values of k .

As $0 \leq k \leq (r-1)^2$, $m/\gcd(m, r-1) \geq r^2$ would imply that $\ell \geq 1$ does not occur. By $m/\gcd(m, r-1) > m/r > r^{h-2}$, this is the case for $h \geq 5$; and also for $h = 3$ and $r \not\equiv 1 \pmod{3}$, as in this case $m = r^2 + r + 1$ and $\gcd(m, r-1) = \gcd(3, r-1) = 1$.

Now suppose $h = 3$, $r \equiv 1 \pmod{3}$. Then $1 \leq \ell \leq 2$, and so $k = 1 + \ell m/\gcd(m, r-1) = 1 + \ell(r^2 + r + 1)/3 \equiv (3 + \ell)/3 \not\equiv 0, 1 \pmod{r}$ as $r > 5$. Let $k! = p^\beta k'$, where $\gcd(k', p) = 1$. Consider the product $\pi = (r^2 - 2r - 1) \dots (r^2 - 2r + 2 - k)(r^2 - 2r + 1 - k)(r^2 - 2r - k)$. As $\binom{r^2 - 2r - 1}{k}$ is an integer, $k! \mid \pi$, so $p^\beta \mid \pi$. Since $(r^2 - 2r + 1)(r^2 - 2r)$ is divisible by r , but $(r^2 - 2r + 1 - k)(r^2 - 2r - k)$ is not, $p^{\beta+1}$ divides $(r^2 - 2r + 1) \dots (r^2 - 2r + 2 - k)$, hence $\binom{(r-1)^2}{k} \equiv 0 \pmod{p}$. Thus the proof is finished. \square

Let us mention some results in connection with Theorem 1.8. As the union of two disjoint blocking sets, a double blocking set of size $2q + 2q^{2/3} + 2q^{1/3} + 2$ was constructed by Davydov, Giulietti, Marcugini and Pambianco [9] in $\text{PG}(2, q = p^3)$ for $p \leq 73$, p prime, and by Polverino and Storme ([15], cited in [7]) in $\text{PG}(2, q = p^{3h})$ for $p^h \equiv 2 \pmod{7}$. Note that Result 3.11 (see later) roughly says that a double blocking set in $\text{PG}(2, q)$ of size at most $2q + q^{2/3}$ contains the union of two disjoint Baer subplanes. These examples show that the term $q^{2/3}$ is of the right magnitude if q is a cube.

Also, relying on [16], the PhD thesis of Van de Voorde [18] implicitly contains the following general result: if B is a minimal blocking set in $\text{PG}(2, q)$, $q = p^h$, that is not a line, and $|B| \leq 3(q - p^{h-1})/2$, then there is a small $\text{GF}(p)$ -linear blocking set that is disjoint from B . It seems that the proof requires the characteristic of the field to be more than five. Note that this implies $\tau_2 \leq 2q + q/p + (q-1)/(p-1) + 1$. For an overview of linear sets, we refer to [14]. We remark that the functions f and g in the above construction are both linear over $\text{GF}(r)$, and hence the arising blocking sets are linear as well.

Finally, let us note that two specific disjoint linear sets were also presented in [4] in order to construct semifields. The rank of those are different from what we need if we want to obtain two disjoint linear blocking sets. However, the construction probably can be modified in a way so that one can use it to find two disjoint blocking sets.

3 Results on the upper chromatic number

Let Π_q be a finite projective plane of order q . Let $\mathcal{C} = \{C_1, \dots, C_m\}$ be a proper coloring, where the point-set of Π_q is partitioned by the color classes C_i , $i = 1, \dots, m = m(\mathcal{C})$. We may assume that $|C_1| \geq |C_2| \geq \dots \geq |C_n| \geq 2$, $|C_{n+1}| = \dots = |C_m| = 1$ for some appropriate $n = n(\mathcal{C})$. A color class of size exactly d will be called a d -class. We say that a color class C_i colors a line ℓ , if $|\ell \cap C_i| \geq 2$. Let $B = B(\mathcal{C}) = \cup_{i=1}^n C_i$. As every line must be colored, B is a double blocking set. We always assume that a proper coloring \mathcal{C} of the plane is given.

By the above notation, \mathcal{C} uses $v - |B| + n$ colors, while a trivial coloring has $v - |\tau_2| + 1$ colors. Thus to achieve this bound, we need to have $n \geq |B| - \tau_2 + 1$. We define the parameter $e = e(\mathcal{C})$, standing for *excess*, which measures how much our coloring is better than a trivial one.

Definition 3.1. Given a proper coloring \mathcal{C} , let

$$e = e(\mathcal{C}) := n - |B| + \tau_2 - 1.$$

To avoid colorings that are worse than the trivial ones, we will usually suppose that $e \geq 0$ (equivalently, $n \geq |B| - \tau_2 + 1$). First we formulate a straightforward observation.

Proposition 3.2. *If \mathcal{C} is a nontrivial proper coloring with $e(\mathcal{C}) \geq 0$, then \mathcal{C} does not contain a monochromatic double blocking set.*

Proof. Suppose to the contrary that \mathcal{C} contains a monochromatic double blocking set S . Then $v - \tau_2 + 1 \leq m(\mathcal{C}) \leq v - |S| + 1 \leq v - \tau_2 + 1$, so $|S| = \tau_2$ and all other color classes are 1-classes, thus \mathcal{C} is trivial, a contradiction. \square

The following lemma shows that we can eliminate all but possibly one 2-classes.

Lemma 3.3. *Let \mathcal{C} be a proper coloring of Π_q . Then there is another proper coloring \mathcal{C}' with the same number of colors such that there is at most one 2-class in \mathcal{C}' . If there is a 2-class in \mathcal{C}' , then its points are essential with respect to $B(\mathcal{C}')$. Moreover, if $\Pi_q = \text{PG}(2, q)$, $\tau_2 < 3q$, and \mathcal{C} is nontrivial, then \mathcal{C}' is also nontrivial.*

Proof. We construct \mathcal{C}' step by step from \mathcal{C} ; the notation always regard to the coloring obtained at the last step. Consider a 2-class $C_i = \{P, Q\}$. Then it colors only one line, namely \overline{PQ} . If \overline{PQ} intersects B in a point R , $R \in C_j$ ($i \neq j$), then remove P from C_i and put it into C_j . As $\overline{PQ} = \overline{PR}$ is now colored by the class C_j , we obtain a proper coloring. Note that C_j originally had at least two points, so we did not create a new 2-class. Repeat this operation until every 2-class colors a line that is a two-secant to B . Now suppose that there are two 2-classes $C_i = \{P_1, P_2\}$ and $C_j = \{Q_1, Q_2\}$ ($i \neq j$) such that $\overline{P_1P_2}$ and $\overline{Q_1Q_2}$ are two-secants to B . Then $R = \overline{P_1P_2} \cap \overline{Q_1Q_2}$ is not in B , so $\{R\} = C_h$ is a singleton color class. Remove P_1 from C_i and Q_1 from C_j , and put both into C_h . Again it is clear that we obtain a proper coloring, and we do not create new 2-classes. Repeating this operation we can decrease the number of 2-classes to at most one. If a 2-class $\{P, Q\}$ remains uneliminated, then \overline{PQ} is a two-secant, hence P and Q are essential. Thus the first part of the lemma is proved.

Now let $\Pi_q = \text{PG}(2, q)$, and suppose to the contrary that the original coloring \mathcal{C} is nontrivial, but we obtain a trivial coloring \mathcal{C}' . Then at the last step we eliminated every color class of size two, and we created a monochromatic double blocking set of size τ_2 . We must have used the first operation at this step (in the second operation no color classes of size more than three are involved), so we put a point from $C_i = \{P, Q\}$ to C_j . As both points of C_i could be used, $C_j \cup \{P\}$ and $C_j \cup \{Q\}$ are both double blocking sets of size τ_2 . Hence $C_i \cup C_j$ is a double blocking set of size $\tau_2 + 1 \leq 3q$ which contains two minimal double blocking sets, in contradiction with Corollary 2.5. \square

By the above lemma, from now on we may rely on the assumption that there is at most one 2-class.

Proposition 3.4. *Suppose that B contains at most one 2-class and $e \geq 0$. Then $n \leq \tau_2/2$.*

Proof. As there is at most one 2-class and all other color classes in B have at least three points, we have $n \leq 1 + (|B| - 2)/3$. By $e \geq 0$, $|B| - \tau_2 + 1 \leq n \leq 1 + (|B| - 2)/3$, hence $|B| \leq 3\tau_2/2 - 1$. Thus we have $n \leq 1 + (|B| - 2)/3 \leq \tau_2/2$. \square

Now we recall and prove our combinatorial result on $\bar{\chi}(\Pi_q)$.

Theorem 1.10. *Let $q \geq 8$. Then for any projective plane Π_q of order q ,*

$$\bar{\chi}(\Pi_q) < q^2 - q - \frac{2c(\Pi_q)}{3} + 4q^{2/3}.$$

Proof. Take a proper coloring \mathcal{C} . We will estimate the number of colors in \mathcal{C} . We may assume that $e(\mathcal{C}) \geq 0$ (otherwise the statement is trivial), moreover, by Lemma 3.3, we may also suppose that there is at most one 2-class in \mathcal{C} . Set $\varepsilon = 4/\sqrt[3]{q}$, and let h denote the number of color classes with at least $K = 6/\varepsilon$ elements ($K \geq 3$ as $q \geq 8$). Let $C = \cup_{i=1}^h C_i$. Recall that n is the number of colors used in B .

First suppose $|C| \leq 2(1 - \varepsilon)q$. Let $P \notin C$. As the number of lines through P that intersect C in at least two points is at most $|C|/2$, there are at least εq lines through P that intersect C in at most one point. Hence the total number of such lines is at least $(q^2 + q + 1 - |C|)\varepsilon q/(q + 1) > \varepsilon q(q - 2)$. On the other hand, color classes of size less than K can color at most $(n - h)\binom{K}{2} \leq (n - h)K^2/2$ lines, thus $(n - h)K^2 \geq 2\varepsilon q(q - 2)$ must hold. Therefore, by Proposition 3.4, $3q/2 \geq \tau_2/2 \geq n \geq 2\varepsilon q(q - 2)/K^2 = \varepsilon^3 q(q - 2)/18$ holds. As $\varepsilon = 4/\sqrt[3]{q}$, this yields $27 \geq 64(q - 2)/q$, in contradiction with $q \geq 8$.

Thus $|C| > 2(1 - \varepsilon)q$ may be supposed. As all but one color classes in B have at least three points, $n \leq |C|/K + (|B| - |C|)/3 + 1$ holds. Since $K \geq 3$, by substituting $|C| = 2(1 - \varepsilon)q$ we increase the right-hand side, so $n \leq 2(1 - \varepsilon)q/K + (|B| - 2(1 - \varepsilon)q)/3 + 1$. Using $|B| \geq \tau_2$, for the total number $m = n + q^2 + q + 1 - |B|$ of colors we get $m \leq q^2 + q + 2 - 2\tau_2/3 - 2(1 - \varepsilon)q(1/3 - 1/K)$. By $\tau_2 = 2q + c(\Pi_q) + 2$, we obtain that

$$n \leq q^2 - q - \frac{2}{3}c(\Pi_q) + \frac{2\varepsilon}{3}q + \frac{2(1 - \varepsilon)}{K}q.$$

As $K = 6/\varepsilon$, we get

$$\bar{\chi}(\Pi_q) < q^2 - q - \frac{2}{3}c(\Pi_q) + \varepsilon q.$$

\square

Attention. From now on, we only consider proper colorings of Desarguesian projective planes; that is, we assume $\Pi_q = \text{PG}(2, q)$, $q = p^h$, p prime.

In the sequel, we show that if τ_2 is small, then a nontrivial coloring cannot have $e \geq 0$. We handle three cases separately, depending on $|B|$ being at least $3q - \alpha$, between $\tau_2 + \xi$ and $3q - \alpha$, or at most $\tau_2 + \xi$, where α and ξ are small constants. In the next proposition we use the well-known fact that if f is a convex function and $x \leq y$, then $f(x + \varepsilon) + f(y - \varepsilon) \geq$

$f(x) + f(y)$ for arbitrary $\varepsilon > 0$. Therefore if the sum of the input x_1, \dots, x_n is fixed and the x_i s are bounded from below, then $\sum_{i=1}^n f(x_i)$ takes its maximum value if all but one of the x_i s meet their lower bound (so the input is spread). However, the function we consider is not entirely convex, so at some point we cannot modify the input by an arbitrarily small ε but only by a large enough value to see that the maximum value is taken if and only if the input is spread.

Proposition 3.5. *Suppose that there is at most one 2-class in B . Let $|B| \geq 3q - \alpha$ for some integer α , $0 \leq \alpha \leq q - 5$, and suppose $\tau_2 \leq c_0q - \beta$, where $c_0 < 8/3$ and $\beta = (2\alpha + 4)/3$. Assume $q \geq q(c_0) = (6c_0 - 11)/(8 - 3c_0)$. Then $e < 0$.*

Proof. Suppose to the contrary that $e \geq 0$. Then $n \geq |B| - \tau_2 + 1 \geq 3q - \alpha - 8q/3 + \beta = (q - \alpha + 4)/3 \geq 3$.

Denote by $\ell(C_i)$ the number of lines colored by C_i , $i = 1, \dots, n$. It is straightforward that $\ell(C_i) \leq \binom{|C_i|}{2}$. On the other hand, counting selected point-line pairs, we get

$$2\ell(C_i) \leq |\{(P, l): P \in l \cap C_i, |l \cap C_i| \geq 2\}| \leq (q+1)|C_i|,$$

whence

$$\ell(C_i) \leq \frac{q+1}{2}|C_i|$$

follows. Therefore $\ell(C_i) \leq \min\left\{\binom{|C_i|}{2}, \frac{q+1}{2}|C_i|\right\} =: f(|C_i|)$. Note that the second upper bound is smaller than or equal to the first one iff $|C_i| \geq q + 2$. As every line must be colored by at least one color class, we have

$$q^2 + q + 1 \leq \sum_{i=1}^n \ell(C_i) \leq \sum_{i=1}^n f(|C_i|),$$

where $\sum_{i=1}^n |C_i| = |B|$ is fixed. We will give an upper bound on the right-hand-side. Extend the function f to \mathbb{R} . Then f is increasing and convex on $[2, q + 2]$, linear on $[q + 2, \infty)$, but it is not convex on $[2, \infty)$. Recall that $|C_1| \geq \dots \geq |C_n| \geq 2$, $n = |B| - \tau_2 + 1 + e \geq |B| - \tau_2 + 1$, and that there is at most one 2-class in B . Note that $3\tau_2 - 2|B| \leq 3c_0q - 3\beta - 6q + 2\alpha = (3c_0 - 6)q - 4 < 2q - 4$.

We claim that $|C_2| \leq q - 1$. If $|C_1| \geq q$, then by $n \geq |B| - \tau_2 + 1$, we have $|C_2| \leq |B| - q - 2 - 3(n - 3) \leq 3\tau_2 - 2|B| - q + 4 < q$. On the other hand, if $|C_1| \leq q - 1$, then $|C_2| \leq |C_1|$ also implies $|C_2| \leq q - 1$. As there is at most one 2-class, $|C_2| \geq 3$ follows from $n \geq 3$. As $2 \leq |C_i| \leq q - 1$ for all $2 \leq i \leq n$ and f is convex on this interval, $\sum_{i=2}^n f(|C_i|)$ achieves its largest possible value if $|C_n| = 2$, $|C_{n-1}| = \dots = |C_3| = 3$, and $|C_2| = |B| - |C_1| - \sum_{i=3}^n |C_i|$. By substituting these values,

$$\sum_{i=1}^n f(|C_i|) \leq \frac{q+1}{2}|C_1| + \sum_{i=2}^n \binom{|C_i|}{2} \leq \frac{q+1}{2}|C_1| + \binom{|C_2|}{2} + (n-3) \binom{3}{2} + \binom{2}{2}.$$

Now we claim that

$$\frac{q+1}{2}|C_1| + \binom{|C_2|}{2} \leq \frac{q+1}{2}(|C_1| + |C_2| - 3) + \binom{3}{2}, \quad (2)$$

which is equivalent to $|C_2|^2 - (q+2)|C_2| + 3q - 3 \leq 0$. It is easy to see that this latter inequality holds for $|C_2| \in [3, q-1]$, so we may use (2). Recall $n \leq \tau_2/2$ (Proposition 3.4) and $3\tau_2 - 2|B| \leq (3c_0 - 6)q - 4$. As $|C_1| + |C_2| \leq |B| - 2 - 3(n-3)$,

$$q^2 + q + 1 \leq \sum_{i=1}^n f(|C_i|) \leq \frac{q+1}{2} (|B| + 4 - 3n) + 3(n-2) + 2 \leq$$

$$\frac{q+1}{2} (|B| + 4 - 3(|B| - \tau_2 + 1)) + 3(\tau_2/2 - 2) + 2 \leq \frac{q+1}{2} (3\tau_2 - 2|B| + 1) + \frac{3}{2}\tau_2 - 4.$$

For aesthetic reasons, we continue with a strict inequality. The last value is less than

$$\frac{q+1}{2} ((3c_0 - 6)q - 3) + \frac{3}{2}c_0q + \frac{5}{2} = \left(\frac{3c_0}{2} - 3\right)q^2 + \left(3c_0 - \frac{9}{2}\right)q + 1.$$

This is equivalent to $(8 - 3c_0)q^2 - (6c_0 - 11)q < 0$, hence, as $c_0 < 8/3$, we obtain $q < (6c_0 - 11)/(8 - 3c_0)$, a contradiction. \square

Next we investigate the case when B is of medium size. We show that in this case there are some large color classes, which bounds the total number of color classes. Note that the next proposition does not use any assumption on $|B|$, however, it is meaningful only if $|B| < 3q$.

Proposition 3.6. *Every color class containing an essential point of B has at least $3q - |B| + 2$ points.*

Proof. Let $P \in B$ be an essential point. Let $|B| = 2(q+1) + k$. Then by Lemma 2.3 there are $q - 1 - k = 3q - |B| + 1$ two-secants through P . The points of a two-secant must have the same color. \square

Remark 3.7. *Proposition 3.6 shows that if $|B| < 3q$, then color classes containing an essential point have at least three points. Thus by Lemma 3.3, every 2-class can be eliminated.*

Proposition 3.8. *Assume that $|B| < 3q$, and suppose that there are no color classes of size two. Then*

$$\left(\frac{2}{3}(|B| - \tau_2) + e + 1\right) (3q - |B| + 2) \leq \tau_2. \quad (3)$$

Proof. Consider a minimal double blocking set $B' \subset B$. Corollary 2.5 yields that B' consists precisely of the set of essential points of B . Thus color classes intersecting B' must have at least $3q - |B| + 2$ points (Proposition 3.6), while color classes disjoint from B' contain at least three points. Thus the total number of color classes in B ,

$$n \leq \frac{|B'|}{3q - |B| + 2} + \frac{|B| - |B'|}{3}.$$

Recall $n = |B| - \tau_2 + 1 + e$ (Definition 3.1). As $|B'| \geq \tau_2$ and $3q - |B| + 2 \geq 3$, we obtain

$$|B| - \tau_2 + 1 + e \leq \frac{\tau_2}{3q - |B| + 2} + \frac{|B| - \tau_2}{3},$$

which is clearly equivalent to the formula stated. \square

Now we recall and prove Theorem 1.11.

Theorem 1.11. *Suppose that $\tau_2(\text{PG}(2, q)) \leq c_0q - 8$, $2 \leq c_0 < 8/3$, and let $q \geq \max\{(6c_0 - 11)/(8 - 3c_0), 15\}$. Then*

$$\bar{\chi}(\text{PG}(2, q)) < v - \tau_2 + \frac{c_0}{3 - c_0}.$$

In particular, $\bar{\chi}(\text{PG}(2, q)) \leq v - \tau_2 + 7$.

Proof. Let \mathcal{C} be a proper coloring of $v - \tau_2 + 1 + e$ colors. Suppose to the contrary that $e \geq c_0/(3 - c_0) - 1$. As $c_0 \geq 2$, this yields $e \geq 1$. By Lemma 3.3, we may assume that there is at most one 2-class in \mathcal{C} .

Suppose that $|B| \geq 3q - 10$. By the assumptions of the present theorem, the assumptions of Proposition 3.5 are also satisfied for $\alpha = 10$. Then we get $e < 0$, a contradiction.

Thus we may assume $|B| \leq 3q - 11$. Then by Remark 3.7, we may use Proposition 3.8 to obtain

$$\left(\frac{2(|B| - \tau_2)}{3} + e + 1 \right) (3q - |B| + 2) < c_0q.$$

We will show that this can not hold. Note that the expression on the left-hand-side is concave in $|B|$, so it is enough to verify that we get a contradiction for the extremal values $|B| = \tau_2$ and $|B| = 3q - 11$. By substituting $|B| = \tau_2 < c_0q$, we easily obtain $(e+1)(3q - c_0q) < c_0q$, thus $e < c_0/(3 - c_0) - 1$, a contradiction. Substituting $|B| = 3q - 11$, using $\tau_2 \leq c_0q - 8$ and $e \geq 1$, we get

$$\left(\frac{2(3q - 11 - c_0q + 8)}{3} + 2 \right) \cdot 13 < c_0q,$$

which results in $84 < 29c_0 < 80$, a contradiction. Thus $e < c_0/(3 - c_0) - 1$. As $c_0 < 8/3$, $e < 7$, hence $e \leq 6$ also follows. \square

To obtain tight results, we need to investigate the case when $|B|$ is close to τ_2 . If such a double blocking set is the union of two disjoint blocking sets (e.g., in $\text{PG}(2, q)$, if q is a square), we easily find two large color classes, so $|B|$ must be big.

Proposition 3.9. *Let \mathcal{C} be a nontrivial proper coloring, and suppose that B contains the union of two disjoint (1-fold) blocking sets, B_1 and B_2 , such that $B_1 \cup B_2$ is a minimal double blocking set. Then $|B| > 12q/5$.*

Proof. We may assume $|B| \leq 3q$. By Corollary 2.5, $B_1 \cup B_2$ is precisely the set of essential points of B . As \mathcal{C} is nontrivial, Proposition 3.2 assures that at least two colors, say, red and green, are used to color the points of $B_1 \cup B_2$. We may assume that there is a red point P in B_1 . Then by Lemma 2.3, there are at least $3q - |B| + 1$ distinct 2-secants to B through P . As B_2 is a blocking set, each of these lines intersects B_2 in precisely one point, which must be red. Therefore there are at least $3q - |B| + 1 > 0$ red points in B_2 . Conversely, starting from a red point in B_2 , we see that there are at least $3q - |B| + 1$ red points in B_1 . Hence the number of red points is at least $2(3q - |B| + 1)$. As this argument is valid for the number of green points as well, $|B| \geq 4(3q - |B| + 1)$ holds, thus $|B| > \frac{12}{5}q$. \square

To finish the proof when q is a square, we need the following results.

Result 3.10 (Ball, Blokhuis [3]). *Let $q \geq 9$, q square. Then $\tau_2(\text{PG}(2, q)) = 2(q + \sqrt{q} + 1)$.*

Result 3.11 (Blokhuis, Storme, Szőnyi [8]). *Let B be a t -fold blocking set in $\text{PG}(2, q)$, $q = p^h$, p prime, $h > 1$, $|B| = t(q + 1) + c$. Let $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$. Then*

1. *If $q = p^{2d+1}$ and $t < q/2 - c_p q^{2/3}/2$, then $c \geq c_p q^{2/3}$, unless $t = 1$ in which case B contains a line, if $|B| < q + 1 + c_p q^{2/3}$.*
2. *If q is a square, $t < q^{1/4}/2$ and $c < c_p q^{2/3}$, then $c \geq t\sqrt{q}$ and B contains the union of t pairwise disjoint Baer subplanes, except for $t = 1$ in which case B contains a line or a Baer subplane.*

Remark 3.12. *In particular, if B is a double blocking set in $\text{PG}(2, q)$, q a square, $q > 256$, and $|B| \leq 2q + 2\sqrt{q} + 11 = \tau_2 + 9$, then B contains two disjoint Baer subplanes.*

Proof. We only verify the respective assumptions of Result 3.11. First of all, $2 = 256^{1/4}/2 < q^{1/4}/2$. Secondly, we need $9 < c_p q^{2/3} - 2\sqrt{q}$. As $q > 256$ and q is a square, we have $q \geq 17^2 = 289$. In the case of $c_p = 1$, we obtain $9.71 < 289^{2/3} - 34 \leq q^{2/3} - 2\sqrt{q}$. In the case of $c_p = 2^{-1/3}$, that is, $p \in \{2, 3\}$, we have $q \geq \min\{3^6, 2^{10}\} = 3^6$, thus $10.28 < 2^{-1/3}81 - 54 \leq c_p q^{2/3} - 2\sqrt{q}$. \square

This is enough to prove Theorem 1.12 in case of q being a square.

Theorem 1.12 (first case). *Let $q > 256$ be a square prime power. Then $\bar{\chi}(\text{PG}(2, q)) = v - \tau_2 + 1 = q^2 - q - 2\sqrt{q}$. Equality can be reached only by a trivial coloring.*

Proof. Result 3.10 yields $\tau_2 = 2q + 2\sqrt{q} + 2$. Let \mathcal{C} be a nontrivial proper coloring of $v - \tau_2 + 1 + e$ colors. Suppose to the contrary that $e \geq 0$. By Lemma 3.3, we may assume that there is at most one 2-class in \mathcal{C} , and the nontriviality of the coloring is also preserved as $\tau_2 = 2q + 2\sqrt{q} + 2 < 3q$.

Suppose that $|B| \geq 3q - 3$. Then $\alpha = 3$ and $c_0 = 2.5$ are convenient in Proposition 3.5: $\tau_2 = 2q + 2\sqrt{q} + 2 \leq 2.5q - 10/3$ for $q \geq 36$, and $q(2.5) = 8$. Thus $e < 0$, a contradiction.

Now suppose $\tau_2 + 6 \leq |B| \leq 3q - 4$. Then by Remark 3.7, we may use Proposition 3.8 to obtain

$$\frac{2}{3}(|B| - \tau_2)(3q - |B| + 2) \leq \tau_2.$$

As the left-hand side is concave in $|B|$, it is enough to obtain a contradiction for the values $|B| = \tau_2 + 6$ and $|B| = 3q - 4$. Substituting either value of $|B|$ we get $4(3q - \tau_2 - 4) \leq \tau_2$, thus $12q/5 - 5 < \tau_2 = 2q + 2\sqrt{q} + 2$, a contradiction even for $q \geq 49$.

Finally, suppose $|B| \leq \tau_2 + 5$. By Remark 3.12, B contains the union of two disjoint Baer subplanes, which is a minimal double blocking set. Thus Proposition 3.9 yields $\tau_2 + 5 \geq |B| > 12q/5$, a contradiction. \square

In Proposition 3.9, we relied on the assumption that a small double blocking set contains two disjoint blocking sets, and this could be used to find large color classes. If q is not a square, we do not know whether small double blocking sets have this property. Thus we need further investigations and the $t \pmod{p}$ result on small t -fold blocking sets to find at least one large color class, and to obtain a result similar to Proposition 3.9.

Result 3.13 (Blokhuis, Lovász, Storme, Szőnyi [7]). *Let B be a minimal t -fold blocking set in $\text{PG}(2, q)$, $q = p^h$, p prime, $h \geq 1$, $|B| < tq + (q + 3)/2$. Then every line intersects B in $t \pmod{p}$ points.*

Proposition 3.14. *Let \mathcal{C} be a nontrivial proper coloring. Let $\xi \in \mathbb{N}$. Suppose $|B| \leq \tau_2 + \xi < 2q + (q + 3)/2$ and $\xi \leq (\tau_2 - 2q)/24$. Then $\tau_2 > 3q/2 + pq/50 - \xi + 1$, where p is the characteristic of the field.*

Proof. As $|B| < 3q$, the set B' of essential points of B is a double blocking set (Corollary 2.5). As \mathcal{C} is nontrivial, B' can not be monochromatic (Proposition 3.2). By merging color classes while preserving this property, we may assume that there are only two color classes inside B , say, red and green, each containing at least one essential point of B . (We do not want to preserve the number of colors this time.) By Result 3.13, if a line l intersects B' in more than two points, then $|l \cap B'| \geq p + 2$. We refer to such lines as long secants. We are about to find a red point on which there are many long secant lines that have more red points than green.

Let $|B'| = b \geq \tau_2 \geq 2(q + 1)$. Denote the set of red and green essential points by B_r and B_g , respectively, and for any line $l \in \mathcal{L}$, let $n_l = |l \cap B'|$, $n_l^r = |l \cap B_r|$ and $n_l^g = |l \cap B_g|$. Clearly $n_l = n_l^r + n_l^g$ for any line l . Using double counting we get $\sum_{l \in \mathcal{L}} n_l = |B'|(q + 1)$, hence

$$\sum_{l \in \mathcal{L}: n_l > 2} n_l \geq \sum_{l \in \mathcal{L}} (n_l - 2) = b(q + 1) - 2v = bq + 2(q + 1) - 2(q^2 + q + 1) \geq (b - 2q)q.$$

Let $\mathcal{L}^r = \{l \in \mathcal{L}: |l \cap B'| > 2, n_l^r > n_l^g\}$, $\mathcal{L}^g = \{l \in \mathcal{L}: |l \cap B'| > 2, n_l^r < n_l^g\}$, and $\mathcal{L}^= = \{l \in \mathcal{L}: n_l^r = n_l^g\}$. Then

$$(b - 2q)q \leq \sum_{l \in \mathcal{L}: n_l > 2} n_l = \sum_{l \in \mathcal{L}^r} (n_l^r + n_l^g) + \sum_{l \in \mathcal{L}^g} (n_l^r + n_l^g) + \sum_{l \in \mathcal{L}^=} (n_l^r + n_l^g) \leq$$

$$\sum_{l \in \mathcal{L}^r} 2n_l^r + \sum_{l \in \mathcal{L}^g} 2n_l^g + \sum_{l \in \mathcal{L}^= } 2n_l^= \leq 4 \cdot \sum_{l \in \mathcal{L}^r \cup \mathcal{L}^= } n_l^r,$$

where we assumed in the last step that the first sum was at least as large as the second (we may interchange the colors without the loss of generality). We say that a line l is red if $n_l^r \geq n_l^g$. Hence by the above inequality there exists a point $P \in B_r$ such that the number of long secant red lines passing through P is at least $(b-2q)q/(4|B_r|) \geq (\tau_2-2q)q/(4|B_r|)$. On these lines there are at least $p/2$ red points besides P . Moreover, on the two-secants to B through P , there are at least $3q - |B| + 1$ red points besides P (see Proposition 3.6). Thus we have $|B_r| \geq (\tau_2 - 2q)pq/(8|B_r|) + 3q - |B| + 2$. As there exists a green essential point, Proposition 3.6 yields that the total number γ of green points in B is at least $3q - |B| + 2$. Therefore, $|B_r| \leq |B| - \gamma \leq 2|B| - 3q - 2$. Thus altogether we have $(\tau_2 - 2q)pq/(8|B_r|) + 3q - |B| + 2 \leq 2|B| - 3q - 2$, hence $(\tau_2 - 2q)pq/(8|B_r|) \leq 3|B| - 6q - 4 < 3(\tau_2 - 2q + \xi)$. Hence

$$\frac{(\tau_2 - 2q)pq}{24(\tau_2 - 2q + \xi)} < |B_r| \leq 2|B| - 3q - 2 = 2\tau_2 - 3q + 2\xi - 2.$$

As $\xi \leq (\tau_2 - 2q)/24$, $24(\tau_2 - 2q + \xi) \leq 25(\tau_2 - 2q)$, thus $pq/50 + 3q/2 - \xi + 1 < \tau_2$. \square

Now we are ready to prove the second (and last) part of Theorem 1.12.

Theorem 1.12 (second case). *Suppose that $q = p^h$, $p \geq 29$ prime, $h \geq 3$ odd. Then $\bar{\chi}(\text{PG}(2, q)) = v - \tau_2 + 1$, and equality can only be reached by a trivial coloring.*

Proof. Theorem 1.8 yields $\tau_2 \leq 2q + 2(q-1)/(p-1)$. As $p \geq 29$, $\tau_2 < 2q + q/14$. Note that $q \geq p^3 > 20000$ is fairly large. Suppose to the contrary that there is a nontrivial proper coloring \mathcal{C} with $e = e(\mathcal{C}) \geq 0$. By Lemma 3.3, we may assume that there is at most one 2-class in \mathcal{C} , and the nontriviality of the coloring is also preserved as $\tau_2 < 3q$.

First suppose that $|B| \geq 3q - 11$. Then $\alpha = 11$ and $c_0 = 2.5$ are convenient in Proposition 3.5: $\tau_2 < 2q + q/14 \leq 2.5q - 26/3$, and $q(2.5) = 8$. Thus $e < 0$, a contradiction.

Now suppose $\tau_2 + 12 \leq |B| \leq 3q - 12$. Then by Remark 3.7, we may use Proposition 3.8 to obtain

$$\frac{2}{3}(|B| - \tau_2)(3q - |B|) < \tau_2.$$

As the left-hand side is concave in $|B|$, it is enough to obtain a contradiction for the values $|B| = \tau_2 + 12$ and $|B| = 3q - 12$. Substituting either value of $|B|$, we get $8(3q - \tau_2 - 12) \leq \tau_2$, thus $24q/9 - 11 < \tau_2 < 2q + q/14$, a contradiction.

Thus $|B| \leq \tau_2 + 11 < 2q + (q+3)/2$. By Result 3.11, we have $(\tau_2 - 2q)/24 > q^{2/3}/24 \geq 29^2/24 > 29$, thus we may apply Proposition 3.14 with $\xi = 11$ to obtain $\tau_2 > 3q/2 + pq/50 - 10 \geq 2q + 2q/25 - 10$. Compared to $\tau_2 < 2q + q/14$, q is large enough to get a contradiction. \square

4 Final remarks

The conditions of Theorem 1.12 on q and p are rather technical, and it is very likely that they are not sharp. Yet some restrictions are necessary. Let P_1, P_2, P_3 be three non collinear points, and let $\ell_1 = \overline{P_2P_3}$, $\ell_2 = \overline{P_1P_3}$, $\ell_3 = \overline{P_1P_2}$. Then the triangle $\ell_1 \cup \ell_2 \cup \ell_3$ is a minimal double blocking set of size $3q$. It is easy to see that the coloring in which the color classes of size at least two are $(\ell_2 \cup \ell_3) \setminus \{P_1\}$ and $\{P_1\} \cup (\ell_1 \setminus \{P_2, P_3\})$ is proper, and it uses $v - 3q + 2$ colors. However, $\tau_2(\text{PG}(2, q)) = 3q$ for $2 \leq q \leq 8$ (see e.g. [3]), thus in these cases Theorem 1.12 fails.

For arbitrary finite projective planes, the results of Theorems 1.11 and 1.12 may be false or hopeless to prove. The problem may be generalized into several directions. For example, one may consider colorings in which every line contains at least t points that have the same color. It is straightforward that we can use $v - \tau_t + 1$ colors. Also, one may investigate colorings such that every line intersects at least s color classes in more than one point. We may obtain such a coloring if we find s disjoint double blocking sets. Moreover, one might combine the two questions and study colorings such that every line intersects at least s color classes in at least t points. The case $t = 2, s = 1$ is what we examined in this paper, other parameters have not been considered.

We may also look for the maximum number of colors under the condition that every line contains at least t points from each color class. This is the same as asking how many disjoint t -fold blocking sets can we use to partition the point-set of a finite projective plane. The case $t = 1$ was studied in [5].

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