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ON LIMITS AND COLIMITS IN THE KLEISLI CATEGORY

by Jenö SZIGETI

1. INTRODUCTION.

Given a triple (monad) $\mathrm{R} = (R, \alpha, \beta)$ on the category $\mathcal{A}$ the standard constructions of the Eilenberg-Moore category $\mathcal{A}^{\mathrm{R}}$ and the Kleisli category $\mathcal{A}_{\mathrm{R}}$ are well known. The computation of $\mathcal{A}^{\mathrm{R}}$-limits easily can be derived from the computation of the corresponding $\mathcal{A}$-limits. On the other hand the case of $\mathcal{A}^{\mathrm{R}}$-colimits proved to be far more complicated. The various $\mathcal{A}^{\mathrm{R}}$-colimits (and $\mathcal{A}(R)$-colimits of R-algebras) were thoroughly investigated in a lot of papers (e.g. in [1-3, 5, 9, 10]). In this way the raising of the similar questions for the Kleisli category $\mathcal{A}_{\mathrm{R}}$ is quite natural. This paper makes an attempt to get closer to the problem of the completeness and the cocompleteness of $\mathcal{A}_{\mathrm{R}}$. The canonical functor $S: \mathcal{A}_{\mathrm{R}} \to \mathcal{A}^{\mathrm{R}}$ will play a central role in our development. It will be proved in 2 that $\dashv S$ is equivalent to the cocompleteness of $\mathcal{A}_{\mathrm{R}}$ under certain circumstances. In 3 we shall use the assumption $S \dashv$ in order to prove completeness for $\mathcal{A}_{\mathrm{R}}$. Mention must be made that these results are powerless in concrete instances. One can regard them as an alternative approach to the problem.

Part 4 deals with the existence of a left adjoint to R, where R is the base functor of some R. The main result of 4 states that if $\mathcal{A}$ has coequalizers of all pairs then the relative adjointness $\dashv_R R$ is equivalent to $\dashv R$.

2. COCOMPLETENESS.

Given a triple $\mathrm{R} = (R, \alpha, \beta)$ on the category $\mathcal{A}$ an adjoint pair $F \dashv U$ consisting of functors $F: \mathcal{A} \to \mathcal{B}$, $U: \mathcal{B} \to \mathcal{A}$ with unit $\epsilon: 1_{\mathcal{A}} \to U \circ F$ and counit $\delta: F \circ U \to 1_{\mathcal{B}}$ is said to be an R-*adjunction on* $\mathcal{A}$ if

$$R = U \circ F, \quad \alpha = \epsilon \quad \text{and} \quad \beta = U \delta F.$$

The Kleisli category $\mathfrak{A}_R$ of R is defined in the following manner (cf. [8, 11, 12]). Objects are the same as in $\mathfrak{A}$, namely $|\mathfrak{A}_R| = |\mathfrak{A}|$. A morphism $r: a \multimap a'$ in $\mathfrak{A}_R$ (the notation $\multimap$ for $\mathfrak{A}_R$-arrows will be used throughout this paper) is given by an $\mathfrak{A}$-morphism $r: a \to R a'$. The $a \multimap a$ unit in $\mathfrak{A}_R$ is $\alpha_a: a \to R a$ and the $\mathfrak{A}_R$-composition ∇ for

$$a \xrightarrow{\ r\ }\!\!\circ\ a' \xrightarrow{\ r'\ }\!\!\circ\ a''$$

is given by $r' \nabla r = \beta_{a''} \circ (R r') \circ r$. The functors of the Kleisli R-adjunction

$$F_R : \mathfrak{A} \to \mathfrak{A}_R \quad \text{and} \quad U_R : \mathfrak{A}_R \to \mathfrak{A}$$

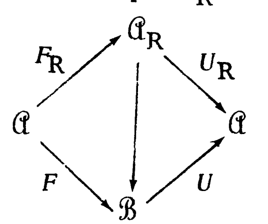
are defined by

$$a \xrightarrow{f} a' \ \overset{F_R}{\longmapsto}\ a \xrightarrow{\alpha_{a'} \circ f}\!\!\circ\ a' \quad \text{and} \quad a \xrightarrow{r}\!\!\circ\ a' \ \overset{U_R}{\longmapsto}\ R a \xrightarrow{\beta_{a'} \circ R r} R a' .$$

U_R is faithful since $U_R r$ uniquely determines r by

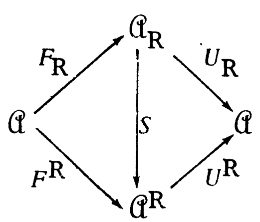
$$(U_R r) \circ \alpha_a = \beta_{a'} \circ (R r) \circ \alpha_a = \beta_{a'} \circ \alpha_{R a'} \circ r = 1_{R a'} \circ r = r .$$

The well known initiality of the Kleisli R-adjunction means that any R-adjunction $F \dashv U$ involves a unique $\mathfrak{A}_R \to \mathfrak{B}$ functor making (2:1) commute.



(2:1)

Take $F \dashv U$ to be the Eilenberg-Moore R-adjunction with $F^R : \mathfrak{A} \to \mathfrak{A}^R$, and $U^R : \mathfrak{A}^R \to \mathfrak{A}$, then there is a unique $S: \mathfrak{A}_R \to \mathfrak{A}^R$ such that (2:2) commutes :



(2:2)

The existence and the uniqueness of such a functor S also follow from the terminal property of the Eilenberg-Moore R-adjunction. A more explicit definition for S is the following

$$a \overset{r}{-\!\!\circ\!\!\rightarrow} a' \quad \overset{S}{\longmapsto} \quad <Ra, \beta_a> \xrightarrow{\beta_{a'} \circ Rr} <Ra', \beta_{a'}> .$$

$U^{\mathrm{R}} \circ S = U_{\mathrm{R}}$ implies that S is faithful just as U_{R}.

2.1. PROPOSITION. *Let* $\mathfrak{A}_{\mathrm{R}}$ *have coequalizers of all pairs (with a common coretraction) then* S *has a left adjoint.*

PROOF. Since $F_{\mathrm{R}} \dashv U_{\mathrm{R}} = U^{\mathrm{R}} \circ S$ an application of Johnstone's adjoint lifting theorem can give $\dashv S$ ($\mathfrak{A}_{\mathrm{R}}$ can be regarded as $\mathfrak{A}^{1}_{\mathrm{R}}$, where 1 is the trivial triple on $\mathfrak{A}_{\mathrm{R}}$). //

The question of $\mathfrak{A}_{\mathrm{R}}$-coproducts doesn't cause difficulties.

2.2. PROPOSITION. *Let* $\mathfrak{A}$ *have coproducts then* $\mathfrak{A}_{\mathrm{R}}$ *has coproducts.*

PROOF. For the objects $a_i \in |\mathfrak{A}_{\mathrm{R}}|$ $(i \in I)$ let $p_i : a_i \to x$ $(i \in I)$ be an $\mathfrak{A}$-coproduct, then $\alpha_x \circ p_i : a_i -\!\!\circ\!\!\rightarrow x$ $(i \in I)$ will be the required $\mathfrak{A}_{\mathrm{R}}$-coproduct. //

In (2.3) a certain converse of Proposition (2.1) will be established.

2.3. THEOREM. *Let* $\mathfrak{A}$ *be cocomplete (with an initial object),* $\mathfrak{A}^{\mathrm{R}}$ *have coequalizers of all pairs (with a common coretraction) and suppose that each partial R-algebra admits a free completion in* $\mathfrak{A}(R)$ *(consequently* $\mathfrak{A}(R)$ *has free algebras). If* S *has a left adjoint then* $\mathfrak{A}_{\mathrm{R}}$ *is cocomplete.*

To prove the above theorem we need the following lemma.

2.4. LEMMA. Let $\mathfrak{A}$ have finite coproducts and suppose that each partial T-algebra admits a free completion in $\mathfrak{A}(T)$ (apart from this $T : \mathfrak{A} \to \mathfrak{A}$ is arbitrary). Then the canonical embedding functor $B^T : \mathfrak{A}(T) \to (T{\downarrow}1_{\mathfrak{A}})$ has a left adjoint.

PROOF. For an object $u : Ta \to b$ in $(T{\downarrow}1_{\mathfrak{A}})$ consider the following partial T-algebra

$$T(a \amalg b) \xleftarrow{Tj_a} Ta \xrightarrow{u} b \xrightarrow{j_b} a \amalg b,$$

where j_a and j_b are coproduct injections. The free completion (2:3) of this partial T-algebra in $\mathfrak{A}(T)$ immediately yields an initial object in the comma category $(Ta \xrightarrow{u} b \downarrow B^T)$. //

$$\begin{array}{ccccc} & & Ta & & \\ & \swarrow^{Tj_a} & & \searrow^{u} & \\ T(a \amalg b) & & & & b \\ & & & & \downarrow^{j_b} \\ & & & & a \amalg b \\ \downarrow^{Ts^*} & & & & \downarrow^{s^*} \\ Tz^* & & \xrightarrow{\quad v^* \quad} & & z^* \end{array} \qquad (2{:}3)$$

PROOF OF (2.3). Since $\mathfrak{A}_R$ has coproducts by (2.2) we have to deal only with coequalizers. Consider a set $r_i : a \multimap a'$ $(i \in I)$ of parallel $\mathfrak{A}_R$-morphisms. Form the coequalizer

$$a \underset{\vdots}{\xrightarrow{r_i}} Ra' \xrightarrow{e} x$$

in $\mathfrak{A}$ and let (2:4) represent an initial object in

$$(Ra' \xrightarrow{e} x \downarrow B^R \circ E^R \circ S),$$

where

$$\mathfrak{A}^R \xrightarrow{E^R} \mathfrak{A}(R) \quad \text{and} \quad \mathfrak{A}(R) \xrightarrow{B^R} (R \downarrow 1_{\mathfrak{A}})$$

are canonical embeddings

$$\begin{array}{ccc} Ra' & \xrightarrow{Re^*} & R^2a^* \\ \downarrow^{e} & & \downarrow^{\beta_{a^*}} \\ x & \xrightarrow{t^*} & Ra^* \end{array} \qquad (2{:}4)$$

The existence of this initial object is clear since $\dashv B^R$ by (2.4) and $\dashv E^R$ by the assumptions that $\mathfrak{A}^R$ has coequalizers of all pairs (with a common coretraction) and that $\mathfrak{A}(R)$ has free algebras (Johnstone's adjoint lifting Theorem works again because of the free R-algebra adjunction is always monadic). We claim that $e^* : a' \multimap a^*$ is the $\mathfrak{A}_R$-coequalizer of the morphisms r_i $(i \in I)$. (2.4) proves that

$$e^* \nabla r_i = t^* \circ e \circ r_i = t^* \circ e \circ r_m = e^* \nabla r_m$$

for all $i, m \in I$. Suppose that $\hat{e} \nabla r_i = \hat{e} \nabla r_m$ $(i, m \in I)$ for an $\mathfrak{A}_R$-morphism $\hat{e} : a' \multimap \hat{a}$. The $\mathfrak{A}$-coequalizer property of e gives a unique $\hat{t} : x \to R\hat{a}$

making (2:5) commute

$$\begin{array}{ccc} R\,a' & \xrightarrow{R\,\hat e} & R^2\,\hat a \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \beta_{\hat a}} \\ x & \xrightarrow{\hat t} & R\,\hat a \end{array} \qquad (2:5)$$

Now

$$<\hat e, \hat t> : \; <a', e, x> \;\to\; B^R\,E^R\,S\,\hat a$$

is in $(R \downarrow 1_{\mathfrak{A}})$, so there exists a unique $k : a^* -\!\circ\!\!\to \hat a$ in $\mathfrak{A}_R$ such that (2:6) commutes.

$$\begin{array}{ccc} & \overset{<e^*,\, t^*>}{\nearrow} & B^R\,E^R\,S\,a^* \\ <a', e, x> & & \downarrow{\scriptstyle B^R E^R S k} \\ & \underset{<\hat e,\, \hat t>}{\searrow} & B^R\,E^R\,S\,\hat a \end{array} \qquad (2:6)$$

Using the facts that S is faithful and e is epimorphic in $\mathfrak{A}$ one can easily obtain that (2:6) and (2:7) are equivalent for a $k : a^* -\!\circ\!\!\to \hat a$. //

$$\begin{array}{ccc} & \overset{e^*}{\nearrow\!\!\circ} & a^* \\ a' & & \circ\!\downarrow{\scriptstyle k} \\ & \underset{\hat e}{\searrow\!\!\circ} & \hat a \end{array} \qquad (2:7)$$

3. COMPLETENESS.

3.1. THEOREM. *Let $\mathfrak{A}$ be complete and suppose that S has a right adjoint, then $\mathfrak{A}_R$ is complete.*

PROOF. Given a functor $D : \mathfrak{D} \to \mathfrak{A}_R$ observe that an $\mathfrak{A}_R$-cone $\xi : a -\!\circ\!\!\to D$ is the same thing as an $\mathfrak{A}$-cone $\xi : a \to U_R \circ D$ and vice versa.

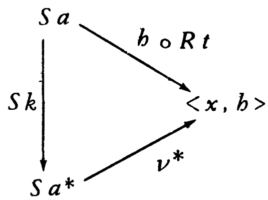
If $q : x \to U_R \circ D$ is a limit cone in $\mathfrak{A}$ then there is a unique $h : Rx \to x$, making the diagram (3:1) commute.

$$\begin{array}{ccc} x & \xrightarrow{q} & U_R \circ D \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \beta D} \\ R\,x & \xrightarrow{R\,q} & R \circ U_R \circ D \end{array} \qquad (3:1)$$

The adjointness $S \dashv$ yields a terminal object $\nu^*: Sa^* \to \langle x, h\rangle$, in $(S{\downarrow}\langle x, h\rangle)$. We claim that

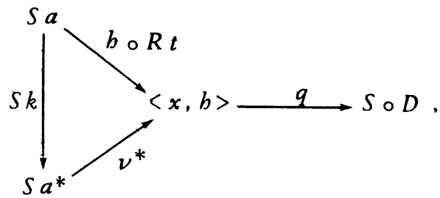
$$\xi^*: a^* \xrightarrow{\alpha_{a^*}} Ra^* \xrightarrow{\nu^*} x \xrightarrow{q} U_R \circ D$$

is an $\mathfrak{A}_R$-limit cone for D. Let $\xi: a \multimap D$ be a cone, then there is a unique $t: a \to x$ with $q \circ t = \xi$. Easily it can be seen that $h \circ Rt: Sa \to \langle x, h\rangle$ is a morphism in $\mathfrak{A}^R$, so there exists a unique $k: a \multimap a^*$ making (3:2) commute.



(3:2)

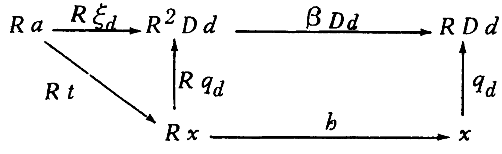
(3:1) makes q to be a $\langle x, h\rangle \to S \circ D$ limit cone in $\mathfrak{A}^R$, consequently for a morphism $k: a \multimap a^*$ (3:2) is equivalent to (3:3).



(3:3)

$$a \overset{\xi_d}{\multimap} Dd \overset{S}{\longmapsto} Sa \xrightarrow{q_d \circ h \circ Rt} SDd \text{ and } a^* \overset{\xi^*_d}{\multimap} Dd \overset{S}{\longmapsto} Sa^* \xrightarrow{q_d \circ \nu^*} SDd$$

are proved by (3:4) and (3:5) respectively.



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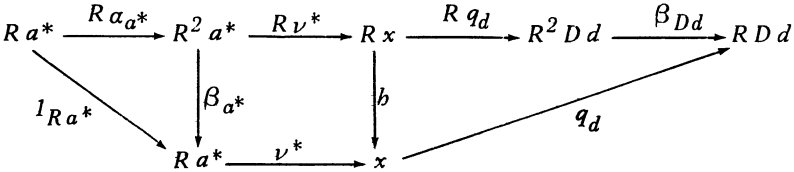
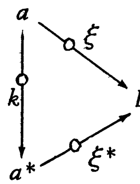


(3:5)

Since S is faithful we obtained that for a $k: a \multimap a^*$:

$$(3{:}2) \Leftrightarrow (3{:}3) \Leftrightarrow (3{:}6).$$



(3:6)

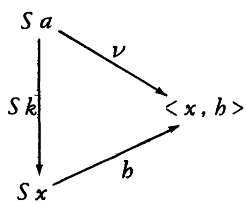
//

3.2. THEOREM. *Let $\mathcal{A}_R$ have coequalizers of all sets of parallel morphisms and suppose that $U_R : \mathcal{A}_R \to \mathcal{A}$ preserves these coequalizers, then S has a right adjoint.*

PROOF. At first we prove that $h : Sx \to \langle x, h\rangle$ is a weak terminal object in $(S{\downarrow}\langle x, h\rangle)$. For an object $\nu : Sa \to \langle x, h\rangle$ in $(S{\downarrow}\langle x, h\rangle)$ define the morphism $k : a \multimap x$ in $\mathcal{A}_R$ as

$$a \xrightarrow{\alpha_a} Ra \xrightarrow{\nu} x \xrightarrow{\alpha_x} Rx.$$

Clearly (3:7) commutes since $h \circ R\nu = \nu \circ \beta_a$.



(3:7)

Let $e^* : x \multimap a^*$ be the $\mathcal{A}_R$-coequalizer of those morphisms $r : x \multimap x$ for which $h \circ (Sr) = h$. Now

$$U_R x \xrightarrow[\vdots]{U_R r} U_R x \xrightarrow{U_R e^*} U_R a^*$$

is a coequalizer diagram in $\mathcal{A}$ by the preservation property of U_R. $RU_R e^*$ is also a coequalizer since $R = U_R \circ F_R$ and F_R preserves coequalizers by $F_R \dashv U_R$. Thus $RU_R e^*$ is proved to be an epimorphism. Accordingly, (3:8) is a coequalizer diagram in $\mathcal{A}^R$, i.e. S preserves the above mentioned (and any other) coequalizer.

Since $h \circ (Sr) = h$ for all the considered «r»'s there exists a unique $\nu^* : Sa^* \to \langle x, h\rangle$ in $\mathcal{A}^R$ with $\nu^* \circ (Se^*) = h$. A standard terminal object

$$\begin{array}{ccccc} U_{\mathrm{R}}x & \xrightarrow{U_{\mathrm{R}}r} & U_{\mathrm{R}}x & \xrightarrow{U_{\mathrm{R}}e^*} & U_{\mathrm{R}}a^* \\ \uparrow{\scriptstyle \beta_x} & & \uparrow{\scriptstyle \beta_x} & & \uparrow{\scriptstyle \beta_{a^*}} \\ R\,U_{\mathrm{R}}x & \xrightarrow[R\,U_{\mathrm{R}}r]{} & R\,U_{\mathrm{R}}x & \xrightarrow[R\,U_{\mathrm{R}}e^*]{} & R\,U_{\mathrm{R}}a^* \end{array} \qquad (3:8)$$

argument can prove that $\nu^*: Sa^* \to \langle x, b\rangle$ is terminal in $(S\downarrow\langle x, b\rangle)$. //

3.3. COROLLARY. *Let $\mathfrak{A}$ be complete, $\mathfrak{A}_{\mathrm{R}}$ have coequalizers of all sets of parallel morphisms and suppose that U_{R} preserves these coequalizers, then $\mathfrak{A}_{\mathrm{R}}$ is complete.* //

4. LEFT ADJOINTS BY DENSITY.

We start this part with two simple but extremely powerful lemmas.

4.1. LEMMA. Let $H: \mathfrak{H} \to \mathfrak{B}$ be a functor and $F: \mathfrak{A} \to \mathfrak{B}$, $U: \mathfrak{B} \to \mathfrak{A}$ be the functors of an adjoint pair $F \dashv U$ with unit $\epsilon: 1_{\mathfrak{A}} \to U \circ F$. Suppose that $(a\downarrow U \circ H)$ has an initial object, then there exists an initial object in $(Fa\downarrow H)$.

PROOF. If $f^*: a \to UH\omega^*$ is initial in $(a\downarrow U\circ H)$, then $\langle H\omega^*, f^*\rangle$ is in $(a\downarrow U)$. The initial property of $\epsilon_a: a \to UFa$ gives a unique morphism $g^*: Fa \to H\omega^*$ in $\mathfrak{B}$ making (4:1) commute.

$$\begin{array}{ccc} & & UFa \\ & \nearrow^{\epsilon_a} & \\ a & & \downarrow{\scriptstyle Ug^*} \\ & \searrow_{f^*} & \\ & & UH\omega^* \end{array} \qquad (4:1)$$

Clearly $\langle\omega^*, g^*\rangle$ is initial in $(Fa\downarrow H)$. //

4.2. LEMMA. Let H be as in (4.1), $D: \mathfrak{D} \to \mathfrak{B}$ be a functor with a colimit cocone $\gamma: D \to b$ and suppose that $(Dd\downarrow H)$ has an initial object for all $d \in |\mathfrak{D}|$. If $\mathfrak{H}$ is $\mathfrak{D}$-cocomplete, then there exists an initial object in $(b\downarrow H)$.

PROOF. For $d \in |\mathfrak{D}|$ let $\langle\omega(d), \eta_d\rangle$ be an initial object in $(Dd\downarrow H)$ and define the functor $D^*: \mathfrak{D} \to \mathfrak{H}$ by

$$d \to d' \quad \overset{D^*}{\longmapsto} \quad \omega(d) \to \omega(d'),$$

where $\omega(d) \to \omega(d')$ is the unique morphism making (4:2) commute.

$$\begin{array}{ccc} Dd & \xrightarrow{\quad \eta d \quad} & H\omega(d) \\ \Big\downarrow & & \Big\downarrow \\ Dd' & \xrightarrow{\quad \eta d' \quad} & H\omega(d') \end{array} \qquad (4{:}2)$$

Thus η becomes a $D \to H \circ D^*$ natural transformation. If $\gamma^* : D^* \to \omega^*$ is a colimit cocone then there is a unique $g^* : b \to H\omega^*$ making (4:3) commute.

$$\begin{array}{ccccc} D & \xrightarrow{\quad \eta \quad} & H \circ D^* & \xrightarrow{\quad H\gamma^* \quad} & H\omega^* \\ & {\scriptstyle\gamma}\searrow & & \nearrow{\scriptstyle g^*} & \\ & & b & & \end{array} \qquad (4{:}3)$$

The purely technical details of the verification that $\langle \omega^*, g^* \rangle$ is initial in $(b \downarrow H)$ are omitted. //

Now we can prove the main result of Section 4.

4.3. THEOREM. *Let $\mathfrak{A}$ have coequalizers of all pairs and $R : \mathfrak{A} \to \mathfrak{A}$ be the base functor of some triple $\mathbf{R} = (R, \alpha, \beta)$. Suppose that R has a left adjoint relative to R, i. e. $(Ra \downarrow R)$ has an initial object for all $a \in |\mathfrak{A}|$. Then there is a left adjoint to R, namely $\dashv R$.*

PROOF. Since $R = U_{\mathbf{R}} \circ F_{\mathbf{R}}$ with the functors of the Kleisli $\mathbf{R}$-adjunction $F_{\mathbf{R}} \dashv U_{\mathbf{R}}$ one can apply (4.1) in order to obtain an initial object in $(F_{\mathbf{R}} R a \downarrow F_{\mathbf{R}})$. $F_{\mathbf{R}} R a = R a$ and

$$Ra \;\underset{R\alpha_a}{\overset{\alpha_{Ra}}{\rightrightarrows}}\; Ra \xrightarrow{\quad 1_{Ra} \quad} a$$

is a coequalizer diagram in $\mathfrak{A}_{\mathbf{R}}$, so (4.2) gives an initial object in $(a \downarrow F_{\mathbf{R}})$. Thus $\dashv F_{\mathbf{R}}$ and $\dashv U_{\mathbf{R}}$ implies $\dashv R$. //

4.4. COROLLARY. *Let $\mathfrak{A}$ have coequalizers of all pairs and $R : \mathfrak{A} \to \mathfrak{A}$ be the base functor of some triple $\mathbf{R} = (R, \alpha, \beta)$. Suppose that for each object $a \in |\mathfrak{A}|$ there is an integer $n \geq 1$ such that $(R^n a \downarrow R)$ has an initial object. Then there is a left adjoint to R, namely $\dashv R$.* //

To illustrate the force of Lemmas 4.1 and 4.2 we give a very simple proof for an important theorem due to W. Tholen (see in [15]).

4.5. THEOREM. *Let H, F, U be as in (4.1) with counit $\delta : F \circ U \to 1_{\mathfrak{B}}$. If (i) or (j) holds, then $\dashv H$ is equivalent to $\dashv U \circ H$.*

(i) $\mathfrak{H}$ has coequalizers of all pairs and each δ_b ($b \in |\mathfrak{B}|$) is a regular epimorphism;

(j) $\mathfrak{H}$ has coequalizers of all pairs with a common coretraction and each δ_b ($b \in |\mathfrak{B}|$) is a regular epimorphism with a kernel pair.

PROOF. $\dashv H \Rightarrow \dashv U \circ H$ is trivial. Set $\dashv U \circ H$. If

$$\bar{b} \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} F\,U\,b \xrightarrow{\delta_b} b$$

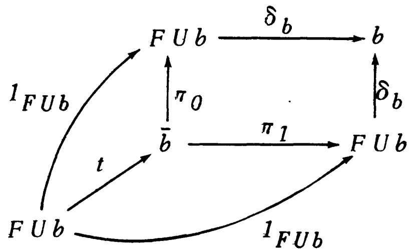
is a coequalizer in $\mathfrak{B}$ then so is (4:4) since $\delta_{\bar{b}}$ is epimorphic.

$$F\,U\,\bar{b} \underset{\pi_1 \circ \delta_{\bar{b}}}{\overset{\pi_0 \circ \delta_{\bar{b}}}{\rightrightarrows}} F\,U\,b \xrightarrow{\delta_b} b \qquad (4:4)$$

By (4.1) there is an initial object in $(F\,U\,\bar{b}\downarrow H)$ and in $(F\,U\,b\downarrow H)$. Let $\gamma : D \to b$ of (4.2) represent the coequalizer (4:4), then (4.2) gives an initial object in $(b\downarrow H)$. If in addition

$$\bar{b} \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} F\,U\,b \xrightarrow{\delta_b} b$$

is a kernel pair, then $(F\,U\,t) \circ F\epsilon_{Ub}$ is a common coretraction of $\pi_0 \circ \delta_{\bar{b}}$ and $\pi_1 \circ \delta_{\bar{b}}$, where $t : F\,U\,b \to \bar{b}$ is the unique morphism making (4:5) commute.



(4:5)

Now $\gamma : D \to b$ (and consequently $\gamma^* : D^* \to \omega^*$) represents a coequalizer of a pair with a common coretraction. Essentially we proved that the full subcategory of $\mathfrak{B}$ consisting of all objects of the form Fa $(a \in |\mathfrak{A}|)$ is a dense subcategory. //

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