Discrete Mathematics 344 (2021) 112445

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

On Turán-good graphs

Dániel Gerbner [1]

Alfréd Rényi Institute of Mathematics, Hungary

ARTICLE INFO

Article history:
Received 23 December 2020
Received in revised form 18 April 2021
Accepted 21 April 2021
Available online 27 April 2021

Keywords:
Extremal graph theory
Turán graph
Generalized Turán problems

ABSTRACT

For graphs H and F, the generalized Turán number $\mathrm{ex}(n, H, F)$ is the largest number of copies of H in an F-free graph on n vertices. We say that H is F-Turán-good if $\mathrm{ex}(n, H, F)$ is the number of copies in the $(\chi(F)-1)$-partite Turán graph, provided n is large enough. We present a general theorem in case F has an edge whose deletion decreases the chromatic number. In particular, this determines $\mathrm{ex}(n, P_k, C_{2\ell+1})$ and $\mathrm{ex}(n, C_{2k}, C_{2\ell+1})$ exactly, if n is large enough. We also study the case when F has a vertex whose deletion decreases the chromatic number.

© 2021 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

A basic result in extremal Combinatorics is Turán's theorem [18]. It states that a K_k-free graph on n vertices cannot have more edges than the Turán graph $T_{k-1}(n)$, which is the complete $(k-1)$-partite graph where each partite class has cardinality $\lfloor n/(k-1)\rfloor$ or $\lceil n/(k-1)\rceil$. In general, Turán theory deals with the function $\mathrm{ex}(n, F)$, which is the largest number of edges in n-vertex F-free graphs. Let $\mathcal{N}(H, G)$ denote the number of copies of H in G. Generalized Turán theory deals with $\mathrm{ex}(n, H, F) := \max\{\mathcal{N}(H, G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\}$, i.e. the largest number of copies of H in F-free graphs on n vertices.

After several sporadic results (see e.g. [2,11,13,15,12,19]), the systematic study of this problem was initiated by Alon and Shikhelman [1]. Since then, this problem has attracted several researchers, see e.g. [3–5,7–10,14,16].

However, there are not many exact results in this area (by exact result we mean that for given H and F, we know the value of $\mathrm{ex}(n, H, F)$ for every n large enough). Most of the exact results are when the Turán graph contains the most copies of H. Győri, Pach and Simonovits [13] examined for what graphs H do we have that $\mathrm{ex}(n, H, K_{k+1}) = \mathcal{N}(H, T_k(n))$. Gerbner and Palmer [9] extended these investigations for arbitrary k-chromatic graphs. Following them, given a graph F with $\chi(F) = k$, we say that H is *F-Turán-good* if $\mathrm{ex}(n, H, F) = \mathcal{N}(H, T_{k-1}(n))$ and H does not contain F. If $F = K_k$, we use the briefer term *k-Turán good*. Let us state the main result of Győri, Pach and Simonovits [13] using this term.

Theorem 1.1 *(Győri, Pach and Simonovits [13]). Let $r \geq 3$ and let H be a $(k-1)$-partite graph with $m > k-1$ vertices, containing $\lfloor m/(k-1)\rfloor$ vertex disjoint copies of K_{k-1}. Suppose further that for any two vertices u and v in the same connected component of H, there is a sequence $A_1, \dots, A_s$ of $(k-1)$-cliques in H such that $u \in A_1$, $v \in A_s$, and for any $i < s$, A_i and A_{i+1} share $k-2$ vertices. Then H is k-Turán-good. Moreover, if n is large enough, the Turán graph is the only K_k-free graph with $\mathrm{ex}(n, H, K_k)$ copies of H.*

Gerbner and Palmer [9] obtained a theorem of a similar flavor.

E-mail address: gerbner.daniel@renyi.hu.

[1] Research supported by the National Research, Development and Innovation Office – NKFIH under the grants FK 132060, KKP-133819, KH130371 and SNN 129364.

https://doi.org/10.1016/j.disc.2021.112445

0012-365X/© 2021 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

Theorem 1.2 *(Gerbner, Palmer [9]). Let H be a k-Turán-good graph. Let H' be any graph constructed from H in the following way. Choose a complete subgraph of H with vertex set X, add a vertex-disjoint copy of K_{k-1} to H and join the vertices in X to the vertices of K_{k-1} by edges arbitrarily. Then H' is k-Turán-good.*

Neither of the above two theorems imply the other. The main difference is that vertices in the additional clique can be connected to anything in Theorem 1.1, but only to vertices of another clique in Theorem 1.2. The trade-off is the necessity of the strong connection property of the cliques in Theorem 1.1. Let us remark that for $k=3$, the assumptions of Theorem 1.1 are nothing else but that H is bipartite and has a matching of size $\lfloor |V(H)|/2 \rfloor$; the property of the sequence of 2-cliques reduces to the property that every connected component is connected.

Observe that on their own, both theorems use K_{k-1} as building blocks, and can be used only for graphs mostly covered by vertex-disjoint copies of K_{k-1}. Therefore, another difference is that in Theorem 1.2 we can start with an arbitrary k-Turán-good graph, and add cliques afterwards. Here we prove such a variant for Theorem 1.1.

Proposition 1.3. *Let H be a k-Turán-good graph with a unique proper $(k-1)$-coloring. Let H' consist of a copy K of K_{k-1} with vertices $v_1, \dots, v_{k-1}$ and H, with additional edges between $V(H)$ and $V(K)$ such that for every $i \le k-1$, there is a copy of K_{k-1} in H' containing v_i, but not containing any v_j for $j > i$. If H' has chromatic number $k-1$, then H' is k-Turán-good.*

Let us show an example where this proposition is stronger than the above theorems. We will start with a slightly unbalanced complete bipartite graph. Ma and Qiu [16] showed that $K_{s,t}$ with $s \le t$ is 3-Turán-good if and only if $t < s + 1/2 + \sqrt{2s+1/4}$. Proposition 1.3 implies that if the vertices of a connected bipartite graph H can be vertex-disjointly covered by one such $K_{s,t}$ and a matching, then H is 3-Turán-good.

Let us turn our attention to F-Turán-good graphs where F is not a clique. We show a weak version of the above results for this case. We say that an edge of a graph G is a color-critical edge if deleting it from G decreases its chromatic number. An m-chromatic graph F with a color-critical edge often behaves similarly to K_m in extremal problems. In particular, Simonovits [17] showed that for n large enough, among all n-vertex F-free graphs the Turán graph $T_{m-1}(n)$ contains the most number of edges, and it was extended by Ma and Qiu [16], who showed that $T_{m-1}(n)$ also contains the most number of K_r for $r < m$. Gerbner [6] proved a stability version.

Lemma 1.4 *(Gerbner [6]). Let F be a k-chromatic graph with a color-critical edge and $r < k$. If G is an n-vertex F-free graph with chromatic number more than $k-1$, then $ex(n, K_r, F) - \mathcal{N}(G, K_r) = \Omega(n^{r-1})$.*

Using this, we can extend the above theorems from K_k to certain graphs with color-critical edges. The main idea is that if an F-free graph does not have too many copies of K_k, then those create only a negligible amount of copies of H.

Theorem 1.5. *Let F be a k-chromatic graph with a color-critical edge such that $\mathrm{ex}(n, K_k, F) = o(n^{k-1})$, and H be a graph that is both F-Turán-good and k-Turán-good. Let H' be any $(k-1)$-colorable graph constructed from H in the following way. Choose a complete subgraph of H with vertex set X, add a vertex-disjoint copy of K_{k-1} to H and join the vertices in X to the vertices of K_{k-1} by edges arbitrarily. Then H' is F-Turán-good. Moreover, if G is an n-vertex F-free graph with chromatic number more than $k-1$, then $\mathrm{ex}(n, H', F) - \mathcal{N}(H', G) = \Omega(n^{|V(H')|-1})$.*

Proposition 1.6. *Let F be a k-chromatic graph with a color-critical edge such that $\mathrm{ex}(n, K_k, F) = o(n^{k-1})$. Let H be a graph that is both F-Turán-good and k-Turán-good with a unique proper $(k-1)$-coloring. Let H' consist of H and a copy K of K_{k-1} with vertices $v_1, \dots, v_{k-1}$, with additional edges between $V(H)$ and $V(K)$ such that for every $i \le k-1$, there is a copy of K_{k-1} in H' containing v_i, but not containing any v_j for $j > i$. If H' has chromatic number $k-1$, then H' is F-Turán-good. Moreover, if G is an n-vertex F-free graph with chromatic number more than $k-1$, then $\mathrm{ex}(n, H', F) - \mathcal{N}(H', G) = \Omega(n^{|V(H')|-1})$.*

Let us note the extra assumption $\mathrm{ex}(n, K_k, F) = o(n^{k-1})$. For $k=3$, an example is $F = C_{2\ell+1}$, as $\mathrm{ex}(n, K_3, C_{2\ell+1}) = O(n^{1+1/\ell})$ due to Győri and Li [12]. Another example is the book B_t, which consists of an edge uv and t other vertices, that are adjacent to both u and v. Alon and Shikhelman [1] showed $\mathrm{ex}(n, K_3, B_t) = o(n^2)$.

Gerbner and Palmer [9] conjectured that paths P_m and even cycles C_{2m} are $C_{2\ell+1}$-Turán-good for any m and ℓ. They proved this conjecture for P_4 and $\ell = 2$. They also showed that if P_{2m} is $C_{2\ell+1}$-Turán-good, then C_{2m} is $C_{2\ell+1}$-Turán-good too. Thus we can fully resolve their conjecture using Theorem 1.5 or Proposition 1.6.

Corollary 1.7. *For any positive integers m and ℓ, P_m and C_{2m} are $C_{2\ell+1}$-Turán-good.*

Gerbner and Palmer [9] also showed that P_4 is B_2-Turán-good. We can generalize it as follows.

Corollary 1.8. *For any positive integers m and t, P_m is B_t-Turán-good.*

Gerbner and Palmer [9] showed an example where a graph H is F-Turán-good, and F does not have a color-critical edge ($H = C_4$ and F is the *2-fan*, two triangles sharing a vertex). Observe that one can add additional edges to the Turán graph without violating the F-free property in this case. However, those edges cannot create additional copies of H. A natural question is for what graphs F can we find an H such that H is F-Turán-good?

We say that a vertex v of a graph G is *color-critical* if by deleting v from G we obtain a graph with smaller chromatic number.

Theorem 1.9. *There exists an F-Turán-good graph if and only if F has a color-critical vertex.*

We prove Proposition 1.3, Theorem 1.5 and Proposition 1.6 in Section 2, and we prove Theorem 1.9 in Section 3. We finish the paper with some concluding remarks in Section 4.

2. Graphs with a color-critical edge

We start with the proof of Proposition 1.3, that we restate here for convenience.

Proposition. *Let H be a k-Turán-good graph with a unique proper $(k-1)$-coloring. Let H' consist of H and a copy K of K_{k-1} with vertices $v_1, \dots, v_{k-1}$, with additional edges between $V(H)$ and $V(K)$ such that for every $i \le k-1$, there is a copy of K_{k-1} in H' containing v_i, but not containing v_j for $j > i$. If H' has chromatic number $k-1$, then H' is k-Turán-good.*

Proof. Let G be a K_k-free graph on n vertices. We will count the copies of H' in G the following way. First we pick a copy K' of K_{k-1}, then a vertex-disjoint copy H_0 of H. Then we pick an actual embedding of H into H_0, and afterwards an actual embedding of K into K' such that the images of the remaining edges of H' are present in G. We claim that $T_{k-1}(n)$ gives the maximum for each of the above four factors, finishing the proof.

We have $\mathcal{N}(K_{k-1}, G) \le \mathcal{N}(K_{k-1}, T_{k-1}(n))$ by a theorem of Zykov [19] (this particular case also follows from Theorem 1.1), thus the number of ways to pick a copy K' of K_{k-1} in G is the largest if G is the Turán graph. Then there are at most $\mathrm{ex}(n-k+1, H, K_k)$ ways to pick a vertex-disjoint copy of H, which is at most $\mathcal{N}(H, T_{k-1}(n-k+1))$ for n large enough, as H is k-Turán-good. Observe that we have equality here in case $G = T_{k-1}(n)$, as removing a maximal clique from the Turán graph gives a smaller Turán graph. The number of ways H can be embedded into H_0 is the number of isomorphisms of H and does not depend on G.

After H is embedded, we claim that there is at most one way to finish the embedding. We pick the images of the vertices v_i from K one by one, in the order of their indices. For each v_i, it is contained in a copy K'' of K_{k-1} that is already embedded. The other $k-2$ vertices of K'' are already embedded, and their images have at most one common neighbor in K', as G is K_k-free. This means we only have one vertex that can be picked as v_i.

Finally, we show that in the Turán graph, there is a way to finish the embedding. The other $k-2$ vertices of K'' that are already embedded must belong to different partite classes of the Turán graph, thus v_i must be from the remaining partite class. We have to show that this way the v_i's are mapped to different vertices. This is where we use the unique coloring property of H. Recall that H' is also $(k-1)$-colorable, and in fact H' is also uniquely $(k-1)$-colorable due to the existence of the $(k-1)$-clique K'' for every v_i. Let j be a color class in this unique coloring, then all the vertices of color j are mapped to the same partite class A_j of the Turán graph. If v_i is of color j in H', then the other $k-2$ vertices of K'' are not of color j, thus they are not in A_j, hence v_i is mapped to the vertex of K' that belong to A_j. As the vertices of K belong to different color classes in H', they are mapped to different partite classes of the Turán graph, finishing the proof. ■

Let us continue with the proof of Theorem 1.5, that we restate here for convenience.

Theorem. *Let F be a k-chromatic graph with a color-critical edge such that $\mathrm{ex}(n, K_k, F) = o(n^{k-1})$, and H be a graph that is both F-Turán-good and k-Turán-good. Let H' be any $(k-1)$-colorable graph constructed from H in the following way. Choose a complete subgraph of H with vertex set X, add a vertex-disjoint copy of K_{k-1} to H and join the vertices in X to the vertices of K_{k-1} by edges arbitrarily. Then H' is F-Turán-good. Moreover, if G is an n-vertex F-free graph with chromatic number more than $k-1$, then* $\mathrm{ex}(n, H', F) - \mathcal{N}(H', G) = \Omega(n^{|V(H')|-1})$.

Proof. By a result of Ma and Qiu [16], if n is large enough, then the maximum number of copies of K_{k-1} in an F-free graph is achieved by the Turán graph $T_{k-1}(n)$. Since H is F-Turán-good, the Turán graph $T_{k-1}(n-k+1)$ has the maximum number of copies of H among F-free graphs on $n-k+1$ vertices. We will show that $T_{k-1}(n)$ has the maximum number of copies of H'.

Let G be an F-free graph on n vertices with the maximum number of copies of H'. If G has chromatic number at most $k-1$, then G is K_k-free, thus we are done by Theorem 1.2. If G has chromatic number more than $k-1$, then by Lemma 1.4 we have $\mathcal{N}(K_{k-1}, T_{k-1}(n)) - \mathcal{N}(K_{k-1}, G) = \Omega(n^{k-2})$.

We follow the proof of Theorem 1.2 from [9]. We take a copy K of K_{k-1} in G, and then a complete subgraph Y of G, disjoint from K, and consider the bipartite subgraph G' of G consisting of the edges between K and Y. It was shown in [9] that if G is K_k-free, then a matching covering Y is missing from G'. It is easy to see that if such a matching is not missing, then not only there is a K_k in G, but there is a K_k with vertices in $Y \cup V(K)$. For the sake of completeness, we repeat the argument here. By Hall's theorem, there is a subset $Y' \subset Y$ such that all the vertices of Y' are connected to all but less than $|Y|'$ vertices of K. Then Y' and those vertices form a clique of size at least k. By the same reasoning, in H' there is a matching missing between H and K.

Let us count first the copies of H' such that there is no K_k in G on the vertex sets of them. For those copies there is a matching missing from G'. Observe that in the Turán graph between a clique of size $k-1$ and a clique of size $|Y|$, only a matching covering the smaller clique is missing. This implies that after picking a copy of H, there are at least as many ways to connect the appropriate subclique of it to K in the Turán graph, as in G (and in the Turán graph, this number is at least 1).

The number of such copies of H' can be counted the following way. First we pick a copy of K_{k-1} at most $\mathcal{N}(K_{k-1}, T_{k-1}(n)) - \Omega(n^{k-2})$ ways, then we pick a copy of H on the remaining $n-k+1$ vertices, and then connect the vertices of the copies of K_{k-1} and H. Finally, we have to divide by the number of times a copy of H' was counted. The number of ways to pick a copy of H on $n-k+1$ vertices is maximized by the Turán graph and is $\Omega(n^{|V(H)|})$, the number of ways to connect the vertices of the copies of K_{k-1} and H is also maximized by the Turán graph, while the last quantity depends only on H'. This implies that the number of such copies of H' is $\mathcal{N}(H', T_{k-1}(n)) - \Omega(n^{|V(H')|-1})$.

Let us continue with the copies of H' that contain a vertex set of K_k in G. As G is F-free, there are $o(n^{k-1})$ copies of K_k in G, thus $o(n^{|V(H')|-1})$ copies of H'. Adding up the two bounds finishes the proof. ■

Let us continue with Proposition 1.6 that we restate here for convenience, We only give a sketch of the proof, as it can be easily obtained by combining the above two proofs. We assume familiarity with those proofs.

Proposition. *Let F be a k-chromatic graph with a color-critical edge such that $\mathrm{ex}(n, K_k, F) = o(n^{k-1})$. Let H be an F-Turán-good graph with a unique proper $(k-1)$-coloring. Let H' consist of H and a copy K of K_{k-1} with vertices $v_1, \dots, v_{k-1}$, with additional edges between $V(H)$ and $V(K)$ such that for every $i \le k-1$, there is a copy of K_{k-1} in H' containing v_i, but not containing v_j for $j > i$. If H' has chromatic number $k-1$, then H' is F-Turán-good. Moreover, if G is an n-vertex F-free graph with chromatic number more than $k-1$, then $\mathrm{ex}(n, H', F) - \mathcal{N}(H', G) = \Omega(n^{|V(H')|-1})$.*

Sketch of proof. Let G be an F-free graph on n vertices. If G has chromatic number $k-1$, then it is K_k-free and we are done. If G has chromatic number at least k, then by Lemma 1.4 G has less copies of K_{k-1} than the Turán graph by $\Omega(n^{k-2})$.

First we count copies of H' in G such that there are no k vertices in that copy that induce a clique in G. For them, we can follow the proof of Proposition 1.3, with one exception: the number of $(k-1)$-cliques is less by $\Omega(n^{k-2})$, which implies that the number of copies of H is less by $\Omega(n^{|V(H')|-1})$.

Then we count the other copies of H': as there are $o(n^{k-1})$ copies of K_k in G, we have that there are $o(n^{|V(H')|-1})$ such copies of H. Adding up the two bounds finishes the proof. ■

3. Graphs with a color-critical vertex

We will use progressive induction. This is a version of induction that can be used to prove combinatorial statements that hold only for n large enough. It was introduced by Simonovits [17]. It was used for a generalized Turán problem in [5]. The statement in [17] is very general, here we state a version adapted for generalized Turán problems.

For a graph G and a subgraph G', we denote by $\mathcal{N}_I(H, G, G')$ the number of copies of H in G that contain at least one vertex from G'. Given H and F, we say that G is an extremal graph if $\mathrm{ex}(n, H, F) = \mathcal{N}(H, G)$.

Lemma 3.1. *Let F and H be graphs and $\mathcal{G}_n$ be a family of n-vertex F-free graphs for every n, such that if $G_1, G_2 \in \mathcal{G}_n$, then $\mathcal{N}(H, G_1) = \mathcal{N}(H, G_2)$. Assume that there is an n_0 such that for every $n \ge n_0$, for every extremal graph G on n vertices, there is a subgraph H' of G with $|V(H')| \le n/2$, such that H' is also the subgraph of some $G_n \in \mathcal{G}_n$ and we have the following: $\mathcal{N}_I(H, G, H') \le \mathcal{N}_I(H, G_n, H')$, with equality only if $G \in \mathcal{G}_n$.*

Then for n large enough, $\mathrm{ex}(n, H, F) = \mathcal{N}(H, G_n)$ for some $G_n \in \mathcal{G}_n$. Moreover, every extremal graph belongs to $\mathcal{G}_n$.

We omit the proof of this specialized version. It follows from the original version [17] in a straightforward way, but it is also easy to see why it holds without knowing the original proof. For small n it is possible that some graph has more copies of H than any $G_n \in \mathcal{G}_n$. However, this means a surplus of constant many copies of H, and then this surplus starts decreasing when $n \ge n_0$, and eventually vanishes. Moreover, this decreasing does not stop at this point, thus for even larger n $G \notin \mathcal{G}_n$ cannot be extremal.

We will also use the following result.

Theorem 3.2 *(Gerbner and Palmer [8]). Let H be a graph and F be a graph with chromatic number k, then* $\mathrm{ex}(n, H, F) \le \mathrm{ex}(n, H, K_k) + o(n^{|V(H)|})$.

The harder part of Theorem 1.9 follows from the next theorem.

Theorem 3.3. *Let H be a complete k-partite graph $K_{b,\dots,b}$ and F be a complete $(k+1)$-partite graph $K_{1,a,\dots,a}$ such that $b > 2a-2$. Then H is F-Turán-good. Moreover, every extremal graph contains $T_k(n)$.*

Proof. We start with describing the proof informally. Let G be an F-free graph on n vertices. We will show that either we can apply Lemma 3.1 to prove the statement, or we can find a complete k-partite graph containing almost all the vertices and edges of G. To do so, we find a copy H' of the complete k-partite graph $K_{c,\dots,c}$ for some c, and show that almost all the vertices of G are connected to all the vertices of $k-1$ partite sets of H'.

Let H_0 denote the complete k-partite graph $K_{b-1,b,b,\dots,b}$.

Claim 3.4. *Let G be an F-free graph and assume that $V(G)$ is partitioned to $V_1,\dots,V_k$ such that for every i, a vertex in V_i is adjacent to less than a other vertices in V_i. Then every copy of H and H_0 in G has a partite set in each V_i.*

Proof. Let us call *extra edges* the edges inside a V_i for some i. If a copy of H contains an extra edge between u and v ($u,v \in V_i$), that means each other vertex of that H is adjacent to at least one of u and v, thus H contains at most $2a-4$ other vertices from V_i. Therefore, if a copy of H (or H_0) in G contains a set U of more than $2a-2$ vertices from a V_i, then there are no extra edges inside U, thus no edges at all inside U. Thus U is a subset of a partite set of H (or H_0), in particular $|U| \le b$. Thus the only way to choose kb (or $kb-1$) vertices from the k partite sets of $T_k(n)$ is to choose b from each (or $b-1$ from one partite set and b from the other partite sets), and they have to form the partite sets of that copy H (or H_0), thus no extra edge is used here. ■

Let $\mathcal{G}_n$ be the family of F-free graphs containing $T_k(n)$, and observe that they each contain the same number of copies of H. Indeed, fix a $G_n \in \mathcal{G}_n$ and a copy of $T_k(n)$ in it, then a vertex is adjacent to less than a vertices in the same partite set, otherwise these $a+1$ vertices with arbitrary a vertices from each other part form a copy of F. Thus we can apply Claim 3.4 to show that the additional edges do not create any copies of H.

Let G be an F-free graph on n vertices with $\mathcal{N}(H,G) = \mathrm{ex}(n,H,F)$ and assume indirectly that progressive induction (Lemma 3.1) cannot be applied to finish the proof, i.e. for any subgraph H' of G with $|V(H')| \le n/2$, we do not have that $\mathcal{N}_I(H,G,H') \le \mathcal{N}_I(H,T_k(n),H')$ with equality only if $G \in \mathcal{G}_n$. Let x denote the minimum number of copies of H that a vertex in the Turán graph $T_k(n)$ is contained in.

Claim 3.5. *Every vertex v is contained in at least x copies of H in G.*

Proof. Otherwise let H' be the graph containing only v, and we can apply Lemma 3.1 to finish the proof, a contradiction. ■

Observe that if the degree of v is $o(n)$, then v is in less than x copies of H. Indeed, every copy of H that contains v also contains at least one vertex in its neighborhood, thus we can count the copies of H containing v by picking a neighbor of v ($o(n)$ ways), then picking $|V(H)|-2$ other vertices ($O(n^{|V(H)|-2})$ ways). Therefore, the number of copies of H containing v is $o(n^{|V(H)|-1})$, while x is $\Theta(n^{|V(H)|-1})$.

Therefore, we can assume that every vertex has linear degree. We will try to apply Lemma 3.1 with H' being the complete k-partite graph $K_{c,\dots,c}$ with $c = k(a-1)+1$. If H' is not a subgraph of G, then we use a result of Alon and Shikhelman [1]. They showed that $\mathrm{ex}(n,T,T') = \Theta(n^{|V(T)|})$ if and only if T' is not a subgraph of T. Obviously, H is a subgraph of a blowup of H', thus in the case G is H'-free, we have $\mathcal{N}(H,G) \le \mathrm{ex}(n,H,H') = o(n^{|V(H)|-1})$, a contradiction. We pick a copy of H' with partite sets $A_1,\dots,A_k$, and with a slight abuse of notation we denote that copy by H' and its vertex set by U', and we also denote a copy of H' in $T_k(n)$ by H'.

To apply Lemma 3.1 in this case, we need to show that $\mathcal{N}_I(H,G,H') \le \mathcal{N}_I(H,T_k(n),H')$, with equality only if $G \in \mathcal{G}_n$. Assume that $\mathcal{N}_I(H,G,H') \ge \mathcal{N}_I(H,T_k(n),H')$. Observe that the number of copies of H containing t vertices from U' is $\Theta(n^{|V(H)|-t})$ in $T_k(n)$, and $O(n^{|V(H)|-t})$ in G. Therefore, we will focus on the main term $n^{|V(H)|-1}$, thus on the copies of H that contain exactly one vertex from U'.

Let G' be the subgraph of G induced on the vertices not in U'. Observe that if a vertex in G' is connected to each A_i by at least a vertices, then they form a copy of F, a contradiction. Thus for each vertex v of G', there is at least one A_i such that v is connected to at most $a-1$ vertices of A_i. Note that v can be connected to each vertex of some other A_j. We say that a copy of H_0 in G' is *nice* if all the vertices of the $k-1$ larger parts of this copy are connected to all the c vertices in some A_j.

Claim 3.6. *All but $o(n^{|V(H_0)|})$ copies of H_0 in G' are nice.*

Proof. Let us consider a copy of H that contains exactly one vertex from U'. It means it contains a copy of the complete k-partite graph $H_0 := K_{b-1,b,b,\dots,b}$ in G'. By Theorem 3.2, G' contains at most $(1+o(1))\mathrm{ex}(n-kc,H_0,K_{k+1})$ copies of H_0. Theorem 1.1 shows that $T_k(m)$ is $(k+1)$-Turán-good for every m, thus we have that $\mathrm{ex}(n-kc,H_0,K_{k+1}) = \mathcal{N}(H_0,T_k(n-kc))$.

It is easy to see that in the Turán graph, every copy of H_0 that avoids the selected copy of H' can be extended to a copy of H with one of c vertices of that copy of H' (those in the same partite class of the Turán graph). We claim that in G, every copy of H_0 can be extended to a copy of H with at most c of the vertices of H'. Indeed, let U'' be the set of vertices in U' that are connected to every vertex in the $(k-1)$ larger partite sets of H_0. If U'' intersects some partite sets of H' in at least a vertices, and has another vertex v in another partite set, then v, those a vertices, and a vertices from each of the $k-1$ larger partite sets of H_0 form a copy of F, a contradiction. If $|U''|>c$, then this is the case. Indeed, as $c>k(a-1)$, one of the partite sets of H' shares at least a elements with U'', and U'' does not fit into a single partite set. Moreover, if $|U''|=c$, then we have the same situation, unless U'' is a partite set of H'.

The main term of $\mathcal{N}_I(H,G,H')$ is at most $(1+o(1))\mathcal{N}(H_0,T_k(n-kc))$ times c, minus the number of those copies of H_0 in G' that are not connected to the c vertices of one part of H' (as they should be counted at most $c-1$ times). If the last term is $\Omega(n^{|V(H_0)|})$, then the main term is smaller than the main term of $\mathcal{N}_I(H,T_k(n),H')$, a contradiction finishing the proof of the claim. ■

Let us return to the proof of the theorem. Consider a copy of H in G'. We say that a vertex v of it is *replaceable* (with respect to that copy) if deleting v we obtain a nice copy of H_0, i.e. all the vertices of the $k-1$ larger parts of that copy of H_0 are connected to the c vertices of one part of H'. If that part is A_i, we say that v is replaceable by A_i.

Consider an arbitrary vertex v in G'. We know that v is in $\Omega(n^{|V(H)|-1})$ copies of H in G, and $O(n^{|V(H)|-2})$ of those copies share a vertex with H'. This implies that there are $\Omega(n^{|V(H)|-1})$ copies of H_0 in G' that can be extended to a copy of H with v. $\Omega(n^{|V(H)|-1})$ of those copies of H_0 can also be extended to a copy of H with any one vertex from A_i for some $i\le k$ by Claim 3.6, i.e. v is replaceable with respect to $\Omega(n^{|V(H)|-1})$ copies of H. Let B_i denote the set of vertices in G' such that there are $\Omega(n^{|V(H)|-1})$ copies of H_0 in G' that can be extended to a copy of H with v or with any vertex of A_i, i.e. v is replaceable by A_i with respect to $\Omega(n^{|V(H)|-1})$ copies of H. So we have $V(G')=B_1\cup\dots\cup B_k$.

Observe that every $v\in B_i$ is connected to less than a vertices of A_i. Indeed, let us consider a copy of H_0 that v extends to a copy of H. If v is connected to a vertices of A_i, then we can take a vertices from each of the $k-1$ larger partite classes of H_0, a neighbors of v from A_i and v to obtain a copy of F.

Claim 3.7. *All but $o(n)$ vertices in B_i are connected to all the vertices in every A_j with $j\neq i$.*

Proof. Consider a vertex $v\in B_i$ and one of the $\Omega(n^{|V(H)|-1})$ copies of H where v is replaceable by A_i. We denote this copy by H^*. If a vertex of H^* is replaceable by A_j with respect to H, then v is connected to all the vertices in A_j and we are done. Otherwise every copy of H_0 obtained by removing a vertex of H^* not in B_i must belong to the $o(n^{|V(H_0)|})$ exceptional copies in Claim 3.6. Observe that only vertices in one part of H^* can belong to B_i, as vertices in other parts are connected to $c\ge a$ vertices of A_i.

We obtain $\Omega(n^{|V(H)|-2})$ copies of H_0 this way, as a copy of H_0 might be obtained $O(n)$ ways. If we have $\Omega(n)$ exceptional vertices, they would belong to $\Omega(n^{|V(H)|-1})=\Omega(n^{|V(H_0)|})$ not nice copies of H_0, a contradiction with Claim 3.6. ■

Let $B_i'=B_i\cup A_i$ for every i.

Claim 3.8. *For every i, every vertex of B_i' is connected to less than a vertices of B_i'.*

Proof. Recall that we started with picking an arbitrary H'. We obtained that $n-o(n)$ vertices of G must be connected to every vertex of $k-1$ partite classes of that H', let Q be their set. Thus Q is partitioned to the sets $Q\cap B_1,\dots Q\cap B_k$. Consider an arbitrary $u\in Q$, that belongs to, say B_1. Then we obtain another copy of H' from the original one if we delete a vertex of A_1 and add u instead. Let us denote this copy by H''. Applying the same for H'', we obtain that $n-o(n)$ vertices of G are connected to every vertex of $k-1$ partite classes of H''. In particular for $j>1$, if $v\in B_j$ is connected to every vertex of $k-1$ partite classes of H'', the missing partite class has to be A_j (which is a partite class of H''). Therefore, v is connected to the first partite class of the new copy of H', in particular to u. Thus every $u\in Q\cap B_1$ is connected to all but $o(n)$ vertices in every B_j for $j>1$.

If a vertex in $B_j\cap Q$ is connected to a vertices in $B_j\cap Q$, then these $a+1$ vertices with a vertices from classes of H' form a copy of F, a contradiction. This implies that for every j, $|B_j\cap Q|=n/k+o(n)$. Indeed, the copies of H inside Q are all formed by taking a partite class from every B_j by Claim 3.4 and the above observation, thus their number is at most $y:=\prod_{j=1}^k\binom{|B_j\cap Q|}{b}\le\mathcal{N}(H,T_k(n))$. It is easy to see that if the sizes of the sets B_j are less balanced, then y decreases by $\Omega(n^{|V(H)|})$. On the other hand, the number of copies of H containing a vertex outside Q is $o(n^{|V(H)|})$, thus the total number of copies of H in G is less than $\mathcal{N}(H,T_k(n))$, a contradiction. We also have by Claim 3.4 that for every copy of H_0 inside Q, its partite sets are contained in distinct B_j's.

Consider a vertex $v\in B_i\setminus Q$. Assume first that for some $j\neq i$, v is connected to at most αn vertices of B_j for some $\alpha<1/k$. Consider the copies of H containing v. There are $o(n^{|V(H)|-1})$ copies of H containing v that also contain another vertex outside Q. Consider those copies of H that have all the vertices in Q (except for v), i.e. a copy of H_0 inside Q that forms a copy of H with v. Then one of the partite classes of that H_0 is inside B_j, thus the vertices are chosen from the αn neighbors of v in B_j. This shows that v is in less than x copies of H altogether, a contradiction with Claim 3.5.

Assume now that a vertex $v \in B'_i$ is connected to a vertices of B_i. Then for $j \neq i$, v ant the other vertices of B'_i each have $n/k - o(n)$ neighbors in B_j. We will build a copy of F. The one-element partite class is v. Its a neighbors in B'_i form another partite class. Then we go through the sets $B_j \cap Q$ $(j \neq i)$ one by one. We always have that the $a+1$ vertices we picked from B'_i have $n/k - o(n)$ neighbors in $B_j \cap Q$, and the already picked other vertices are connected to all of those, except for $o(n)$. Therefore, we can always pick a vertices from $B_j \cap Q$ that are connected to all the vertices picked earlier. This way we obtain a copy of F, a contradiction. ■

The above claim together with Claim 3.4 implies that in a copy of H, vertices of B'_i cannot belong to two different partite classes, i.e. every B'_i contains a partite class. Let G'' be the graph we obtain by deleting the edges inside B_i for every i. Then $\mathcal{N}(H,G) = \mathcal{N}(H,G'') \leq \mathcal{N}(H,T_k(n))$, where the inequality follows from the facts that G'' is K_{k+1}-free and H is $(k+1)$-Turán-good. ■

Now we can prove Theorem 1.9, which we restate here for convenience.

Theorem. *There exists an F-Turán-good graph if and only if F has a color-critical vertex.*

Proof. Assume first that F does not have a color-critical vertex and let $k = \chi(F)$. Let $T'_{k-1}(n)$ be obtained from $T_{k-1}(n)$ by taking a vertex v from a largest partite set of $T_{k-1}(n)$, and connect it to every other vertex. Then deleting v from $T'_{k-1}(n)$ we obtain a $(k-1)$-partite graph. As deleting any vertex from F we obtain a k-chromatic graph, $T'_{k-1}(n)$ is F-free.

We claim that for any graph H, for n large enough, either $\mathcal{N}(H,T'_{k-1}(n)) > \mathcal{N}(H,T_{k-1}(n))$ or $\mathcal{N}(H,T'_{k-1}(n)) = \mathcal{N}(H,T_{k-1}(n)) = 0$. Indeed, if there is an H in $T_{k-1}(n)$, then there is one avoiding v, but using a vertex u from the same partite set, and a vertex w from another partite set. Then we can replace w with v to obtain a copy of H that is in $T'_{k-1}(n)$, but not in $T_{k-1}(n)$.

If $\mathcal{N}(H,T_{k-1}(n)) = 0$, then the Turán graph may be extremal if H contains F, but then H is not F-Turán-good. If $\mathcal{N}(H,T'_{k-1}(n)) > \mathcal{N}(H,T_{k-1}(n))$, then the Turán graph is not extremal, finishing the proof.

Assume now that F has a color-critical vertex. Then F is a subgraph of a complete k-partite graph $K_{1,a,\dots,a}$, thus Theorem 3.3 finishes the proof. ■

4. Concluding remarks

• We showed that if a k-chromatic graph F has a color-critical edge and $\mathrm{ex}(n,K_k,F) = o(n^{k-1})$, then several k-Turán-good graphs are also F-Turán-good. The proof deals only with those copies of K_k that are in the vertex set of a copy of H. This suggests to study a local variant of generalized Turán problems. We say that a subgraph G' of G is F-free with respect to G if there is no copy of F induced on $V(G')$. How many subgraphs of an n-vertex graph G can be isomorphic to H and be F-free with respect to G at the same time?

If $F = K_k$ and the largest number as an answer to the above question is obtained when $G = T_{k-1}(n)$, then it is immediate that H is k-Turán-good. The argument used in the proof of Theorem 1.5 shows that for any other F with a color-critical edge and $\mathrm{ex}(n,K_k,F) = o(n^{k-1})$, we have that H is also F-Turán-good.

• A theme of this paper is to extend some results on k-Turán good graphs to F-Turán good graphs when F is a k-chromatic graph with a color critical edge. This motivates the question: is every k-Turán-good graph also F-Turán-good?

• Maybe even more is true than what is suggested in the previous paragraph. Let us call a graph H weakly F-Turán-good if the number of copies of H is maximized by a complete multipartite graph among F-free graphs on n vertices, provided n is large enough. Is every weakly K_k-Turán-good graph also weakly F-Turán-good?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] N. Alon, C. Shikhelman, Many T copies in H-free graphs, J. Comb. Theory, Ser. B 121 (2016) 146–172.
[2] B. Bollobás, E. Győri, Pentagons vs. triangles, Discrete Math. 308 (19) (2008) 4332–4336.
[3] D. Chakraborti, D.Q. Chen, Exact results on generalized Erdős-Gallai problems, arXiv preprint, arXiv:2006.04681, 2020.
[4] Z. Chase, A proof of the Gan-Loh-Sudakov conjecture, arXiv preprint, arXiv:1911.08452, 2019.
[5] D. Gerbner, Generalized Turán problems for small graphs, arXiv preprint, arXiv:2006.16150, 2020.
[6] D. Gerbner, Counting multiple graphs in generalized Turán problems, arXiv preprint, arXiv:2007.11645, 2020.
[7] D. Gerbner, E. Győri, A. Methuku, M. Vizer, Generalized Turán numbers for even cycles, J. Comb. Theory, Ser. B 145 (2020) 169–213.
[8] D. Gerbner, C. Palmer, Counting copies of a fixed subgraph in F-free graphs, Eur. J. Comb. 82 (2019) 103001.
[9] D. Gerbner, C. Palmer, Some exact results for generalized Turán problems, arXiv preprint, arXiv:2006.03756, 2020.
[10] L. Gishboliner, A. Shapira, A generalized Turán problem and its applications, in: Proceedings of STOC 2018 Theory Fest: 50th Annual ACM Symposium on the Theory of Computing, June 25-29, 2018, Los Angeles, CA, 2018, pp. 760–772.

[11] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, J. Comb. Theory, Ser. B 102 (2012) 1061–1066.
[12] E. Győri, H. Li, The maximum number of triangles in C_{2k+1}-free graphs, Comb. Probab. Comput. 21 (2011) 187–191.
[13] E. Győri, J. Pach, M. Simonovits, On the maximal number of certain subgraphs in K_r-free graphs, Graphs Comb. 7 (1) (1991) 31–37.
[14] E. Győri, N. Salia, C. Tompkins, O. Zamora, The maximum number of P_l copies in P_k-free graphs, Acta Math. Univ. Comen. 88 (3) (2019) 773–778.
[15] H. Hatami, J. Hladký, D. Král', D. Norine, A. Razborov, On the number of pentagons in triangle-free graphs, J. Comb. Theory, Ser. A 120 (2012) 722–732.
[16] J. Ma, Y. Qiu, Some sharp results on the generalized Turán numbers, Eur. J. Comb. 84 (2020) 103026.
[17] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pp. 279–319.
[18] P. Turán, Egy gráfelméleti szélsőértékfeladatról, Mat. Fiz. Lapok 48 (1941) 436–452.
[19] A.A. Zykov, On some properties of linear complexes, Mat. Sb. 66 (2) (1949) 163–188.