Unique reducibility of multiple blocking sets

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Abstract. A weighted t-fold $(n-k)$-blocking set B of $\mathrm{PG}(n,q)$ always contains a minimal weighted t-fold $(n-k)$-blocking set. We prove that, if $|B| < (t+1)q^{n-k} + \theta_{n-k-1}$, then the minimal weighted t-fold $(n-k)$-blocking set contained in B is unique.

Mathematics Subject Classification (2010). 51E21, 51E20.

Keywords. Projective space, multiple blocking sets.

1 Introduction

A *t-fold $(n-k)$-blocking set* of $\mathrm{PG}(n,q)$ is a set of points which meets every k-dimensional subspace in at least t points. To exclude the trivial cases we will always suppose that $0 < k < n$. If the points of the set are not all different, so the set is a *multiset* of points, then it is called a *weighted* t-fold $(n-k)$-blocking set. A *weight function* of $\mathrm{PG}(n,q)$ is a mapping from the point set of $\mathrm{PG}(n,q)$ to the set of nonnegative integers. For a point P the integer $w(P)$ is the *weight* of P. There is a natural correspondence between multisets and weight functions of $\mathrm{PG}(n,q)$: let the weight of a point be the multiplicity of that point in the set. For a weight function w, the weight of a set M of points is by definition the sum of the weights of all its points, denoted by $w(M)$, and $w(\mathrm{PG}(n,q)) =: |w|$ can be called the *total weight* of w. The multiset associated to a weight function w is a t-fold $(n-k)$-blocking set if and only if the weight of every k-dimensional subspace is at least t. If this is the case, then we will call the weight function w a *t-fold $(n-k)$-blocking set* for short.

*The author was supported by OTKA Grant K81310.

If w is a t-fold $(n-k)$-blocking set, then a point P is called a *non-essential* point of w, if the weight of every k-subspace containing P is at least $t+1$ and $w(P) \geq 1$. Then the weight function w' defined by

$$w'(Q) = \begin{cases} w(Q) & \text{if } Q \neq P, \\ w(P) - 1 & \text{if } Q = P \end{cases}$$

is also a t-fold $(n-k)$-blocking set.

If w and w' are weight functions, and $w'(P) \leq w(P)$ for all points $P \in \mathrm{PG}(n,q)$, then we will say that w' is *contained* in w, and denote this by $w' \leq w$.

The t-fold $(n-k)$-blocking set w is said to be *minimal* if $w' \equiv w$ for any t-fold $(n-k)$-blocking set w' contained in w.

A t-fold $(n-k)$-blocking set is not minimal if and only if it has non-essential points. If we start reducing the weight of the non-essential points one by one, always checking carefully that the resulting set/weight function is still a t-fold $(n-k)$-blocking set, then after some steps we will arrive at a minimal t-fold $(n-k)$-blocking set. It is a natural question to ask if there are conditions which guarantee the uniqueness of this minimal t-fold $(n-k)$-blocking set. Here, two weight fuctions w' and w'' are considered to be different if there is a point P, such that $w'(P) \neq w''(P)$.

In [12] such a condition is given for non-weighted 1-fold 1-blocking sets of $\mathrm{PG}(2,q)$.

Result 1.1. (Szőnyi, [12]) *A non-weighted* 1*-fold* 1*-blocking set of* $\mathrm{PG}(2,q)$*, with size smaller than* $2q+1$ *contains a unique minimal 1-fold 1-blocking set.*

This result was recently generalized to non-weighted 1-fold $(n-k)$-blocking sets of $\mathrm{PG}(n,q)$ in [9].

Result 1.2. (Lavrauw, Storme and Van de Voorde, [9]) *A non-weighted* 1*-fold* $(n-k)$*-blocking set of* $\mathrm{PG}(n,q)$*, with size smaller than* $2q^{n-k}$ *contains a unique minimal* 1*-fold* $(n-k)$*-blocking set.*

Using the standard notation $\theta_m = \frac{q^{m+1}-1}{q-1}$ for the number of points of an m-dimensional subspace of $\mathrm{PG}(n,q)$, our result is the following.

Theorem 1.3. *A weighted t-fold $(n-k)$-blocking set of* $\mathrm{PG}(n,q)$, *with total weight smaller than*

$$(t+1)q^{n-k} + \theta_{n-k-1}$$

contains a unique minimal weighted t-fold $(n-k)$-blocking set.

Note that Theorem 1.3 is stronger than Result 1.2. Examples in the last section show that the bound is sharp if $t = 1$, or if $k = n - 1$.

2 t-fold $(n-k)$-blocking sets containing two minimal t-fold $(n-k)$-blocking sets

Let w be a t-fold $(n-k)$-blocking set. We will now define a new weight function s_w on the points of $\mathrm{PG}(n,q)$. For a point P let $s_w(P)$ be the largest integer for which the weight function w' defined by

$$w'(Q) = \begin{cases} w(Q) & \text{if } Q \neq P, \\ w(P) - s_w(P) & \text{if } Q = P \end{cases}$$

is also a t-fold $(n-k)$-blocking set. Then $w(P) \geq s_w(P) \geq 0$, so if $w(P) = 0$, then $s_w(P) = 0$. It is also clear that w is minimal if and only if $s_w \equiv 0$.

Lemma 2.1. *For a t-fold $(n-k)$-blocking set w and $P \in \mathrm{PG}(n,q)$ the following are true*:

(a) $s_w(P) = \min\{w(P), \min_{P \in \Pi_k}(w(\Pi_k) - t)\}$, *where Π_k runs along the k-dimensional subspaces containing P*;

(b) $s_w(P) = \max_{w' \leq w}\{w(P) - w'(P)\}$, *where w' runs along the t-fold $(n-k)$-blocking sets contained in w.*

Lemma 2.2. *If w is a t-fold $(n-k)$-blocking set which contains two different minimal t-fold $(n-k)$-blocking sets, then there is a weight function $v \leq w$ and a line l^* with the following properties*:

(a) $v(\Pi_k) \geq t$ *for any k-subspace Π_k not containing l^**;

(b) $v(\Pi_k) \geq t-1$ *for any* k*-subspace* Π_k *containing* l^*;

(c) *there is a* k*-subspace* Π_k^* *containing* l^*, *for which* $v(\Pi_k^*) = t-1$;

(d) $|w| \geq |v| + 2$.

Proof. Let w' and w'' be two different minimal t-fold $(n-k)$-blocking sets contained in w. Then there is a point $P^* \in \mathrm{PG}(n,q)$, such that $w'(P^*) > w''(P^*)$. Define $\tilde{w}$ as follows:

$$\tilde{w}(Q) = \begin{cases} w(Q) & \text{if } Q \neq P^*, \\ w'(P^*) & \text{if } Q = P^*. \end{cases}$$

Then $\tilde{w}$ is a t-fold $(n-k)$-blocking set, $w', w'' \leq \tilde{w}$, and Lemma 2.1(b) yields that $s_{\tilde{w}}(P^*) \geq \tilde{w}(P^*) - w''(P^*) = w'(P^*) - w''(P^*) > 0$. (*)

As $\tilde{w}$ contains the minimal t-fold $(n-k)$-blocking set w', we can start reducing the weight of the points with $\tilde{w}(P) > w'(P)$, one at a time, until we arrive at w'. Formally, let $\tilde{w} = w_1 \geq w_2 \geq \cdots \geq w_m = w'$ be a sequence of t-fold $(n-k)$-blocking sets, such that for $i \in \{1, 2, \ldots, m-1\}$ the t-fold $(n-k)$-blocking sets w_i and w_{i+1} only differ in one point P_i, and $w_{i+1}(P_i) = w_i(P_i) - 1$. Clearly $P_i \neq P^*$, and the points P_i are not necessarily all different. It is also clear that $\tilde{w} \neq w'$, because $\tilde{w} = w'$ would mean that w'' is contained in w', which is a contradiction, so $m \geq 2$ follows.

By Lemma 2.1(a), $s_{w_{i+1}} \leq s_{w_i}$, in fact, for any point Q, either $s_{w_{i+1}}(Q) = s_{w_i}(Q)$, or $s_{w_{i+1}}(Q) = s_{w_i}(Q) - 1$. For the point P^* we have $s_{\tilde{w}}(P^*) > 0$ by (*), and $s_{w'}(P^*) = 0$ by the minimality of w'. So there will be an $i \in \{1, 2, \ldots, m-1\}$ such that $s_{w_i}(P^*) = 1$ and $s_{w_{i+1}}(P^*) = 0$. The weight functions w_i and w_{i+1} only differ in the point P_i. Then by Lemma 2.1(a) there is a k-space Π_k^* which contains P_i and P^*, and has weight $w_i(\Pi_k^*) = t+1$. Also by Lemma 2.1(a) this yields $s_{w_i}(P_i) \leq 1$, and as $w_{i+1}(P_i) = w_i(P_i) - 1$, so P_i is a non-essential point of w_i, then $s_{w_i}(P_i) = 1$ follows. Thus for any k-dimensional subspace Π_k, which contains P^* and/or P_i we have $w_i(\Pi_k) \geq t+1$.

Let l^* be the line connecting P_i and P^*, and define v to be the following weight function:

$$v(Q) = \begin{cases} w_i(Q) & \text{if } Q \notin \{P^*, P_i\}, \\ w_i(Q) - 1 & \text{if } Q \in \{P^*, P_i\}. \end{cases}$$

Clearly $|w| \geq |w_i| = |v| + 2$, and v is a weight function contained in w. The weight of a k-subspace Π_k is $w_{i-1}(\Pi_k) - |\Pi_k \cap \{P^*, P_i\}|$. Thus, v, l^* and Π_k^* satisfy the properties given in the lemma. □

3 t-fold nuclei

If $t = 1$, $n = 2$, $k = 1$, then Lemma 2.2 yields that if w is a 1-fold 1-blocking set of $\mathrm{PG}(2, q)$ containing two different minimal 1-fold 1-blocking sets, then w contains a weight function v, which defines a blocking set of the affine plane $\mathrm{AG}(2, q) := \mathrm{PG}(2, q) \setminus l^*$. Thus $|w(\mathrm{PG}(2, q))| \geq s(q) + 2$, where $s(q)$ denotes the size of the smallest 1-blocking set of $\mathrm{AG}(2, q)$. There are several independent proofs for $s(q) = 2q - 1$, from which Result 1.1 follows (see Jamison [8], Brouwer and Schrijver [5], Blokhuis [2], Szőnyi [12]).

In [2], $s(q) = 2q - 1$ is proved as a corollary of a theorem on *nuclei* of point sets. Now we generalize the notion of *nucleus* to multisets/weight functions.

Definition 3.1. (1) Let S be a multiset of $\mathrm{PG}(n, q)$. A point $P \notin S$ will be called a *t-fold nucleus* of S if every line through P meets S in at least t points, counted with multiplicities.

(2) Let w be a weight function of $\mathrm{PG}(n, q)$. A point $P \in \mathrm{PG}(n, q)$ with $w(P) = 0$ will be called a *t-fold nucleus* of w if every line through P has weight at least t.

For S to have nuclei, clearly $|S| \geq t\theta_{n-1}$ is needed. Let $|S| = t\theta_{n-1} + r$, $r \geq 0$.

Note that for $|S| = t\theta_{n-1} - r$, $r \geq 0$, a 'symmetric' version of the definition can be: a point $P \notin S$ is a t-fold nucleus of S, if every line through P meets S in at most t points, counted with multiplicities.

The notion of *nucleus* was first introduced by Mazzocca for affine sets for $n = 2$, $t = 1$ and $r = 0$. Blokhuis extended the notion to $r \geq 0$ in [2] and to $t \geq 1$ in [3], and Sziklai generalized the definition for sets of the projective space $\mathrm{PG}(n, q)$ in [11]. (The 'symmetric' version was introduced in [7] and [11].)

Denote by $N^t(S)$ the set of t-fold nuclei of S, and let p be the characteristic of the field $\mathrm{GF}(q)$.

Result 3.2. (Sziklai, [11]) *Let S be a set of points in* $\mathrm{PG}(n,q)$ *with* $|S| = t\theta_{n-1} + r$, $r \geq 0$. *Let H_∞ be a given hyperplane,* $|S \cap H_\infty| = m_\infty$. *Then*

$$|N^t(S) \setminus H_\infty| \leq (r+1)(q-1),$$

provided that $\binom{t\theta_{n-1}+r-m_\infty}{r+1} \neq 0 \pmod p$.

Result 3.2 was proved in the case when $m_\infty = 0$, $n = 2$ by Blokhuis and Wilbrink ($r = 0$, $t = 1$, see [4]) and by Blokhuis (for $r \geq 0$, $t = 1$, see [2], and for $r \geq 0$, $t \geq 1$ see [3]). The 'symmetric' version was also settled by Sziklai in [11].

As Result 3.2 is not applicable when $\binom{t\theta_{n-1}+r-m_\infty}{r+1} = 0 \pmod p$, to obtain an upper bound in this case, Ball presented the following theorem.

Result 3.3. (Ball, [1]) *Let S be a set of points in* $\mathrm{PG}(n,q)$ *with* $|S| = t\theta_{n-1} + r$, $r \geq 0$, *and let H_∞ be a given hyperplane,* $|S \cap H_\infty| = m_\infty$. *Then*

$$|N^t(S) \setminus H_\infty| \leq (r+1+j)(q-1),$$

provided that the binomial coefficient

$$\binom{t\theta_{n-1}+r-m_\infty}{r+1+j} \neq 0 \pmod p$$

for some $j \geq 0$.

The proof of Result 3.2 and 3.3 can be easily copied for multisets/weight functions and we obtain the following lemma.

Lemma 3.4. *Let w be a weight function on* $\mathrm{PG}(n,q)$ *and H_∞ a given hyperplane with* $w(H_\infty) = m_\infty$. *Suppose that* $w(\mathrm{PG}(n,q)) = t\theta_{n-1} + r$, *with* $r \geq 0$. *Then if*

$$\binom{t\theta_{n-1}+r-m_\infty}{r+1+j} \neq 0 \pmod p$$

for some $j \geq 0$, *then the number of t-fold nuclei of w in* $\mathrm{PG}(n,q) \setminus H_\infty$ *is at most* $(r+1+j)(q-1)$.

Proof. If the binomial coefficient is nonzero, then $w(\mathrm{PG}(n,q) \setminus H_\infty) > 0$, so the number of t-fold nuclei in $\mathrm{PG}(n,q) \setminus H_\infty$ is at most $q^n - 1$. Thus the statement is trivially true for $r + 1 \geq \theta_{n-1}$, so from now on we will suppose $r < \theta_{n-1} - 1$.

Identify the points of $\mathrm{AG}(n,q) := \mathrm{PG}(n,q) \setminus H_\infty$ with the elements of $\mathrm{GF}(q^n)$, and the points of H_∞ with the θ_{n-1}-st roots of unity of $\mathrm{GF}(q^n)$ in the usual way. The points of $\mathrm{PG}(n,q)$ will be denoted by capital letters, and the corresponding elements of $\mathrm{GF}(q^n)$ by the same lowercase letters. Then for points $A \neq B \in \mathrm{AG}(n,q)$, the line AB contains the ideal point $C \in H_\infty$ if and only if $(a-b)^{q-1} = c$ holds.

Let $\mathcal{S} = \{a_1, a_2, \ldots, a_{t\theta_{n-1}+r-m_\infty}\} \cup \{c_1, \ldots, c_{m_\infty}\}$ be the multiset of elements of $\mathrm{GF}(q^n)$ corresponding to the points of nonzero weight of $\mathrm{PG}(n,q) \setminus H_\infty$ and H_∞ respectively, such that $a \in \mathcal{S}$ has multiplicity $w(A)$ in $\mathcal{S}$ for the corresponding point $A \in \mathrm{PG}(n,q)$.

Let X and Y be variables, and define

$$\mathcal{B}(X) = \{(X - a_i)^{q-1} | i = 1, \ldots, t\theta_{n-1} + r - m_\infty\} \cup \{c_1, \ldots, c_{m_\infty}\},$$

and

$$F(Y, X) = \prod_{b \in \mathcal{B}(X)} (Y - b).$$

Then

$$F(Y, X) = \sum_{j=0}^{t\theta_{n-1}+r} (-1)^j \sigma_j(\mathcal{B}(X)) Y^{t\theta_{n-1}+r-j},$$

where $\sigma_j(\mathcal{B}(X))$ denotes the jth elementary symmetric polynomial of the set $\mathcal{B}(X)$.

Suppose that $x \in \mathrm{GF}(q^n)$ is an element corresponding to a t-fold nucleus of w. Then $\mathcal{B}(x)$ contains every θ_{n-1}-st root of unity with multiplicity at least t, so

$$F(Y, x) = (Y^{\theta_{n-1}} - 1)^t (Y^r + \text{terms of lower degree}).$$

As $r < \theta_{n-1} - 1$, the coefficients of the terms

$$Y^{t\theta_{n-1}-1}, Y^{t\theta_{n-1}-2}, \ldots, Y^{(t-1)\theta_{n-1}+r+1}$$

are 0 in $F(Y, x)$. Thus $\sigma_{r+1+j}(\mathcal{B}(x)) = 0$ for $0 \leq j \leq \theta_{n-1} - r - 2$.

The degree of $\sigma_{r+1+j}(\mathcal{B}(X))$ as a polynomial of X is at most $(r+1+j)(q-1)$, with equality precisely if the binomial coefficient

$$\binom{t\theta_{n-1}+r-m_\infty}{r+1+j}$$

does not vanish. In this case $\sigma_{r+1+j}(\mathcal{B}(X))$ is not the zero polynomial, and every nucleus is a root of it, hence the number of nuclei is at most its degree: $(r+1+j)(q-1)$. □

We will now use Lemma 3.4 for $n=2$, $j=0$ and $m_\infty = t-1$.

Lemma 3.5. *Suppose that v is a weight function of* $\mathrm{PG}(2,q)$ *such that there is a line l_∞, with $v(l_\infty)=t-1$, while all other lines have weight at least t. Then $|v| \geq (t+1)q-1$.*

Proof. Assume first that $t \leq q-2$. Suppose on the contrary that v is such a weight function, yet the total weight of v is less than $(t+1)q-1$. We may suppose $|v| = (t+1)q-2$ (or else increase the weight of some of the points of $\mathrm{PG}(2,q)\setminus l_\infty$). All lines other than l_∞ have weight at least t, which means that all the points of $\mathrm{PG}(2,q)\setminus l_\infty$ with weight 0 are t-fold nuclei of v. As $v(\mathrm{PG}(2,q)\setminus l_\infty) = (t+1)q-2-(t-1) = tq+q-t-1$, $\mathrm{PG}(2,q)\setminus l_\infty$ has at most $tq+q-t-1$ points with positive v weight (and exactly this many if every point of $\mathrm{PG}(2,q)\setminus l_\infty$ has weight ≤ 1). So v has at least $q^2-(tq+q-t-1) = q^2-tq-q+t+1$ t-fold nuclei.

We will use Lemma 3.4 to prove that this is not possible. As

$$|v| = (t+1)q-2 = t(q+1)+q-t-2$$

and

$$\binom{t(q+1)+q-t-2-(t-1)}{q-t-2+1} = \binom{tq+q-t-1}{q-t-1} \neq 0 \pmod{p}$$

by Lucas' theorem, so Lemma 3.4 yields that the number of t-fold nuclei of v is at most $(q-t-1)(q-1) = q^2-tq-2q+t+1$, a contradiction. The same arguments prove that, if $|v| = (t+1)q-1$, then $v(P) \leq 1$ for all points $P \in \mathrm{PG}(2,q)\setminus l_\infty$.

For $t \geq q-1$, the assertion can be proved by summing the weights of all lines through a carefully selected point P. If $P \in \mathrm{PG}(2,q)\setminus l_\infty$ and $v(P)=0$, then $|v| \geq t(q+1) = tq + t \geq tq + q - 1$. If $P \in l_\infty$ and $v(P) = 0$, then $|v| \geq tq + t - 1$ and so if $t \geq q$, then we are done. If $t = q - 1$ and all points of $\mathrm{PG}(2, q) \setminus l_\infty$ have positive weight, then $v(\mathrm{PG}(2, q) \setminus l_\infty) \geq q^2$, so $|v| \geq q^2 + t - 1 > (t+1)q - 1$. With this we have proved that if we can select a point $P \in \mathrm{PG}(2, q)$ with $v(P) = 0$, then the assertion is true.

Assume now that $v(P) > 0$ for every point, let $m = \min_P v(P)$ and define a new weight function $\tilde{v}$, by $\tilde{v}(P) := v(P) - m$. Then $\tilde{v}(l_\infty) = t - m(q+1) - 1$ and $\tilde{v}(l) \geq t - m(q+1)$ for any line $l \neq l_\infty$. If $t - m(q+1) \leq q - 2$ then we can use the first part of the proof to prove $|\tilde{v}| \geq (t - m(q+1) + 1)q - 1$. If $t - m(q+1) \geq q - 1$ then we can use the second part, as there will be a point with zero $\tilde{v}$ weight. Then

$$|v| = |\tilde{v}| + m(q^2+q+1) \geq (t - m(q+1) + 1)q - 1 + m(q^2+q+1) = (t+1)q - 1 + m.$$

Hence the result is established. □

4 Proof of the main theorem

Theorem 1.3. *A weighted t-fold $(n-k)$-blocking set of $\mathrm{PG}(n, q)$, with total weight smaller than*

$$(t+1)q^{n-k} + \theta_{n-k-1}$$

contains a unique minimal weighted t-fold $(n-k)$-blocking set.

Proof. Assume that w is a weighted t-fold $(n-k)$-blocking set of $\mathrm{PG}(n, q)$ which contains two different minimal t-fold $(n-k)$-blocking sets. We will prove $|w| \geq (t+1)q^{n-k} + \theta_{n-k-1}$. By Lemma 2.2 there is a weight function $v \leq w$, a line l^* and a k-subspace Π_k^* containing l^*, such that

(a) $v(\Pi_k) \geq t$, for every k-subspace Π_k not containing l^*;

(b) $v(\Pi_k) \geq t - 1$ for every k-subspace Π_k containing l^*;

(c) $v(\Pi_k^*) = t - 1$;

(d) $|w| \geq |v| + 2$.

Case 1

Assume first that $k=1$. Then $\Pi_k^*=l^*$ is a line, and $v(l^*)=t-1$, while the v weight of any other line is at least t. If $n=2$, then $|v|\geq (t+1)q-1$ by Lemma 3.5, which proves the theorem in this case. Now assume $n\geq 3$ and let Π be a plane containing the line l^*. Then the weight function v restricted to the plane Π fulfills the requirements of Lemma 3.5, so $v(\Pi)\geq (t+1)q-1$. This is true for all the planes containing the line l^*, so clearly $|v|\geq \theta_{n-2}\cdot((t+1)q-1-(t-1))+t-1=(t+1)q^{n-1}+\theta_{n-2}-2$.

Case 2

For $n\geq 3$ and $k\geq 2$ we will use induction on n to prove that

$$|v|\geq (t+1)q^{n-k}+\theta_{n-k-1}-2.$$

Case 2a Let $V\in\Pi_k^*\setminus l^*$ be a point with $v(V)=0$. Consider the quotient space $\mathrm{PG}(n,q)/V\cong\mathrm{PG}(n-1,q)$, and the weight function $\tilde{v}$ induced by v on $\mathrm{PG}(n-1,q)$. Clearly $\tilde{v}(\mathrm{PG}(n-1,q))=v(\mathrm{PG}(n,q))$. The plane $\langle V,l^*\rangle$ corresponds to a line, and a k-space containing V corresponds to a $(k-1)$-space. It is not hard to check that $\tilde{v}$ fulfills requirements (a)-(c) with $\langle V,l^*\rangle/V$ as l^* and Π_k^*/V as Π_{k-1}^*, and so by induction

$$\tilde{v}(\mathrm{PG}(n-1,q))\geq (t+1)q^{n-k}+\theta_{n-k-1}-2.$$

Case 2b Suppose now that for all $P\in\Pi_k^*\setminus l^*$: $v(P)>0$, but there is a point $v(V)=0$. Then $t-1\geq\theta_k-(q+1)$. Increase the weight of one point ($\neq V$) of l^* by one to obtain the new weight function v', which is now a t-fold $(n-k)$-blocking set of $\mathrm{PG}(n,q)$. We will prove that $|v'|\geq tq^{n-k}+\theta_{n-k}-1$. This is generally not true for t-fold $(n-k)$-blocking sets of $\mathrm{PG}(n,q)$, only if t is large enough.

Assume, on the contrary, that $|v'|\leq tq^{n-k}+\theta_{n-k}-2$. Then we can find a line Σ_1 containing V, such that

$$v'(\Sigma_1)\leq\frac{t-(q^{k-1}+q^{k-2}+\cdots+q)}{q^{k-1}},$$

because if all lines through V had v' weight more than

$$\frac{t-(q^{k-1}+q^{k-2}+\cdots+q)}{q^{k-1}},$$

then all these weights would be at least $\geq \dfrac{t-(q^{k-1}+q^{k-2}+\cdots+q)}{q^{k-1}}+\dfrac{1}{q^{k-1}}$, and then the total weight of v' would be

$$|v'| \geq \left(\frac{t-q^{k-1}-q^{k-2}-\cdots-q}{q^{k-1}}+\frac{1}{q^{k-1}}\right)\cdot\theta_{n-1}$$
$$= tq^{n-k}+\left(\frac{t}{q^{k-1}}-\frac{q^k+q^{k-1}+\cdots+q^2}{q^{k-1}}\right)\theta_{n-2}-\frac{q^{k-1}+q^{k-2}+\cdots+q}{q^{k-1}}+\frac{\theta_{n-1}}{q^{k-1}}$$
$$> tq^{n-k}+\frac{q^{n-1}+q^{n-2}+\cdots+q^k}{q^{k-1}} = tq^{n-k}+\theta_{n-k}-1.$$

We will now prove that if $1 \leq j \leq k-2$ and Σ_j is a j-space with

$$v'(\Sigma_j) \leq \frac{t-(q^{k-j}+q^{k-j-1}+\cdots+q)}{q^{k-j}},$$

then we can find a $(j+1)$-space $\Sigma_{j+1} \supset \Sigma_j$, with

$$v'(\Sigma_{j+1}) \leq \frac{t-(q^{k-j-1}+\cdots+q)}{q^{k-j-1}}.$$

If this were not true, then we would have

$$|v'| > \left(\frac{t-(q^{k-j-1}+\cdots+q)}{q^{k-j-1}}-v'(\Sigma_j)\right)\cdot\theta_{n-j-1}+v'(\Sigma_j)$$
$$\geq \left(\frac{t-(q^{k-j-1}+\cdots+q)}{q^{k-j-1}}-\frac{t-(q^{k-j}+q^{k-j-1}+\cdots+q)}{q^{k-j}}\right)\cdot\theta_{n-j-1}$$
$$+\frac{t-(q^{k-j}+q^{k-j-1}+\cdots+q)}{q^{k-j}} = tq^{n-k}+\theta_{n-k}+1.$$

Thus we can find a $(k-1)$-space Σ_{k-1}, with $v'(\Sigma_{k-1}) \leq \frac{t-q}{q}$. But all k-spaces containing Σ_{k-1} have v' weight at least t, so

$$|v'| \geq (t-\frac{t}{q}+1)\cdot\theta_{n-k}+\frac{t-q}{q} = tq^{n-k}+\theta_{n-k}-1,$$

a contradiction.

Case 2c There is one more case remaining to be proved: if $v(P) > 0$ for all points $P \in \mathrm{PG}(n,q)$. Then let $m := \min_P v(P)$ and let $\tilde{v} := v-m$. Then

$\tilde{v}$ fulfills requirements (a)-(c) with $\tilde{t} := t - m \cdot \theta_k$. Cases 2a and 2b prove $|\tilde{v}| \geq \tilde{t}q^{n-k} + \theta_{n-k} - 2$ and then

$$\begin{aligned} |v| = |\tilde{v}| + m \cdot \theta_n &\geq (t - m \cdot \theta_k)q^{n-k} + \theta_{n-k} - 2 + m \cdot \theta_n \\ &= tq^{n-k} + \theta_{n-k} - 2 + m\theta_{n-k-1}. \end{aligned}$$

□

5 Examples

In this section we investigate the sharpness of Theorem 1.3. We are looking for weighted t-fold $(n-k)$-blocking sets of size $(t+1)q^{n-k} + \theta_{n-k-1}$, which contain two different minimal t-fold $(n-k)$-blocking sets.

5.1 The case $t = 1$

Example 1 Let Π^1 and Π^2 be two $(n-k)$-dimensional subspaces of $\mathrm{PG}(n, q)$ meeting in an $(n-k-1)$-dimensional subspace. Then $B := \Pi^1 \cup \Pi^2$ contains two different minimal 1-fold $(n-k)$-blocking sets (Π^1 and Π^2), and $|B| = 2q^{n-k} + \theta_{n-k-1}$. ■

Corollary 5.1. *Theorem 1.3 is sharp, if* $t = 1$.

The following proposition is a corollary of Theorem 1.3, but in fact equivalent to it if $t = 1$ and $k = 1$. Corollary 5.3 can also be found in [13].

Proposition 5.2. *Let B be a minimal 1-fold $(n-1)$-blocking set of $\mathrm{PG}(n, q)$, and $P \in B$. Then there are at least $\geq 2q^{n-1} + \theta_{n-2} - |B|$ tangents thorugh P.*

Proof. Suppose that there are k tangents through P. Take points P_1, P_2, ..., P_k, one from each of the tangents, $P_i \neq P$. Clearly $(B \setminus \{P\}) \cup \{P_1, \ldots, P_k\}$ is a 1-fold $(n-1)$-blocking set. It contains a minimal 1-fold $(n-1)$-blocking set B', and $B \neq B'$. Thus $B \cup \{P_1, \ldots, P_k\}$ contains two different minimal 1-fold $(n-1)$-blocking sets, so $|B| + k \geq 2q^{n-1} + \theta_{n-2}$. □

Corollary 5.3. *Let B be any 1-fold $(n-1)$-blocking set of* $\mathrm{PG}(n,q)$*, and $P \in B$ an essential point of B. Then there are at least $\geq 2q^{n-1} + \theta_{n-2} - |B|$ tangents thorugh P.*

Construction 1 Let B be a 1-fold $(n-1)$-blocking set which has a point $P \in B$, through which there are exactly $2q^{n-1} + \theta_{n-2} - |B|$ tangents to B. Then adding a point to every tangent will result in a 1-fold $(n-1)$-blocking set of size $2q^{n-1} + \theta_{n-2}$, which contains two different minimal 1-fold $(n-1)$-blocking sets. □

Construction 2 Embed construction 1 in an $(n-k+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ to obtain 1-fold $(n-k)$-blocking sets of size $2q^{n-k} + \theta_{n-k-1}$, which contain two different minimal 1-fold $(n-k)$-blocking sets. □

Note that blocking sets used in the above construction exist: the so called Rédei type blocking sets always contain points which are on exactly $2q^{n-1} + \theta_{n-2} - |B|$ tangents (see [10]).

5.2 The case $t \geq 2$

We will use the following notation: for the multisets B_1 and B_2, with associated weight functions w_1 and w_2 respectively, $B_1 \cup B_2$ will denote the multiset defined by the weight function $\max\{w_1, w_2\}$, while $B_1 + B_2$ will denote the multiset defined by the weight function $w_1 + w_2$.

Note that the proof of Lemma 3.5 yields that for $n = 2$, $k = 1$ it is not possible to have $v(\mathrm{PG}(2,q)) = (t+1)q - 1$, if $t \geq q+1$, and so the proof of Theorem 1.3 yields that the bound cannot be sharp if $t \geq q+1$. Also from the proofs of Lemma 3.5 and Theorem 1.3 it follows that if $t \leq q-2$ and B is a weighted t-fold $(n-k)$-blocking set which contains two different minimal t-fold $(n-k)$-blocking sets and $|B| = (t+1)q^{n-k} + \theta_{n-k-1}$, then only points on one line (the line l^*) can be multiple points.

Example 2 Let Π be a plane of $\mathrm{PG}(n,k)$, let $l_1, l_2, \ldots, l_t$ be different lines in Π through a common point P, and l_{t+1} a further line of Π, with $P \notin l_{t+1}$. Then the multiset $B := (l_1 + l_2 + \cdots + l_t) \cup l_{t+1}$ is a t-fold 1-blocking set in $\mathrm{PG}(n,q)$, $|B| = t(q+1) + (q+1-t) = (t+1)q + 1$, and $l_1 + l_2 + \cdots + l_t$ and $l_1 \cup (l_2 + \cdots + l_t) \cup l_{t+1}$ are two minimal t-fold 1-blocking sets contained in B; the latter one differs from B only in the point P. □

Corollary 5.4. *Theorem 1.3 is sharp if $k = n-1$, $2 \le t \le q$.*

The following proposition is again a corollary of Theorem 1.3, which is in fact equivalent to it if $k = 1$. For $n = 2$ and with an upper bound on the size of B, it can also be found in [6].

Proposition 5.5. *Let B be a minimal t-fold $(n-1)$-blocking set of $\mathrm{PG}(n,q)$, and $P \in B$. Then there are at least $\ge (t+1)q^{n-1}+\theta_{n-2}-|B|$ t-secants through P.*

Proof. Suppose that there are k t-secants through P. Take points $P_1, P_2, \dots, P_k$, one from each of the t-secants, $P_i \neq P$. Clearly the t-fold $(n-1)$-blocking set $B \setminus \{P\} + \{P_1, \dots, P_k\}$ contains a minimal t-fold $(n-1)$-blocking set B', and $B \neq B'$. Thus $B + \{P_1, \dots, P_k\}$ contains two different minimal t-fold $(n-1)$-blocking sets, so $|B| + k \ge (t+1)q^{n-1} + \theta_{n-2}$. □

Construction 3 Let B be a minimal t-fold $(n-1)$-blocking set which has a point $P \in B$, through which there are exactly $(t+1)q^{n-1} + \theta_{n-2} - |B|$ t-secants to B. Then adding a point to every t-secant will result in a t-fold $(n-1)$-blocking set of size $(t+1)q^{n-1} + \theta_{n-2}$ and containing two different minimal t-fold $(n-1)$-blocking sets. □

Construction 4 Embed Construction 3 in an $(n-k+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ to obtain t-fold $(n-k)$-blocking sets of size $(t+1)q^{n-k} + \theta_{n-k-1}$, which contain two different minimal t-fold $(n-k)$-blocking sets. □

For $n = 2$, $k = 1$ and $2 \le t \le q$ one can find t-fold 1-blocking sets in $\mathrm{PG}(2,q)$ which have points that are on exactly $(t+1)q+1-|B|$ t-secants to B: take the sum of t Rédei type blocking sets which have a common Rédei line, and share exactly one point, that is not on the Rédei line. Example 2 is a special case of this: the sum of t lines sharing a common point. Then, with Construction 4, we get examples for $n \ge 3$, $k = n-1$ and $1 \le t \le q$. Unfortunately, for $t \ge 2$, $n \ge 3$ and $k = 1$, in the minimal t-fold $(n-1)$-blocking sets examined by the author all points have at least $t\theta_{n-1} - (q+1-t)q^{n-2} - |B|$ t-secants to B. Thus it may be conjectured that the correct bound in Theorem 1.3 should be

$$t\theta_{n-k} + (q+1-t)q^{n-k-1}.$$

Acknowledgment

I am very grateful to Leo Storme for his valuable suggestions and help.

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