SMALL POINT SETS OF PG(n, p^{3h}) INTERSECTING EACH LINE IN $1 \bmod p^h$ POINTS

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ABSTRACT. The main result of this paper is that point sets of PG(n, q), $q = p^{3h}$, $p \geq 7$ prime, of size less than $3(q^{n-1}+1)/2$ intersecting each line in 1 modulo $\sqrt[3]{q}$ points (these are always small minimal blocking sets with respect to lines) are linear blocking sets. As a consequence, we get that minimal blocking sets of PG(n, p^3), $p \geq 7$ prime, of size less than $(3p^{3(n-1)}+1)/2$ with respect to lines are always linear.

1. INTRODUCTION

Throughout this paper PG(n, q) will denote the n-dimensional projective space over the Galois field of order q, where $q = p^h$, $p \geq 7$ prime. An $(n-k)$-*blocking set* B in PG(n, q) $(0 < k < n)$ is a set of points intersecting each k-dimensional subspace. The smallest $(n-k)$-blocking sets are the $(n-k)$-dimensional subspaces, see [11]. An $(n-k)$-blocking set containing an $(n-k)$-dimensional subspace is called *trivial*. A point P of B is *essential* to B, if there exists a k-space through P, called the *tangent* of B at P, intersecting B in P only. The blocking set B is *minimal*, if each point of it is essential. Finally, B is *small*, when $|B| < 3(q^{n-k}+1)/2$.

Small minimal $(n-k)$-blocking sets are of special interest, since there is hope to classify them. Lines intersect small $(n-k)$-blocking sets in either 0 or 1 modulo p points, see [17], [16], [18]. A wide class of small minimal blocking sets, called *linear* ones, were constructed by Lunardon [9], Polito and Polverino [12]. Sziklai's [16] *linearity conjecture* says that they are the only examples.

The main result of this paper is that point sets of PG(n, p^{3h}), $n > 2$, $p \geq 7$ prime, with cardinality less than $3(q^{n-1}+1)/2$ intersecting each 1-space (i.e. line) in 1 modulo p^h points are linear blocking sets. As a consequence, we get that small minimal blocking sets of PG(n, p^3), $n > 2$, $p \geq 7$ prime, are linear. This confirms Sziklai's conjecture in this particular case. For planar point sets, this was already proved by Polverino [14] and Polverino and Storme [13], see Remark 2.6.

We deal with the case $n = 3$ separately. Our proof also yields characterizations of Rédei type blocking sets in PG$(3, q^3)$. In the case $n = 3$, we give an explicit description of the

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non-Rédei example. Similar results for $(n-k)$-blocking sets in $\mathrm{PG}(n, q^3)$ were obtained independently by Lavrauw, Storme, Van de Voorde [7] and also in Harrach, Metsch [5].

2. Small minimal blocking sets

There has been a lot of attention paid on small minimal (planar) blocking sets. Bruen showed that a non-trivial blocking set has size at least $q + \sqrt{q} + 1$. When q is a square, minimal blocking sets of this size exist; they are *Baer subplanes* (subplanes of order $\sqrt{q}$), see [6].

For the construction of small linear blocking set we need the definition of linear point sets, see Lunardon [8], [10].

Definition 2.1. ([9], [12]) (1) A point set S of $\mathrm{PG}(n, q^t)$ is said to be *linear* if there is a projective space $\mathrm{PG}(n', q^t)$ containing $\mathrm{PG}(n, q^t)$ such that S is the projection of a subgeometry $\mathrm{PG}(n', q) \subset \mathrm{PG}(n', q^t)$ from a suitable subspace (vertex) onto $\mathrm{PG}(n, q^t)$.

(2) A point set S of $\mathrm{PG}(n, q^t)$ is said to be *linear* if the $(n+1)-$dimensional $\mathrm{GF}(q^t)$-vectorspace V defining $\mathrm{PG}(n, q^t)$ has a $\mathrm{GF}(q)$-linear subspace W such that a point of $\mathrm{PG}(n, q^t)$ belongs to S if and only if it is defined by a vector of W.

In [10] Lunardon, Polito and Polverino prove the equivalence of these two definitions. Linear blocking sets are obtained by the following construction.

Construction 2.2. (Linear blocking sets) *Let $0 < k < n$ be integers and assume that $\mathrm{PG}(n, q^t)$ is embedded in $\mathrm{PG}(t(n-k), q^t)$. Furthermore, let P be a $(t(n-k)-n-1)$-dimensional subspace not intersecting $\mathrm{PG}(n, q^t)$. Then the projection of any subgeometry (disjoint from P and) isomorphic to $\mathrm{PG}(t(n-k), q)$ from P onto $\mathrm{PG}(n, q^t)$ is a small minimal $(n-k)$-blocking set in $\mathrm{PG}(n, q^t)$.* □

In [8] Lunardon showed that small minimal planar blocking sets of Rédei type are linear. For higher dimensions, this was proved by Storme and Sziklai, see [15]. An $(n-k)$-blocking set B of $\mathrm{PG}(n, q)$ is called a *Rédei type blocking set*, if there is a hyperplane meeting B in $|B| - q^{n-k}$ points. Such a hyperplane is called a *Rédei hyperplane* of the set.

Result 2.3. *Let B be a minimal $(n-k)$-blocking set in $\mathrm{PG}(n, q)$, $q = p^m$, p prime, of size less than $3(q^{n-k}+1)/2$.*

(1) ([17], [18]) *Then each subspace of dimension at least k intersects B in 1 modulo p points.*
(2) (Sziklai [16]) *Let e be the largest integer such that B intersects each k-space in 1 modulo p^e points (from above $e \geq 1$), then $e|m$. Furthermore, if the k-space L intersects B in $p^e + 1$ points, then $L \cap B$ is isomorphic to $\mathrm{PG}(1, p^e)$.* □

Denote by $[l_q(n, k, e), u_q(n, k, e)]$ the smallest interval containing the sizes of all the small minimal $(n-k)$-blocking sets for which e is the integer defined in Result 2.3 (2). In [18] it is proved that these intervals are disjoint, furthermore, if $e'|m$ and $e' < e$, then $u_q(n, k, e) < l_q(n, k, e')$. Thus a minimal $(n-k)$-blocking set whose size belongs to the interval $[l_q(n, k, e), u_q(n, k, e)]$ intersects each k-space in 1 mod p^e points.

For point sets of size less than $3(q^{n-k}+1)/2$ (small point sets) the converse of Result 2.3 (1) is also true:

Result 2.4. ([18]) *Assume that B is a point set in $\mathrm{PG}(n,q)$, $q = p^m$, $2 < p$ prime, with $|B| < 3(q^{n-k}+1)/2$. Then B is a minimal $(n-k)$-blocking set if and only if B intersects each k-space in 1 modulo p points.* □

Now we discuss small minimal blocking sets in $\mathrm{PG}(2,q)$. The famous result of Blokhuis, [1] shows that for $q = p$ prime, there are no small minimal non-trivial blocking sets in $\mathrm{PG}(2,p)$ at all. If $q = p^2$, p prime, then small minimal non-trivial blocking sets in $\mathrm{PG}(2,p^2)$ are Baer subplanes, see [17].

Result 2.5. (Polverino [14], Polverino and Storme [13]) *A non-trivial blocking set in $\mathrm{PG}(2,p^{3h})$, $p \geq 7$, meeting every line in $1 \bmod p^h$ points is either a Baer subplane (and h is even) or one of the following type:*

(1) a minimal blocking set of size $p^{3h}+p^{2h}+1$, projectively equivalent to the set $\{(x, Tr(x), 1) || x \in \mathrm{GF}(p^{3h})\} \cup \{(x, Tr(x), 0) || x \in \mathrm{GF}(p^{3h}) \setminus \{0\}\}$, where Tr is the trace function from $\mathrm{GF}(p^{3h})$ to $\mathrm{GF}(p^h)$ (i.e. $Tr : \mathrm{GF}(p^{3h}) \to \mathrm{GF}(p^h) : x \mapsto x + x^{p^h} + x^{p^{2h}}$);
(2) a minimal blocking set of size $p^{3h}+p^{2h}+p^h+1$, projectively equivalent to the set $\{(x, x^{p^h}, 1) || x \in \mathrm{GF}(p^{3h})\} \cup \{(x, x^{p^h}, 0) || x \in \mathrm{GF}(p^{3h}) \setminus \{0\}\}$.

The next remark summarizes some properties of the blocking sets of Result 2.5. For more details the reader is referred to [14] and [13].

Remark 2.6. ([14],[13]) All possible types of blocking sets in Result 2.5 are linear (and hence each line intersects it in a linear point set).

The Baer subplane has $p^{3h}+p^{3h/2}+1$ points and every line meets it in 1 or $p^{3h/2}+1$ points.

The minimal blocking set of size $p^{3h}+p^{2h}+1$ has a unique point lying on p^h+1 lines containing $p^{2h}+1$ points of the blocking set. The minimal blocking set of size $p^{3h}+p^{2h}+p^h+1$ has exactly one $(p^{2h}+p^h+1)$-secant. In both cases, all other lines are 1-secants or (p^h+1)-secants, the points of the blocking set on a (p^h+1)-secant form a subline of order p^h.

From here on a blocking set will always mean an $(n-1)-$blocking set of $\mathrm{PG}(n,q)$.

3. Small blocking sets in $\mathrm{PG}(n, p^{3h})$

3.1. **The main result.** The smallest minimal blocking sets of $\mathrm{PG}(n,q)$, $q = p^{3h}$, are trivial (see e.g. [11]), these are the blocking sets of the interval belonging to $e = 3h$. When q is a square (hence $2|h$), then there are point sets intersecting each line in 1 modulo $p^{3h/2}$ points and in this case they give the second interval ($e = 3h/2$). These blocking sets are certain Baer-cones, see [19]. The next interval is when $e = h$ and

our main aim is to characterize point sets of size at most $u_q(n,1,h)$ intersecting each line in 1 mod p^h points; these are minimal blocking sets intersecting each line in 1 modulo p^h points.

Result 3.1. (Polverino [14]) *A small minimal blocking set in $\mathrm{PG}(n,q)$, $q = p^{mh}$ intersecting each line in $1 \bmod p^h$ points has size at most $u_q(n,1,h) < q^{n-1}+q^{n-2}p^{2h}+q^{n-2}(p^h+3)$.*

It is easy to see that certain linear blocking sets satisfy our condition.

Example 3.2. *Embed* $\mathrm{PG}(n, p^{3h})$ *in* $\mathrm{PG}(t(n-1), p^{3h})$*, where* $t = 2, 3$ *when* $2|h$ *and* $t = 3$ *otherwise. Furthermore, let* P *be a* $(t(n-1)-n-1)$*-dimensional subspace not intersecting* $\mathrm{PG}(n, p^{3h})$*. Then the projection of any subgeometry isomorphic to* $\mathrm{PG}(t(n-1), p^{3h/t})$ *from* P *onto* $\mathrm{PG}(n, p^{3h})$ *is a small minimal blocking set in* $\mathrm{PG}(n, p^{3h})$ *intersecting each line in* 1 *modulo* p^h *points.* □

The main result of our paper is that we show that these are the only minimal blocking sets intersecting each line in 1 modulo p^h points.

Theorem 3.3. *Let* B *be a point set of* $\mathrm{PG}(n, q)$*,* $q = p^{3h}$*,* $p \geq 7$ *prime, intersecting each line in* $1 \bmod p^h$ *points, and* $|B| < \frac{3}{2}(q^{n-1} + 1)$*. Then* B *is a linear blocking set.*

As a corollary, this shows that small minimal blocking sets of $\mathrm{PG}(n, q)$, $q = p^3$, $p \geq 7$ are linear. As a further corollary minimal blocking sets of size at most $l_q(n, 1, s)$ are proven to be linear, where s is the largest integer so that $s < h$ and $s|3h$.

3.2. **Proof of Theorem 3.3 for** $n = 3$**.** *Throughout this section it will be assumed that* B *is a point set of* $\mathrm{PG}(3, q)$*,* $q = p^{3h}$*,* $p \geq 7$ *prime, intersecting each line in* $1 \bmod p^h$ *points, and* $|B| < \frac{3}{2}(q^2 + 1)$*.*

The following lemma follows from Result 3.1.

Lemma 3.4. $|B| < p^{6h} + p^{5h} + p^{4h} + 3p^{3h}$. □

The next lemma is crucial when characterizing the above point sets.

Lemma 3.5. *A plane* π *either intersects* B *in a small minimal blocking set, or contains more than* $p^{4h} - p^{3h}$ *points from* B*.*

Proof. Let $x = |B \cap \pi|$, where π is a plane of $\mathrm{PG}(3, p^{3h})$. Let b_i be the number of lines of π meeting B in exactly i points. As π has $b := q^2 + q + 1$ lines and $r := q + 1$ lines on each point, standard counting arguments give the following three equations.

$$\begin{aligned}\sum_i b_i &= b\\ \sum_i b_i i &= xr\\ \sum_i b_i i(i-1) &= x(x-1)\end{aligned}$$

Combining these we find

$$\sum_i b_i(i-1)(i-p^h-1) = x(x-1) - (p^h+1)xr + (p^h+1)b.$$

As every line meets B in $1 \mod p^h$ points, the left hand side is non-negative. As the right-hand side is quadratic in x and negative for $x = \frac{3}{2}q + 1$ and $x = qp^h - q$, the assertion follows. □

Corollary 3.6. *On any line of* $\mathrm{PG}(3, p^{3h})$ *there has to be a plane which intersects* B *in a small minimal blocking set.*

PROOF. Suppose on the contrary, that all the planes on l contain more than $p^{4h} - p^{3h}$ points from B. Then counting the points of B on these planes we get

$$|B| > (p^{3h}+1)(p^{4h} - p^{3h} - |l \cap B|) + |l \cap B| > p^{7h} - p^{6h} - |l \cap B| p^{3h},$$

but $|l \cap B| \leq p^{3h}+1$, which contradicts the bound of Lemma 3.4, as $p \geq 7$. □

Corollary 3.7. *For l an arbitrary line of* PG$(3, p^{3h})$, *$l \cap B$ is a linear set of size 1, p^h+1, $p^{2h}+1$, $p^{2h}+p^h+1$, $p^{3h}+1$, or $p^{3h/2}+1$ if $2|h$.*

PROOF. From above we have that every secant of B has to be a secant of a small minimal planar blocking set. The secants of these sets are described in Remark 2.6. □

The following is a technical lemma which will be useful for us.

Lemma 3.8. *(1) On a $(p^h+1)-$secant there are less than 4 planes intersecting B in more than $p^{4h} - p^{3h}$ points.*

(2) On a $(p^{2h}+1)-$ or a $(p^{2h}+p^h+1)-$secant there are less than $2p^h$ planes intersecting B in more than $p^{4h} - p^{3h}$ points.

(3) On a line totally contained in B there are less than $p^{2h}+3p^h$ planes containing further points of B.

PROOF.

(1) Counting the number of points on the planes through a $(p^h+1)-$secant l, with K the number of planes on l which intersect B in more than $p^{4h} - p^{3h}$ points, gives

$$|B| > K(p^{4h} - p^{3h} - p^h - 1) + (p^{3h} + 1 - K)(p^{3h} + p^{2h} - p^h).$$

For $K \geq 4$ this is in contradiction with Lemma 3.4.

(2) Counting the number of points on the planes through a $(p^{2h}+1)-$ or a $(p^{2h}+p^h+1)-$secant gives

$$|B| > K(p^{4h} - p^{3h} - p^{2h} - p^h - 1) + (p^{3h} + 1 - K)p^{3h}.$$

For $K \geq 2p^h$ this is in contradiction with Lemma 3.4.

(3) A plane on a line totally contained in B and containing a further point of B intersects B in at least $p^{4h}+p^h+1$ points, as B intersects every line in $1 \bmod p^h$ points. Having at least $p^{2h}+3p^h$ such planes on a line totally contained in B would lead to a contradiction with Lemma 3.4. □

In case $2|h$ we will now characterize the blocking sets having a $(p^{3h/2}+1)-$secant.

Lemma 3.9. *If $2|h$ and B has a $(p^{3h/2}+1)-$secant, then a line can intersect B in 1, $p^{3h/2}+1$ or $p^{3h}+1$ points only.*

PROOF. If B has a $(p^{3h/2}+1)-$secant, then by Corollary 3.6 and Result 2.5 there has to be a plane π intersecting B in a Baer subplane. Through a point $P \in \pi \cap B$, there are $p^{3h/2}+1$ $(p^{3h/2}+1)-$secants in π. Suppose now that there is a line l through P, not in π which intersects B in p^h+1, $p^{2h}+1$ or $p^{2h}+p^h+1$ points. The planes containing l and a $(p^{3h/2}+1)-$secant have to intersect B in more than $p^{4h}-p^{3h}$ points (see Result 2.5 that a small minimal planar blocking set having a $(p^{3h/2}+1)-$secant can have tangents or $(p^{3h/2}+1)-$secants only), but there are $p^{3h/2}+1$ such planes, which is in contradiction with (1) and (2) of Lemma 3.8. □

Corollary 3.10. *If B has a $(p^{3h/2}+1)-$secant, then B meets all lines in $1 \bmod p^{3h/2}$ points. In [19] it was proved that such point sets are so-called Baer cones, which are in fact projections of a $\mathrm{PG}(2(n-1), p^{3h/2})$ subgeometry and thus are linear.*

For the rest of this section we will assume that B has no $(p^{3h/2}+1)-$secants, and thus no Baer plane sections. Thus all lines intersect B in a linear set of size 1, p^h+1, $p^{2h}+1$, $p^{2h}+p^h+1$, or $p^{3h}+1$. A plane can intersect B in a line, a small minimal blocking set described in (1) or (2) of Result 2.5 or in more than $p^{4h}-p^{3h}$ points.

Definition 3.11. We will call a point $P \in B$ *special*, if there is a plane π through P for which the following holds: $\pi \cap B$ is the small minimal blocking set described in (1) of Result 2.5, and P is the point of this point set playing the special role.

The following lemma summarizes some properties of the special points of B.

Lemma 3.12. *(1) On every $(p^{2h}+1)-$secant there is exactly one special point.*

(2) The lines through a special point can be tangents, lines totally contained in B, or $(p^{2h}+1)-$secants only.

(3) Two special points are always connected by a line contained in B.

Proof. (1) The $(p^{2h}+1)$-secant of the Rédei type blocking set of size $(p^{3h}+p^{2h}+1)$ on the line $y=0$ is just $A=\{x : Tr(x)=0\} \cup \{(\infty)\}$, with special point (∞). Let G be the subgroup of $\mathrm{PGL}(2,p^{3h})$ leaving A invariant. The stabilizer $G_{(\infty)}$ is transitive on $A \setminus \{(\infty)\}$, since it contains all translations by a trace zero element. If there were two special points then G itself would also be transitive on A, so $|G|$ would be divisible by $|A| = p^{2h}+1$. This is impossible, since the g.c.d. of $p^{2h}+1$ and $(p^{3h}+1)p^{3h}(p^{3h}-1)$ is at most 2.

(2) Let P be a special point, l a $(p^{2h}+1)-$secant through P. According to Lemma 3.8, more than $p^{3h}+1-2p^h$ of the planes on l intersect B in the small minimal blocking set (1) of Result 2.5, thus more than $(p^{3h}+1-2p^h)p^h+1$ of the lines through P have to be $(p^{2h}+1)-$secants.

If m is a $(p^{2h}+p^h+1)-$secant on P, then because of Corollary 3.6 there has to be a plane on m in which there are $(p^h+1)-$secants on P.

Now let m be a $(p^h+1)-$secant on P. Assume that a plane π on m intersects B in the small minimal blocking set (1) of Result 2.5. From Remark 2.6 it is clear that in this blocking set, on a special point there are tangents or $(p^{2h}+1)-$secants only. Thus the special point of $\pi \cap B$ has to be a point Q, different from P and the line PQ is a $(p^{2h}+1)-$secant of $\pi \cap B$. But this would be in contradiction with (1), because P and Q would be two special points of the line PQ. Thus all the planes on m intersect B in the small minimal blocking set (2) of Result 2.5 or in more than $p^{4h}-p^{3h}$ points. By (1) of Lemma 3.8, there can be at most 3 planes meeting B in more than $p^{4h}-p^{3h}$ points, and thus there can be at most 3 planes on m containing $(p^{2h}+1)-$secants on P, which means that the number of $(p^{2h}+1)-$secants on P can be at most $3p^{3h}$, a contradiction. Thus there are no $(p^h+1)-$ or $(p^{2h}+p^h+1)-$secants on P.

(3) is a direct consequence of (1) and (2). $\square$

Lemma 3.12 (1) was also proved by Fancsali and Sziklai, see [3] and [4] where a more general case is discussed.

Proposition 3.13. *If π is a plane of $\mathrm{PG}(3,q)$ such that $|\pi \cap B| > p^{4h} + 3p^{3h}$, then B is of Rédei type, where π is a Rédei plane.*

Proof. First observe that there are no special points outside π, because if $S \in B$ were a special point, $S \notin \pi$, then according to Lemma 3.12, all lines connecting S with a point of $B \cap \pi$ would intersect B in at least $p^{2h}+1$ points and counting the points of B on the lines through S would give $|B| > (p^{4h}+3p^{3h})p^{2h}+1$, contradicting Lemma 3.4.

Now we will prove that there are no $(p^h+1)-$secants in π. Suppose on the contrary that l is a $(p^h+1)-$secant in π. If a plane through l intersects B in a small minimal blocking set, it has to be the one given in (2) of Result 2.5, as there are no special points outside π. But then even if all the planes through l (other than π) would intersect B in small minimal blocking sets, we would get to a contradiction with Lemma 3.4, because counting the points of B in these planes would give $|B| > p^{4h}+3p^{3h}+p^{3h}(p^{3h}+p^{2h})$. Thus $\pi \cap B$ has no $(p^h+1)-$secants.

Let $l \not\subset \pi$ be a line meeting π in the point $P \in \pi \setminus B$. Assume $|l \cap B| > 1$, that is (as there are no special points outside π) $|l \cap B| = p^h+1$ or $p^{2h}+p^h+1$. Let α be a plane on l, and let $m := \alpha \cap \pi$. If l is a $(p^h+1)-$secant of B and m a $(p^{2h}+p^h+1)-$secant, then the plane α meets B in more than $p^{4h}-p^{3h}$ points, because in a small minimal planar blocking set every $(p^h+1)-$secant has to meet the $(p^{2h}+p^h+1)-$secant in a point belonging to the set (see Result 2.5), but $P = l \cap m \notin B$. With similar arguments $|\alpha \cap B| > p^{4h}-p^{3h}$ if l is a $(p^h+1)-$secant and m a $(p^{2h}+1)-$secant, and clearly $|\alpha \cap B| > p^{4h}-p^{3h}$ if l is a $(p^{2h}+p^h+1)-$secant and m a $(p^{2h}+1)-$secant or a $(p^{2h}+p^h+1)-$secant.

By Lemma 3.8, as l is a $(p^h+1)-$secant or a $(p^{2h}+p^h+1)-$secant, then there are less than $2p^h$ planes on l intersecting B in more than $p^{4h}-p^{3h}$ points. But then by these reasonings there are less than $2p^h$ lines on P meeting $B \cap \pi$ in more than one point. As every line of π on P contains at most $p^{2h}+p^h+1$ points of B, we have $|B \cap \pi| < p^{3h}+1+2p^h(p^{2h}+p^h)$, but this is in contradiction with the lower bound on $B \cap \pi$.

Thus for the point $P \in \pi \setminus B$, all the lines through P, but not in π are tangents to B, and this means that $|B \setminus \pi| = p^{6h}$, and so B is of Rédei type with π a Rédei plane. $\square$

Corollary 3.14. *If there is a line contained in B and a special point of B not on this line, then B is of Rédei type.*

Proof. If a plane contains a line of B and a special point of B not on the line, then it contains at least $p^{2h}(p^{3h}+1)+1$ points of B, because by (2) of Lemma 3.12 any line on a special point which intersects B in at least 2 points, has to intersect it in at least $p^{2h}+1$ points. $\square$

The following is a technical lemma, which will be useful for us.

Lemma 3.15. *Let $P \in B$ be a non-special point and t a tangent on P. Denote by N the number of planes on t which intersect B in the small minimal blocking set of type (1) of Result 2.5 and M is the number of planes on t which intersect B in a line. Suppose that*

$M \leq p^h$ and $N \leq p^{2h}$. Then all the planes on t intersect B in small minimal blocking sets and

$$|B| = (p^{3h}+1)(p^{3h}+p^{2h}+p^h)+1-M(p^{2h}+p^h)-Np^h.$$

Proof. Having a plane on t which intersects B in more than $p^{4h}-p^{3h}$ points would result in

$$|B| > p^{4h}-p^{3h}+p^{3h}(p^{3h}+p^{2h}+p^h)+1-p^h(p^{2h}+p^h)-p^{2h}p^h,$$

which is in contradiction with the bound of Lemma 3.4. Thus $B = N(p^{3h}+p^{2h})+Mp^{3h}+(p^{3h}+1-N-M)(p^{3h}+p^{2h}+p^h)+1$. □

Lemma 3.16. *There has to be at least one line contained in B.*

Proof. Suppose on the contrary that there are no lines totally contained in B. Then by (3) of Lemma 3.12, there can be at most one special point in B. Let P be a nonspecial point of B and t a tangent on P. By Lemma 3.15,

$$|B| = (p^{3h}+1)(p^{3h}+p^{2h}+p^h)-Np^h+1,$$

with $N \leq 1$ the number of special points in B.

Now if $N=1$ then let l be a $(p^h+1)-$secant of B in a plane which intersects B in the small minimal blocking set (1) of Result 2.5, while if $N=0$ then let l be any $(p^h+1)-$secant of B. Counting the points of B in the planes on l yields that one plane π has to intersect B in exactly $p^{4h}+p^{3h}+p^{2h}+p^h+1$ points. By the choice of l, there is no special point in π. From this it follows, that there are no $(p^{2h}+1)-$secants on π. There are no tangents on π either, because having a tangent t would lead to a contradiction with Lemma 3.15 (with $P := t \cap B$, $N \leq 1$, $M=0$, and $\pi \cap B$ not being a small minimal blocking set).

Thus through a point of π not belonging to B there can be $(p^h+1)-$secants or $(p^{2h}+p^h+1)-$secants in π only. Denote by L the number of $(p^{2h}+p^h+1)-$secants in π on a point $Q \in \pi \setminus B$. We have:

$$|B \cap \pi| = L(p^{2h}+p^h+1)+(p^{3h}+1-L)(p^h+1),$$

from which $L=1$. Now denote by K the number of $(p^{2h}+p^h+1)-$secants in π. Double-counting the number of pairs (Q,m), $Q \in \pi \setminus B$, m a $(p^{2h}+p^h+1)-$secant on Q, we get:

$$(p^{6h}+p^{3h}+1-|\pi \cap B|)\cdot 1 = K \cdot (p^{3h}-p^{2h}-p^h),$$

which has no integer solutions for K. □

Proposition 3.17. *If there are at least two lines contained in B, then B is of Rédei type.*

Proof. Any two lines totally contained in B must intersect, as two skew lines would contradict (3) of Lemma 3.8. Let l_1 and l_2 be lines contained in B and let $P = l_1 \cap l_2$. If there is a special point in $B \setminus \{P\}$ or if P is special and there are further lines in B that are not on P, then by Corollary 3.14, B is of Rédei type.

Case 1: Suppose now that P is the only special point of B and all the lines of B go through P. Let Q be any point on a line of B through P. From Lemma 3.15,

$$|B| = (p^{3h}+1)(p^{3h}+p^{2h}+p^h)+1-p^{2h}-p^h.$$

Now let R be any point of B which is on a $(p^{2h}+1)-$secant through P. Then again from Lemma 3.15

$$|B| = (p^{3h}+1)(p^{3h}+p^{2h}+p^h)+1-p^h,$$

but this is a contradiction. Thus there are no $(p^{2h}+1)-$secants through P, but then P is not a special point.

Case 2: Suppose now that there are no special points in B at all and again $P = l_1 \cap l_2$, where l_1 and l_2 are lines contained in B. If there is a $(p^h+1)-$ or a $(p^{2h}+p^h+1)-$secant on P then by (1) and (2) of Lemma 3.8 we can find a $(p^h+1)-$secant l on P which is not in the plane of l_1 and l_2. Because the planes $\langle l, l_1\rangle$ and $\langle l, l_2\rangle$ both contain at least $p^{4h}+p^{3h}+1$ points of B, we have $|B| \geq 2(p^{4h}+p^{3h}-p^h)+(p^{3h}-1)(p^{3h}+p^{2h})+p^h+1$, which is in contradiction with Lemma 3.4. Thus there are no $(p^h+1)-$ or $(p^{2h}+p^h+1)-$secants on P and B has to be a cone with vertex P. The base of this cone has to be a plane section of B, but from Lemma 3.4 $|B| \geq p^{3h}(p^{4h}-p^{3h})+1$ is not possible, and thus the base has to be a small minimal blocking set, which is either a line, or has to be of type (2) of Result 2.5. This planar blocking set is of Rédei type, and so the cone is of Rédei type also. □

Proposition 3.18. *B is either of Rédei type, or is a blocking set with the following properties:*

- *$|B| = p^{6h}+p^{5h}+p^{4h}+p^{3h}+1$;*
- *There is exactly one line l contained in B. There are p^h+1 special points in B and all are on the line l.*
- *On a nonspecial point of l there are tangents and $(p^h+1)-$secants only. On a special point of l there are tangents and $(p^{2h}+1)-$secants only.*
- *There are $p^{2h}+p^h+1$ planes on l containing further points of B. These planes meet B in $p^{4h}+p^{3h}+1$ points.*
- *On a $(p^h+1)-$secant meeting the line l, there is one plane meeting B in $p^{4h}+p^{3h}+1$ points (the plane on l), and all other planes intersect B in the small minimal blocking set (2) of Result 2.5.*

Proof. By Lemma 3.16, Proposition 3.17 and Corollary 3.14, we can assume that there is exactly one line l totally contained in B and all the special points of B (if there are any) are on l. If there are at least $4p^h$ special points on l, then a plane on l which contains further points of B will contain at least $4p^hp^{2h}+(p^{3h}+1-4p^h)p^h+1$ points of B, and thus by Proposition 3.13, B is of Rédei type.

Suppose now, that the number of special points is less than $4p^h$. Let P be any non-special point of the line l containing the special points and let t be a tangent of P such that the plane on t and l intersects B in the points of l only. By Lemma 3.15,

$$|B| = p^{3h}(p^{3h}+p^{2h}+p^h)+p^{3h}+1.$$

Now let P be a point of B not on the line l, and t a tangent of P. Again by Lemma 3.15, we have

$$|B| = (p^{3h}+1)(p^{3h}+p^{2h}+p^h)-Np^h+1,$$

with N the number of special points in B. From this $N = p^h+1$.

Let π be a plane on the line l and containing further points of B. As there are p^h+1 special points on l, counting the points of $B \cap \pi$ on the lines through a point of $\pi \cap B$ not on l we have: $|B \cap \pi| \geq (p^h+1)p^{2h} + (p^{3h}-p^h)p^h + 1$. Counting the points of B in the planes on any $(p^h+1)-$secant m of π, we have $|B| \geq p^{3h}(p^{3h}+p^{2h}) + |B \cap \pi|$, because there are no special points outside π, and so the small sections on m can be of type (2) of Result 2.5 only. From the size of B comes that $|B \cap \pi| = p^{4h}+p^{3h}+1$ and that equality has to hold above. From this it is clear that a point of $\pi \setminus l$ is connected to the special points of l by $(p^{2h}+1)-$secants, and to the non- special points by $(p^h+1)-$secants. It is also clear, that on a $(p^h+1)-$secant which intersects l, all the planes not containing l will intersect B in the small minimal blocking set (2) of Result 2.5. Counting the points of B in the planes on l, we see that there have to be exactly $p^{2h}+p^h+1$ planes containing $p^{4h}+p^{3h}+1$ points of B, and all other planes are tangent planes. □

Remark 3.19. *The blocking set with the properties above is not a Rédei type blocking set. The Rédei plane would have to contain $|B| - p^{6h} = p^{5h}+p^{4h}+p^{3h}+1$ points and (as it is proved in the proof of Proposition 3.13) would have to contain all the special points of B. But the planes containing the special points of B all contain $p^{4h}+p^{3h}+1$ points of B.*

Notation: Let V be the $\mathrm{GF}(p^{3h})$-vectorspace defining $\mathrm{PG}(3,p^{3h})$. For every line e put $e^B := e \cap B$. Suppose that P is a point of B and $e_1,\dots,e_s$ are lines on B such that all sets e_i^B are sublines isomorphic to $\mathrm{PG}(1,p^h)$. Let $v \in V$ be any vector representing P. Then V has a unique $\mathrm{GF}(p^h)$-subspace V_i of rank two containing v and representing exactly the points of e_i^B. Consider the $\mathrm{GF}(p^h)$-span of the vectors in $V_1 \cup \dots \cup V_s$. The set of all points of $\mathrm{PG}(3,p^{3h})$ generated by vectors in this $\mathrm{GF}(p^h)$ span will be denoted by $\langle e_1^B,\dots,e_s^B\rangle_h$. Notice that this definition does not depend on the choice of the vector v representing P. If the V_i subspaces are $\mathrm{GF}(p^h)$-independent, then $\langle e_1^B,\dots,e_s^B\rangle_h$ will be referred to as an s-dimensional $\mathrm{GF}(p^h)$-linear subspace.

Lemma 3.20. *Suppose that B is as described in Proposition 3.18. Let P be a point not on l and consider two (p^h+1)-secants l_1 and l_2 on P such that l_1 meets l. Then $\langle l_1^B, l_2^B\rangle_h$ is contained in B.*

Proof. Case 1: l_2 is skew to l. Then the plane $\langle l_1, l_2\rangle$ meets B in a small blocking set and thus the assertion follows by inspection of the small blocking sets. Alternatively, the small blocking set is $\mathrm{GF}(p^h)$-linear, which also proves the claim.

Case 2: l_2 meets l, that is the plane $\pi = \langle l_1, l_2\rangle$ contains l. Then $E_1 := l \cap l_1$ and $E_2 := l \cap l_2$ are non- special points of l. It suffices to show for all points $R \in l_2^B$ that the set $E_1R \cap \langle l_1^B, l_2^B\rangle_h$ is contained in B. This holds for $R = P$ and $R = E_2$ (because $E_1E_2 = l$ is contained in B). Suppose therefore that $R \neq P, E_2$.

As stated in Proposition 3.18, all planes on l_2 other than π intersect B in small minimal blocking sets (2) of Result 2.5. Thus we can find a point E_3 outside π such that $l_3 := PE_3$ and E_3R are $(p^h+1)-$secants. By Proposition 3.18, also E_1E_3 is a $(p^h+1)-$secant.

From Case 1 we see that $\langle l_1^B, l_3^B\rangle_h$ is contained in B. As E_1E_3 contains the points E_1, E_3 of this set, it follows that

$$(E_3E_1)^B \subseteq \langle l_1^B, l_3^B\rangle_h$$

Similarly

$$(E_3R)^B \subseteq \langle l_2^B, l_3^B\rangle_h \quad \text{and} \quad (E_1R)^B \subseteq \langle (E_3E_1)^B, (E_3R)^B\rangle_h.$$

Hence $(E_1R)^B \subseteq \langle l_1^B, l_2^B, l_3^B \rangle_h$. As E_1R is also contained in π, it follows that $(E_1R)^B \subseteq \langle l_1^B, l_2^B \rangle_h$. □

Theorem 3.21. *Let B be a point set of $\mathrm{PG}(3, q)$, $q = p^{3h}$, $p \geq 7$ prime, intersecting each line in $1 \bmod p^h$ points, and $|B| < \frac{3}{2}(q^2 + 1)$. Then B is a linear blocking set.*

Proof. Clearly by Result 2.4, B is a small minimal blocking set of $\mathrm{PG}(3, p^{3h})$. If $2|h$ and B has a $(p^{3h/2} + 1)-$secant, then by Corollary 3.10 and [19] we are done. Suppose now that B has no $(p^{3h/2} + 1)-$secants. If B is of Rédei type, then B is linear by [15].

Suppose therefore, that B is the point set described in Proposition 3.18. We will be using the properties of B given there.

Let P be any point of B not on the line l containing the special points, and let π be the plane on P and l. Take any two $(p^h + 1)-$ secants e_1, e_2 through P in π, let $E_1 := e_1 \cap l$ and $E_2 := e_2 \cap l$. By the previous lemma, we have $\langle e_1^B, e_2^B \rangle_h \subseteq B$. Let e_3 be a third $(p^h + 1)-$secant of π on P meeting the set $\langle e_1^B, e_2^B \rangle_h$ only in point P, and let $E_3 := e_3 \cap l$.

We will now prove that $\langle e_1^B, e_2^B, e_3^B \rangle_h$ is also contained in B. Because of Lemma 3.20, $\langle e_1^B, e_3^B \rangle_h$ and $\langle e_2^B, e_3^B \rangle_h$ are contained in B, thus for any point $R \in e_3^B$ it is true that $\langle e_1^B, e_2^B, e_3^B \rangle_h \cap RE_1 \subseteq RE_1 \cap B$ and $\langle e_1^B, e_2^B, e_3^B \rangle_h \cap RE_2 \subseteq RE_2 \cap B$, with equality iff $R \notin l$, because in this case RE_1 and RE_2 are $(p^h+1)-$secants of $B \cap \pi$. Applying Lemma 3.20 to R and the $(p^h+1)-$secants RE_1 and RE_2, we have that $\langle (RE_1)^B, (RE_2)^B \rangle_h \subset B$. Every point of $\langle e_1^B, e_2^B, e_3^B \rangle_h$ is contained in one of the sets $\langle (RE_1)^B, (RE_2)^B \rangle_h$ with $R \in e_3^B$, and thus $\langle e_1^B, e_2^B, e_3^B \rangle_h \subset B$ follows.

Thus we have found a 3-dimensional $\mathrm{GF}(p^h)$-linear subspace containing P and contained in B. The number of $(p^h + 1)-$secants a 3-dimensional subspace can generate on a point is at most $p^{2h} + p^h + 1$, but in π the number of $(p^h + 1)-$secants on P is $p^{3h} - p^h$ (see Proposition 3.18) and thus there have to be further $(p^h + 1)-$secants of π on P. Take one and denote it by e_4, and let $E_4 := e_4 \cap l$. We will prove $\langle e_1^B, e_2^B, e_3^B, e_4^B \rangle_h \subset B \cap \pi$. By Lemma 3.20 we have that $\langle e_1^B, e_4^B \rangle_h$, $\langle e_2^B, e_4^B \rangle_h$ and $\langle e_3^B, e_4^B \rangle_h$ are contained in B. Thus for any point $R \in e_4^B \setminus E_4$ the set $\langle e_1^B, e_2^B, e_3^B, e_4^B \rangle_h$ meets the lines RE_1, RE_2 and RE_3 in the sets $RE_1 \cap B$, $RE_2 \cap B$ and $RE_3 \cap B$ respectively (these are all $(p^h + 1)-$secants). Clearly from the reasonings of the previous paragraph $\langle (RE_1)^B, (RE_2)^B, (RE_3)^B \rangle_h \subset B$ if $R \in e_4^B$. (Note that $(RE_3)^B \not\subset \langle (RE_1)^B, (RE_2)^B \rangle_h$, but we don't need it in the proof.) From this $\langle e_1^B, e_2^B, e_3^B, e_4^B \rangle_h \subset B \cap \pi$ clearly follows.

The number of $(p^{2h}+1)-$secants on P in π is p^h+1 and the number of $(p^h+1)-$secants on P in π is $p^{3h} - p^h$, thus the lines on P in π can contain at most $(p^{3h} - p^h) + (p^h+1)(p^h+1)$ sublines, and this proves $\langle e_1^B, e_2^B, e_3^B, e_4^B \rangle_h = B \cap \pi$.

Now let α be a plane on e_1 different from π. By the properties of B stated in Proposition 3.18, $\alpha \cap B$ is the small minimal blocking set (2) of Result 2.5. This is a linear blocking set, thus there are $(p^h + 1)-$secants e_5 and e_6 on P such that $\langle e_1^B, e_5^B, e_6^B \rangle_h = \alpha \cap B$. We will now prove that $\langle e_1^B, e_2^B, e_3^B, e_4^B, e_5^B, e_6^B \rangle_h \subset B$.

There is exactly one $(p^{2h} + p^h + 1)-$secant on α, and we may suppose that P is not contained in it (if it were, then choose another point as P). Thus for any point $R \in \alpha \cap B$ the line PR is a $(p^h+1)-$secant. By Lemma 3.20, $\langle (PR)^B, e_i^B \rangle \subset B$ for all $i = 1, \ldots, 4$ and all $R \in \alpha \cap B$, $R \neq E_1$. But then the lines RE_i all meet the set $\langle e_1^B, e_2^B, e_3^B, e_4^B, e_5^B, e_6^B \rangle_h$ in exactly the points of $RE_i \cap B$, as these are all $(p^h + 1)-$secants of B. Applying the reasonings of this proof in the previous paragraphs on R in place of P, we come to $\langle (RE_1)^B, (RE_2)^B, (RE_3)^B, (RE_4)^B, \rangle_h \subset B$. But from this $\langle e_1^B, e_2^B, e_3^B, e_4^B, e_5^B, e_6^B \rangle_h \subset B$ follows.

Thus B contains a 6-dimensional $\mathrm{GF}(p^h)$-linear subspace. By observation of ranks it is clear that such a point set is blocking all the lines of $\mathrm{PG}(3, p^{3h})$, and so if B contained further points, it would be in contradiction with the minimality of B. □

3.3. **Proof of Theorem 3.3 for arbitrary** $n \geq 4$. *Throughout the section it will be assumed that B is a point set of* $\mathrm{PG}(n, q)$, $q = p^{3h}$, $p \geq 7$, $n \geq 4$, *with* $|B| \leq \frac{3}{2}(q^{n-1} + 1)$ *and intersecting every line of* $\mathrm{PG}(n, q)$ *in* $1 \bmod p^h$ *points.*

Our technique will be to prove that the plane sections of such point set are always linear, and then prove the linearity of the whole set similarly as in the case $n = 3$.

For the size of B, we will again be using the upper bound which follows from Result 3.1, that is $|B| < q^{n-1} + q^{n-2}p^{2h} + q^{n-2}p^h + 3q^{n-2}$.

Lemma 3.22. *A 3-dimensional subspace of* $\mathrm{PG}(n, q)$ *either intersects B in a small minimal blocking set, or contains more than* $q^2p^h - q^2$ *points from B.*

PROOF. The 3-dimensional subspace has $b := (q^2 + 1)(q^2 + q + 1)$ lines and $r := q^2 + q + 1$ lines on every point. With these values for b and r, equation (1) in the proof of Lemma 3.5 remains true in our situation. As the right-hand side of this equation is negative for $x = \frac{3}{2}q^2 + 1$ and $x = q^2p^h - q^2$, the assertion follows. □

Corollary 3.23. *On any plane of* $\mathrm{PG}(n, p^{3h})$ *there has to be a 3-dimensional subspace which intersects B in a small minimal blocking set.*

PROOF. If all 3-spaces on a plane π contained more than $q^2p^h - q^2$ points from B, then counting the points of B in these 3-spaces we would get

$$|B| \geq (q^{n-3} + q^{n-4} + \cdots + 1)(q^2p^h - q^2 - |\pi \cap B|) + |\pi \cap B|,$$

which is in contradiction with Result 3.1, because $|\pi \cap B| \leq q^2 + q + 1$. □

Corollary 3.24. *Every plane of* $\mathrm{PG}(n, p^{3h})$ *intersects B in a linear point set.*

PROOF. By Corollary 3.23 every plane π is contained in a 3-dimensional space which intersects B in a small minimal blocking set. From Theorem 3.21 we have that the intersection is a linear point set, and thus $\pi \cap B$ is also a linear point set. □

Corollary 3.25. *An arbitrary line can intersect B in 1,* $p^h + 1$, $p^{2h} + 1$, $p^{2h} + p^h + 1$, $p^{3h} + 1$, *or* $p^{3h/2} + 1$ *(if* $2|h$*) points.*

In case $2|h$ we will now characterize the blocking sets which have a $(p^{3h/2} + 1)-$secant.

Lemma 3.26. *If* $2|h$ *and B has a* $(p^{3h/2} + 1)-$*secant, then B intersects every line in* $1 \bmod p^{3h/2}$ *points*

PROOF. If a $(p^{3h/2}+1)-$secant m were in the same plane with a $(p^h+1)-$, a $(p^{2h}+1)-$ or a $(p^{2h} + p^h + 1)-$ secant l, then by Lemma 3.23 there would be a 3-dimensional subspace intersecting B in a small minimal blocking set and having a $(p^{3h/2} + 1)-$secant and a

$(p^h+1)-$, a $(p^{2h}+1)-$ or a $(p^{2h}+p^h+1)-$ secant. This would be in contradiction with Theorem 3.21. Suppose now that m and l are skew, and take $P \in l \setminus B$ and π the plane on m and P. All the lines through P in π have to be tangents of B which is a contradiction. □

Corollary 3.27. *If B has a $(p^{3h/2}+1)-$secant, then it is a so-called* Baer cone *characterized in* [19]. *Such point sets are known to be linear.*

From now on we will suppose that B has no $(p^{3h/2}+1)-$secants.

A point set S of $\mathrm{PG}(n,q)$ will be called a *projected* $\mathrm{PG}(s,p^h)$, if there is a projective space $\mathrm{PG}(n',q)$ containing $\mathrm{PG}(n,q)$ such that S is the projection of a subgeometry $\mathrm{PG}(s,p^h) \subset \mathrm{PG}(n',q)$ from a suitable vertex onto $\mathrm{PG}(n,q)$. A point of S will be called *single projected* or *multiple projected* accordingly. The next statements follow from simple rank argument.

Lemma 3.28. *Let S be a projected* $\mathrm{PG}(s,p^h)$ *contained in a t-dimensional subspace π of* $\mathrm{PG}(n,q)$. *Then*

(1) *$s \leq 3t+2$;*

(2) *If $s = 3a+b$ with $0 \leq b \leq 2$, then S meets every $(t-a)-$dimensional subspace of π in a projected* $\mathrm{PG}(d,p^h)$ *with $d \geq b$.*

By Corollary 3.24 every plane intersects B in a projected $\mathrm{PG}(m,p^h)$. Clearly $m \leq 2$ would not block all the lines of the plane, and by (1) of Lemma 3.28 $m \leq 8$.

Corollary 3.29. *An arbitrary plane of* $\mathrm{PG}(n,p^{3h})$ *intersects B in a projected* $\mathrm{PG}(m,p^h)$, *with $3 \leq m \leq 8$.* □

Corollary 3.30. *Let π be a plane of* $\mathrm{PG}(n,p^{3h})$ *intersecting B in a projected* $\mathrm{PG}(m,p^h)$.

(1) *Every point of a $(p^h+1)-$secant of B in π is the projection of one point only.*
(2) *If $m \geq 6$, then π is totally contained in B.*
(3) *If $m \geq 5$, then B has no $(p^h+1)-$secants in π.*
(4) *If $m = 4$, then in π there are no tangents on a single-projected point and at most p^h+1 of the lines through a single-projected point can be long secants (that is secants containing at least $p^{2h}+1$ points of $B \cap \pi$).*

Proof. (1) If a $\mathrm{PG}(s,p^h)$ with $s \geq 1$ is projected onto a $(p^h+1)-$secant, then the preimages of the points of the $(p^h+1)-$secant give a partition of the space $\mathrm{PG}(s,p^h)$ into p^h+1 non-empty subspaces, which is only possible if $s=1$.

(2) Clear from (2) of Lemma 3.28.

(3) Clear from (2) of Lemma 3.28 and (1) of this lemma.

(4) By Lemma 3.28 every line contains a projected $\mathrm{PG}(d,p^h)$ with $d \geq 1$. For a single projected point $P \in \pi \cap B$ and $l_1, \dots, l_{p^{3h}+1}$ the lines on P in π, the preimages of the sets $B \cap l_i$ are subspaces of $\mathrm{PG}(4,p^h)$ of dimension at least 1 and meeting in one point (the preimage of P). Counting the points of $\mathrm{PG}(4,p^h)$ in these subspaces yields that either one line l_i contains a projected 3-dimensional subspace, and all others contain projected lines, or p^h+1 have projected planes and all others lines. □

Lemma 3.31. *Let π be a plane of $\mathrm{PG}(n,q)$ intersecting B in a projected $\mathrm{PG}(m,p^h)$. Let H be a plane of $\mathrm{PG}(m,p^h)$ and $P' \in H$ a point which is projected onto $P \in \pi$. Suppose that the lines of H through P' are projected onto the lines l_i, $i = 1,\dots,p^h+1$, which are lines of π through the point P. Then either $|l_i \cap B| \geq p^{2h}+1$ for each l_i or this is true only for at most two of these lines.*

Proof. By Corollary 3.30, we may assume that $m \leq 4$. If P is not a single projected point, then having a $(p^h+1)-$secant on it would be in contradiction with (1) of Corollary 3.30, thus we may assume that P is a single projected point with preimage P'. If $m=3$ then clearly there can be at most one long secant on a single projected point. Thus we can also assume $m=4$, and the projection can be viewed as the projection of a $\mathrm{PG}(4,p^h)$ subgeometry embedded into $\mathrm{PG}(4,p^{3h})$, with the vertex of the projection being a line v. The lines in π through P correspond to 3-spaces on the plane $\langle v,P'\rangle$. By the proof of Corollary 3.30 (4), we may also assume that p^h+1 of these 3-spaces meet the $\mathrm{PG}(4,p^h)$ in planes (denote these by π'_i, $i=1,\dots,p^h+1$), while all others meet it in lines. Any two of these π'_i planes have only the point P' in common. We will prove that the union of the π'_i planes is a cone of $\mathrm{PG}(4,p^h)$ with vertex P' and a regulus of a 3-dimensional space Σ as base. This finishes the proof, as a plane H on P' which meets at least three of the planes π'_i (that is: has at least three lines on long secants through P) will meet the 3-dimensional space Σ in a line meeting at least three lines of the regulus, and thus will have to meet all the lines of the regulus, from which it is clear that the H has to meet all the planes π'_i.

Let r'_1 and r'_2 be any lines of the planes π'_1 and π'_2 respectively, such that neither of them contains P' and that their extensions over $\mathrm{GF}(q)$ meet the plane $\langle v,P'\rangle$ in points R_1,R_2. Let Σ be the 3-dimensional $\mathrm{GF}(p^h)$-space spanned by r'_1 and r'_2. Clearly Σ does not contain P' (because then it would contain the planes π'_1 and π'_2, but these planes have only one point in common), and thus intersects the planes π'_i, $i=3,\dots,p^h+1$ in lines r'_i; and so Σ^*, the extension of Σ, does not contain P'. Thus Σ^* meets $\langle v,P'\rangle$ in a line t on R_1 and R_2. For every i, the extension of the line r'_i will also meet t, because it is contained in a 3-dimensional subspace on $\langle v,P'\rangle$, and thus has to meet it, but is also contained in Σ^*, so it can meet it in a point of t only.

We will prove that the lines r'_i are the only lines of Σ with the property that their extension meets the line t. If a line has this property, then it is projected onto a line on P. It is either projected onto a $(p^h+1)-$secant on P, but these all have to contain P' (and $P' \notin t$), or is contained in one of the π'_i subplanes. But these subplanes meet Σ only in the lines r'_i.

Let $\mathcal{R}$ be a regulus of Σ determined by any three of the lines r'_i. The extensions of these three lines also determines a regulus $\mathcal{R}^*$ of Σ^* and $\mathcal{R}^*$ will contain the extensions of the elements of $\mathcal{R}$. The line t will be an element of the opposite regulus of $\mathcal{R}^*$. Thus the extensions of all the elements of $\mathcal{R}$ meet the line t. But then this means that $\mathcal{R} = \{r'_1,\dots,r'_{p^h+1}\}$ is the regulus we are looking for. □

Lemma 3.32. *If there is a $(p^h+1)-$secant on a point $P \in B$, then the number of $(p^h+1)-$secants on P is at least $p^{h(3n-4)} - p^{h(3n-6)}$.*

Proof. Let l be a $(p^h+1)-$secant on the point $P \in B$. By Corollary 3.29 and (3) of Corollary 3.30, a plane on l meets B in a projected $\mathrm{PG}(3,p^h)$ or a projected $\mathrm{PG}(4,p^h)$. In the first case it is easy to check that there are at least $p^{2h}-1$ further $(p^h+1)-$secants on P. In the latter case according to (4) of Corollary 3.30 at least $p^{3h}-p^h$ of the lines on P are $(p^h+1)-$secants. Thus in any plane on l there are at least $p^{2h}-1$ further $(p^h+1)-$secants on P. As there are $q^{n-2}+q^{n-3}+\cdots+q+1$ planes on a line of $\mathrm{PG}(n,q)$, we have that the number of $(p^h+1)-$secants on P is at least $(p^{2h}-1)(p^{3h(n-2)}+p^{3h(n-3)}+\cdots+1) \geq p^{(3n-4)h}-p^{(3n-6)h}$. □

We are now ready to prove the main theorem. We will again use the notation $\langle e_1^B,\ldots,e_s^B\rangle_h$ given before Lemma 3.20.

Theorem 3.33. *Let B be a point set of $\mathrm{PG}(n,q)$, $q=p^{3h}$, $p \geq 7$, $n \geq 4$, with $|B| < \frac{3}{2}(q^{n-1}+1)$ and intersecting every line of $\mathrm{PG}(n,q)$ in $1 \bmod p^h$ points. Then B is a linear point set.*

Proof. By Result 2.4 B is a minimal blocking set of $\mathrm{PG}(n,q)$. We may assume, that B is not a hyperplane (which is clearly linear). If $2|h$ and B has a $(p^{3h/2}+1)-$secant, then by Corollary 3.27, B is a linear point set. Now we may assume that B has no $(p^{3h/2}+1)-$secants, and so by Corollary 3.29 every plane meets B in a projected $\mathrm{PG}(m,p^h)$ with $3 \leq m \leq 8$. If B has a $(p^{2h}+1)-$ or a $(p^{2h}+p^h+1)-$secant, then B has to have $(p^h+1)-$secants also, or else all the planes on such a secant would meet B in a projected $\mathrm{PG}(5,p^h)$, which would be in contradiction with the size of B.

Let $P \in B$ be a point of a $(p^h+1)-$secant. By Lemma 3.32, there are many $(p^h+1)-$secants on P. Let e_1 and e_2 be two $(p^h+1)-$secants of B meeting in the point P. The plane containing e_1 and e_2 meets B in a projected $\mathrm{PG}(m,p^h)$, and by (1) of Corollary 3.30, the point P is a single-projected point. Thus e_1^B and e_2^B are projections of intersecting lines of $\mathrm{PG}(m,p^h)$. But then the subplane $\langle e_1^B,e_2^B\rangle_h$ generated by them is the image of the plane of $\mathrm{PG}(m,p^h)$ on the pre-images, and thus is contained in the projection.

Now suppose that $e_1,e_2,\ldots,e_s$ are $(p^h+1)-$secants through $P \in B$, such that $e_i^B \notin \langle e_1^B,\ldots,e_{i-1}^B\rangle_h$ for $i=2,\ldots,s$ and $\langle e_1^B,\ldots,e_s^B\rangle_h \subset B$. If $s<3(n-1)$ then we can find further $(p^h+1)-$secants through P, as the subspace $\langle e_1^B,\ldots,e_s^B\rangle_h$ has at most $p^{h(s-1)}+p^{h(s-2)}+\cdots+1$ $(p^h+1)-$secants through P, and from Lemma 3.32 there are more. Let e_{s+1} be any further $(p^h+1)-$secant through P not contained in $\langle e_1^B,\ldots,e_s^B\rangle_h$.

Let Σ be a 3-dimensional $\mathrm{GF}(p^h)$-linear subspace of $\langle e_1^B,\ldots,e_s^B,e_{s+1}^B\rangle_h$ on $\langle e_1^B,e_{s+1}^B\rangle_h$. Σ meets $\langle e_1^B,\ldots,e_s^B\rangle_h$ in a subplane on e_1 which contains by Lemma 3.31 further lines f_i $(i=1,\ldots,p^h-2)$ which are all $(p^h+1)-$secants of B going through P. From the reasonings above, the subplane $\langle e_{s+1},e_1\rangle_h$ and the subplanes $\langle e_{s+1},f_i\rangle_h$ are all contained in B. Suppose now that $Q \in \Sigma$ is not on any of these planes. Again by Lemma 3.31, among the p^h further $\mathrm{GF}(p^h)$-linear subplanes on the line PQ in Σ we can find a subplane which intersects two of the subplanes $\langle e_{s+1},f_i\rangle_h$ in sublines which are both $(p^h+1)-$secants of B. But then the subplane generated by these two $(p^h+1)-$secants is contained in B and Q is an element of this subplane.

With this we have proved that any 3-dimensional $\mathrm{GF}(p^h)$-linear subspace of $\langle e_1^B,\ldots,e_s^B,e_{s+1}^B\rangle_h$ on e_1^B and e_{s+1}^B is contained in B, thus $\langle e_1^B,\ldots,e_s^B,e_{s+1}^B\rangle_h$ is contained in B.

From this it is clear that B contains a projected $\mathrm{PG}(3(n-1), p^h)$. This projected subgeometry is a blocking set of $\mathrm{PG}(n, q)$, and so it is equal to B by the minimality of B. $\square$

3.4. **Constructions for $n = 3$.** In Theorem 3.3 we have proved, that a small minimal blocking set of $\mathrm{PG}(3, q^3)$ which meets every line in 1 mod q points and has no $(q^{3/2}+1)-$secants is a projected $\mathrm{PG}(6, q)$ subgeometry. Two wellknown such blocking sets are the cones over the blocking sets (1) and (2) of Result 2.5, which are of Rédei type. Now we present a construction for the blocking set with properties given in Proposition 3.18. The construction is of special interest, because it gives a blocking set of $\mathrm{PG}(3, q^3)$ that is linear, but not of Rédei type. We will use Construction 2.2 with $n = 3$, $k = 1$, $t = 3$.

Construction 3.34. Let $\mathrm{PG}(6, q)$ be embedded in $\mathrm{PG}(6, q^3)$ as a subgeometry. Suppose that $\Sigma = \mathrm{PG}(3, q)$ is a 3-dimensional subspace of the embedded subgeometry, denote by $e(\Sigma)$ the unique $\mathrm{PG}(3, q^3) \subset \mathrm{PG}(6, q^3)$ subspace which contains Σ (the extension of Σ). Let $\mathcal{R}$ be a regulus of Σ. The extensions of the lines of $\mathcal{R}$ are elements of a regulus $\mathcal{R}^*$ of $e(\Sigma)$. Let v be a line of the opposite regulus of $\mathcal{R}^*$ such that v is skew to Σ (that is v is not the extension of an element of $\mathcal{R}^{OPP}$). Let Q be a further point of $\mathrm{PG}(6, q^3) \setminus \mathrm{PG}(6, q)$ such that Q is not contained in the extension of any of the 5-dimensional subspaces of $\mathrm{PG}(6, q)$ containing Σ. We can find such a point, because the number of 5-dimensional subspaces of $\mathrm{PG}(6, q)$ containing Σ is $q^2 + q + 1$, the extension of such a 5-dimensional subspace contains $q^{15} + q^{12}$ points from $\mathrm{PG}(6, q^3) \setminus e(\Sigma)$ and thus even if these were all different points, the extensions would be covering at most $(q^2 + q + 1)(q^{15} + q^{12}) + |e(\Sigma)|$ points, but the number of points in $\mathrm{PG}(6, q^3) \setminus \mathrm{PG}(6, q)$ is larger than q^{18}.

We will now prove, that if π is the plane on v and Q, then the projection of the embedded subgeometry $\mathrm{PG}(6, q)$ from the plane π onto an arbitrary 3-dimensional subspace $\mathrm{PG}(3, q^3)$ disjoint from π will be a minimal blocking set B having the properties described in Proposition 3.18.

The points of Σ are distributed on the q^3+1 planes on v in $e(\Sigma)$. As Σ is a plane blocking set of $e(\Sigma)$ (see Lemma 3.28), from counting arguments it follows that $q+1$ of the planes on v meet Σ in lines (the elements of $\mathcal{R}$) and all other planes on v meet Σ in exactly one point. If m is a line of $\mathrm{PG}(6, q)$ with $e(m) \cap \pi = P \in v$, then clearly $m \subset \Sigma$ and so $m \in \mathcal{R}$. Suppose now that m is a line of $\mathrm{PG}(6, q)$ with $e(m) \cap \pi = P$, $P \notin v$. The extension of the subspace generated by Σ and m contains π, and thus contains Q, which is in contradiction with the choice of Q, because this subspace has dimension at most 5.

With this we have proved that the lines of $\mathcal{R}$ are the only lines of $\mathrm{PG}(6, q)$ with the property that their extension meets π. From this it follows, that there are exactly $q + 1$ multiple projected (special) points in B and so $|B| = q^6 + q^5 + q^4 + q^3 + 1$. By the reasonings above it is clear that the special points are on a line which is totally contained in B, while all other points are non-special. The other properties can be derived from these. $\square$

Remark 3.35. Starting with the same line v, but choosing the point Q to be a point contained in the extension of a 5-space on Σ, but not contained in the extension of a 4-space on Σ will result in a Rédei type blocking set of size $q^6 + q^5 + q^4 + 1$, which is not a cone. For a complete characterization of the projections of $\mathrm{PG}(6, q)$ into $\mathrm{PG}(3, q^3)$ see [5].

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