Multidiszciplináris tudományok, 11. kötet. (2021) 5 sz. pp. 373-385 **https://doi.org/10.35925/j.multi.2021.5.41**

INTEGRATION OF FCA WITH FUZZY LOGIC: A SURVEY

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Abstract
The theory of formal concept analysis(FCA), which was developed in the early 1980s (Ganter and Wille, 1999), has evolved into an effective technique for data analysis, knowledge discovery and information retrieval. The study on expanding the theory of FCA to deal with uncertain and imperfect data has made considerable progress in recent years. In this paper, we will introduce a survey of the research papers on integrating FCA with fuzzy logic. The key goal is to investigate and compare different fuzzy FCA approaches that have been proposed and to clarify relationships between these approaches, as well as we will introduce a survey of the research papers on employing FCA with fuzzy logic in knowledge discovery in databases and data mining, information retrieval and ontology engineering.

Keywords: *FCA, formal concept analysis, fuzzy FCA, fuzzy logic*

1. Introduction

Formal concept analysis (FCA) was developed as a mathematical theory in the early 1980s (Ganter and Wille, 1999). FCA is concerned with a specific type of data analysis. FCA's classical setting takes the context as a table with rows for objects, columns for attributes, and table inputs containing 1s and 0s based on whether or not an object has the attribute. Therefore, the classical FCA is better for attributes that are crisp. At the same time, attributes can be fuzzy instead of crisp. If we consider fuzzy attributes, such as (“cold”, “hot”), the table entries will be a truth degree to which an object has a fuzzy attribute. For example, when asking about the weather temperature like 5 °C, we properly get a response like a cold or a little bit cold or very cold and so on, so it is based on personal feelings. Like these concepts have been established in fuzzy logic by (Lotfi Aliasker Zadeh, 1965) to allocate a truth degree to an object depending on whether or not the object has a fuzzy attribute. Degrees are calculated using an L-scale of truth degrees. The real unit interval [0, 1], is one of the most L-scale favorite choices. Back to our previous example, we can say that the weather with 5 °C has on “cold” truth degree of 0.6, indicating that the temperature is almost cold. As a consequence, instead of values from 0 or 1, as in the classical FCA, the entries of a table representing objects and attributes become degrees from the unit interval [0,1].

Many approaches were proposed to deal with fuzzy attributes. This paper's essential goal is to investigate and compare different fuzzy FCA approaches that have been proposed and to clarify relationships between these approaches.

This paper is organized as follows. Section 2 will give a theoretical introduction to the concept of fuzzy sets and fuzzy logic. In Section 3, we will provide a brief background of the basic notations of FCA theory. In section 4, we will discuss the most important methods for integrating FCA with fuzzy logic that have been introduced, as well as we will introduce a brief survey on using formal concept analysis with fuzzy logic in several domains like information retrieval, knowledge discovery in databases, data mining, and ontology engineering.

2. Theoretical introduction of the concept of fuzzy set and fuzzy logic

Essentially, fuzzy logic (FL) is a multi-valued logic (MVL) established in 1965 by Lotfi a. Zadeh on the basis of the mathematical theory of fuzzy sets which is a generalization concept of classical set theory (Zadeh, 1965). Fuzzy logic (FL) is considered an extension of classical logic (Boolean logic). In classical logic, only two values, such as 0 (False) or 1 (True), are allowed as the value for a variable; whereas fuzzy logic is a method of multi-valued logic in which every value is appropriate as long as it falls within the range [0, 1].

Either in a particular situation or the decision-taking process, real-world conditions are complicated and not precise. The absence of correct and precise knowledge, as well as the quantification of linguistic variables, can cause ambiguity in the decision-taking process (Tarannum and Jabin, 2018). In order to deal with these types of situations, Lotfi a. Zadeh proposed the principle of fuzzy logic in 1965 (Zadeh, 1965). Fuzzy logic is a logic of approximation that enables different degrees of truth. It can accommodate the idea of partial truth, or values that are somewhere between true and false (the values in the unit interval [0, 1]) (Zadeh, 1996). Fuzzy logic enables thinking in terms of linguistic concepts like tall, very tall, semi-tall, and so on, as well as grading how much a linguistic concept belongs to a particular set. Rather than forcing an element to be a member or not a member of a set, Lotfi a. Zadeh proposed the concept of partial set membership, in which the presence or absence of a related member of a set is determined by its degree of membership (Zadeh, 1965).

The membership function is a function in the real unit interval [0,1], in which the relevant element is included in the set or not depending on its degree of belongingness (Zadeh, 1996). Belongingness in a fuzzy set is characterized by the membership or non-membership function. The membership value in a crisp set is either 0 (False) or 1 (True), and in a fuzzy set, the membership is any value between 0 and 1. It is important to mention here that fuzzy logic simulates human thinking ability and uses approximate information to find a precise solution. Because of this possibility, systems built on fuzzy logic are characterized by their simplicity, ease of build, and testing compared to ordinary systems.

2.1. Fuzzy sets

The perception of fuzzy sets (Zadeh, 1965) arose from classical sets' extension to include ambiguous concepts and uncertain boundaries. In some cases, it is not really apparent if an object x relates to a set A or not. As a result, the membership can be determined by a membership degree, which can be calculated by agreeing on a value from the interval [0,1].

Let A be a fuzzy set in a universe of discourse U; a fuzzy set A can be defined as a set of ordered pairs.

$$A = \{(x, \quad \mu_A(x)) \mid x \in U \ \wedge \ \mu_A(x) \in [0,1]\} \tag{1}$$

where $\mu_A : U \to [0,1]$ is a membership function that fits each element of U with its membership value from the interval $[0, 1]$ to the set A.

The case $\mu_A(x) = 0$ claims that the existence of element x is improbable to be in the fuzzy set A, and $\mu_A(x) = 1$ says that is simply no doubt that x is a member of the fuzzy set A. A higher value of $\mu_A(x)$ for an element $x \in A$ means that x has a higher membership degree in the fuzzy set A.

2.1.1. Characteristics of fuzzy sets

In this part, we will explain the fuzzy set's key features (Hudec, 2016).

1. Cardinality

Let A be a fuzzy set in a finite universe of discourse U, the formula defines the scalar cardinality of A is:

$$card(A) = |A| = \sum_{x \in U} \mu_A(x) \tag{2}$$

where $card(A)$ is defined in (2), and $card(U)$ represents the number of elements in U.

2. Support

For a fuzzy set A, the support property is the crisp set that can be defined as follows:

$$supp(A) = \{x \in U \mid \mu_A(x) > 0\} \tag{3}$$

3. Core

For a fuzzy set A, the core property is the crisp set that can be defined as follows:

$$core(A) = \{x \in U \mid \mu_A(x) = 1\} \tag{4}$$

4. Height

The highest value for the degree of membership of all elements of a fuzzy set is defined as a height.

$$Hgt(A) = sup\{\mu_A(x) | \ x \ \in U\} = sup(\mu_A(U)) \tag{5}$$

where, sup is the supremum of a fuzzy set.

5. α-cuts

The $\alpha - cut$ is one of the most significant notions used in the fuzzy sets. The following is how $\alpha - cut$ $A^{(\alpha)}$ defined:

$$A^{(\alpha)} = \{x \in U \mid \mu_A(x) \geq \ \alpha\} \tag{6}$$

where $\alpha \ \in [0,1]$

For a fuzzy set A, the $\alpha - cut$ is considered as a crisp set that contains all the elements of U whose membership degrees in A greater than or equal to the defined value of α. the $\alpha - cut$ property is helpful in several cases, such as when dealing with elements that belong to a fuzzy set in a significant way.

2.1.2. Fuzzy sets operations

Let assume two fuzzy sets $A = \{(x, \ \mu_A(x)) \mid x \ \in U \ \}$ and $B = \{(x, \ \mu_B(x)) \mid x \ \in U\}$ over the same universe of discourse U, operations on the membership functions of fuzzy sets A and B determine the operations for them.

1. Equality

Let A and B are fuzzy sets, the two sets A and B are equal, if for $\forall x \ \in \ U$:

$$\mu_A(x) = \ \mu_B(x) \tag{7}$$

2. Union

$$\mu_{A\cup B}(x) = max(\mu_A(x)\,, \qquad \mu_B(x)) \tag{8}$$

3. Intersection

$$\mu_{A\cap B}(x) = min(\mu_A(x)\,, \qquad \mu_B(x)) \tag{9}$$

4. Inclusion

Let A and B are fuzzy sets, A is included in B, if for every $x \in U$ holds

$$\mu_A(x) \leq \mu_B(x) \tag{10}$$

5. Complement

Let A and A' be a fuzzy set. They can be considered as complements if:

$$\mu_A(x) = 1 - \mu_{A'}(x) \tag{11}$$

3. Formal Concept Analysis (FCA): Background

FCA was developed as a mathematical theory in the early 1980s (Ganter and Wille, 1999), which is a type of data analysis that takes the context as a table with rows for objects, columns for attributes, and table inputs containing 1s and 0s based on whether or not an object has an attribute. The relationship between the objects and their attributes is depicted as a formal context. A formal context is a triple (G, M, I) where G is a set of objects and M is a set of attributes, and $I \subseteq G \times M$ is a binary relation between G and M. The relation gIm is met if and only if attribute m is true to the object g, where $g \in G\,, m \in M$. A formal concept is defined as a pair (A, B), where $A \subseteq G$ (the set of all objects sharing all the attributes from B) and $B \subseteq M$ (the set of all attributes shared by all objects from A).

For each $A \subseteq G$ a derivation operator defined as:

$$A' = \{m \in M \mid gIm\; for\; \forall\, g \in A\} \tag{12}$$

And for each $B \subseteq M$

$$B' = \{g \in G \mid gIm\; for\; \forall\, m \in B\} \tag{13}$$

This implies that A' refers to the set of all M attributes shared by all A objects, while B' refers to the set of all G objects sharing all B attributes.

Therefore, for a formal context (G, M, I), a formal concept is defined as a pair (A, B), where,

$$A = B'\,, B = A' \text{ is satisfied.}$$

A is considered an extent of the concept, and B is called an intent part for a formal concept (A, B).

On the set of formal concepts $C\,(A, B)$, a partial ordering relationship is defined as follows:

$$(A_1, B_1) \leq (A_2, B_2) = A_1 \subseteq A_2 (or \text{ equivalently } B_1 \subseteq B_2\,)$$

The structure $(C(A, B), \leq)$ is called the concept lattice. The so-called key theorem of concept lattices in (Wille, 1982) describes the basic structure of concept lattices.

4. Fuzzy Concepts with FCA

In the classical FCA (classical setting of FCA), that takes a context like a table from rows((objects) and columns (attributes or properties), the attributes are Boolean if an object does or does not have an attribute, there are one or zero entries on the table based on its content, respectively. Therefore, classical FCA is better for crisp attributes. At the same time, most attributes are fuzzy instead of crispy. If the attributes being considered are fuzzy, such as (“cold”, “hot”), in this case, the table entries will be a truth degree to which an object has a fuzzy attribute. Zadeh referred to ideas with a graded form as fuzzy concepts (Zadeh, 1965). He concluded that these concepts are a rule and not an anomaly in the way people transmit information. Further, he debated that mathematically modelling such concepts is necessary for decision-making, pattern recognition, etc. The concept of a fuzzy set was suggested by Zadeh, which gave rise to the domain of fuzzy logic.

A fuzzy set A in a universe U is formally defined as a mapping that assigns a truth degree $A(u) \in L$ for each $u \in U$, L is a partially ordered set of truth degree with the lowest and highest elements of L being 0 and 1, respectively (Bělohlávek et al., 2005a). A complete resituated lattice $\boldsymbol{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ reflects the configuration of truth degrees supported by logical connectives, such that:

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (with the lowest and highest elements of L being 0 and 1, respectively), i.e., a partially ordered set in which arbitrary suprema ($\bigvee$) and infima ($\bigwedge$) exist.
- $\langle L, \otimes, 1\rangle$ *is a commutative monoid, i.e.,* $\otimes$ *is a binary operation satisfying* $a \otimes (b \otimes c) = (a \otimes b) \otimes c, a \otimes b = b \otimes a,$ *and* $a \otimes 1 = a.$
- $\otimes$ *and* $\rightarrow$ *satisfy the adjointness property* $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c;$ *for each* $a, b, c \in L.$
- L^X *is denoted the set of all fuzzy sets in* X *(universe of discourse), its means,* A *of* X *to* L*. where* $A \in L^X$ *and* $x \in L$*, a set* ${}^{x}A = \{x \in X \mid A(x) \geq a\}$ *is called an* $a - cut\ of\ A$

Fuzzy logic tends to be a prominent alternative for expanding FCA to graded concepts (Fuzzy Concepts). Burusco and Fuentes-Gonzalez were the first to suggest expanding FCA with fuzzy concepts in their paper (Burusco Juandeaburre and Fuentes-González, 1994); in their theory, they presented some basic concepts. Further, Belohlávek (1998) and Pollandt (1997) developed these basic concepts; in their method, they used residuated structures of truth degree rather than presenting basic concepts. Rather than presenting simple concepts, Belohlavek (Belohlávek, 1998) emphasizes analyzing similarities in fuzzy concept lattices. Fuzzy closure operators investigated by Belohlavek (Bêlohlávek, 2001), which are the fundamental structures underlying fuzzy concept lattices, alongside fuzzy Galois connections. These findings have since been referenced in several articles on the topic.

The author Belohlavek (2001) reveals that “fuzzy Galois connections” and “classical Galois connections” have a relationship. He shows how a fuzzy concept lattice can be treated as a traditional concept lattice by using this result. The researchers (Belohlavek, Vychodil) provide an overview and analysis of the various methods to fuzzy concept lattices that had been developed up to that point (Bêlohlávek and Vychodil, 2005c). In his paper (Belohlavek et al., 2010), the author debates the latest mathematical problems regarding algorithms for searching fuzzy concepts.

A description of a formal fuzzy concept lattice was given by (Bêlohlávek, 1999, Bělohlávek et al., 2005) $L - context$ (fuzzy formal context) is atriple (X, Y, I), X (set of objects) and Y (set of attributes), where I is L-relation (fuzzy relation) among the sets $X\ and\ Y$, i.e. $I : X \times Y \rightarrow L$. Each pair (x, y) is given a truth degree $I\ (x, y) \in L$, where $x \in X\ and\ y \in Y$ and L denotes the set of values of a complete residuated lattice $\boldsymbol{L}$. The degree to which attribute y belongs to object x is defined by the element $I(x, y)$.

Based on (Belohlavek, 1999), A and B are fuzzy sets, where, $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ described as follows:

$$A^{\uparrow}(y) = \bigwedge_{x \in X} (A(x) \to I(x,y)) \tag{14}$$

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} (B(y) \to I(x,y)) \tag{15}$$

where $x \in X$, $y \in Y$ and $\to$ serve as the truth function of fuzzy implication

Employing the fundamental principles of fuzzy logic (Hájek, 1998, Belohlavek, 2002). It is clear to see that $A^{\uparrow}(y)$ denote to the truth degree of all objects from A have y in common, $B^{\downarrow}(x)$ denote to the truth degree of x cover all attributes from B. In other terms, (14) and (15) accurately generalize (12) and (13) (Bêlohlávek, 1999). Placing

$$\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\ \}$$

$\mathcal{B}(X,Y,I)$ is the set of all formal fuzzy concepts (all pairs $\langle A,B\rangle$), A is a fuzzy set of all objects (extent) which have all attributes from B, B is a fuzzy set of all attributes (intent) which are shared by all objects from A. Besides, we define $\leq$ that models super concept-subconcept hierarchy in $\mathcal{B}(X,Y,I)$:

$$\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_1 \supseteq B_2) \tag{16}$$

For $\langle A_1,B_1\rangle$, $\langle A_2,B_2\rangle \in \mathcal{B}(X,Y,I)$, equipped with $\leq$, $\langle \mathcal{B}(X,Y,I), \leq\ \rangle$ It is called a complete lattice.

In addition to the previously mentioned methods presented for the theory of fuzzy concept lattice, part 4.1, we will discuss the most important methods for integrating FCA with fuzzy logic that have been introduced. Part 4.2, we will survey some papers that proposed to deal with the issues that may appear while using fuzzy concept lattice. Part 4.3, in this part, we will introduce a brief survey on using formal concept analysis with fuzzy concepts in several domains like knowledge discovery in databases, data mining, information retrieval, and ontology engineering.

4.1. Comparison of the different methods to integrate FCA with fuzzy logic

In this part, we will go through and compare the most important methods to integrate FCA with fuzzy logic that has been proposed.

4.1.1. One-sided fuzzy concept lattice

The "One-sided fuzzy concept lattice" method has been proposed by (Ben Yahia and Jaoua, 2001) and (Krajci, 2003) independently. Both Yahia and Krajci definitions provide the same results for (X,Y,I^{-1}), $I^{-1} \in L^{X\times Y}$ described by $I^{-1}(x,y) = I(x,y)$, that's mean, the methods are identical when it comes to the role of objects and attributes. The authors also take $L = [0,1]$.

For a fuzzy formal context (L-context), the authors defined two mapping operators, (a) $f: 2^X \to L^Y$ by $f(A)(y) = \wedge_{x\in A} I(x,y)$, where $A \sqsubseteq X$ (the set of objects), $f(A) \in L$ (fuzzy set of attributes). And (b) $h: L^Y \to 2^X$ by $h(B) = \{x \in X \mid each\ y \in Y: B(y) \leq I(x,y)\}$, where $B \in L^Y$(fuzzy set of attributes) and $h(B) \in X$(the set of objects).

Then, the authors put.

$$\mathcal{B}_{f,h}(X,Y,I) = \{\langle A,B\rangle \in 2^X \times L^Y \mid f(A) = B, h(B) = A\}$$

They demonstrated that $\mathcal{B}_{f,h}(X,Y,I)$ outfitted with the partial order $\leq$ as described in (16) a complete lattice (Yahia and Krajci refer to this as a one-sided fuzzy concept lattice). Note that the extents and the intents of concepts from $\mathcal{B}_{f,h}(X,Y,I)$ are crisp sets and fuzzy sets, respectively.

4.1.2. Crisply generated fuzzy concepts

The authors (Bělohlávek et al., 2005b) suggested the following solution when dealing with the issue of a potentially large number of formal concepts in Pollandt's and Belohlávek 's $\boldsymbol{L}-concept$ lattice $\mathcal{B}(X,Y,I)$. Rather than taking the entire $\mathcal{B}(X,Y,I)$, the authors suggested just a subset of it $\mathcal{B}_c(X,Y,I) \subseteq \mathcal{B}(X,Y,I)$

$\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ is namely crispy if there is a crisp set $B_c \subseteq Y$ (of attributes) exists, such that $A = B_c^{\downarrow}$ (thus, $B = B_c^{\downarrow\uparrow}$). Then, the complete lattice of crispy generated fuzzy concept represented by $\mathcal{B}_c(X,Y,I) = \{\langle A,B\rangle \in \mathcal{B}(X,Y,I) \mid exists\, B_c \subseteq Y : A = B_c^{\downarrow}\}$.

Notice that in Yahia and Krajci's approach, “One-sided fuzzy concept lattice”, extents are fuzzy sets and intents are crisp sets, while in the approach “crisply generated fuzzy concepts”, both extents and intents are fuzzy sets in general. In the “One-sided fuzzy concept lattice” $\mathcal{B}_{f,h}(X,Y,I)$ outfitted with the partial order $\leq$ as described in (16) is a complete lattice which is isomorphic to $\mathcal{B}_c(X,Y,I)$ outfitted with the partial order inherited from $\mathcal{B}(X,Y,I)$. Furthermore, for the corresponding concepts, $\langle A,B\rangle \in \mathcal{B}_{f,h}(X,Y,I)$ and $(\mathrm{C},\mathrm{D}) \in \mathcal{B}_c(X,Y,I)$ such that $A = C$, $B = D^{\downarrow\uparrow}$ there is an isomorphism.

4.1.3. Generalized concept lattice

In (Krajci, 2005), the author investigates what is known as a “generalized concept lattice”. Krajci proposes that in general, three sets of truth degrees be considered, namely, L_X (set of objects), L_Y (set of attributes) and L table entries (degrees to which objects have attributes).

For X is the set of objects and Y is the set of attributes, a fuzzy context can be considered as a triple (X,Y,I), where I refer to L −relation between the sets X and Y, that means $I \in L^{X\times Y}$. Moreover, the author suggests that $L_X\ and\ L_Y$ are complete lattices and L is a partially ordered set. All partial orders on (L_X, L_Y, and L) are represented by $\leq$. To define arrow-operators, the author assumes that an operation exists: $\otimes$: $L_X \times L_Y \rightarrow L$ satisfies:

$$a_1 \leq a_2 \Rightarrow a_1 \otimes b < a_2 \otimes b, \tag{17}$$

$$b_1 \leq b_2 \Rightarrow a \otimes b_1 < a \otimes b_2, \tag{18}$$

$$\text{If } a_j \otimes b \leq c \text{ for each } j \in J \text{ then } \left(\textstyle\bigvee_{j\in J} a_j\right) \otimes b \leq c, \tag{19}$$

$$\text{If } a \otimes b_j \leq c \text{ for each } j \in J \text{ then } a \otimes \left(\textstyle\bigvee_{j\in J} b_j\right) \leq c, \tag{20}$$

This is for each index set J and for all $a, a_j \in L_X, b, b_j \in L_Y and\ c \in L$. In other words, we have a three-tiered structure of truth degrees $(L_1, L_2, L, \otimes, \leq, ...)$. Such a structure is known as Krajci's structure if it is satisfying (17) – (20).

Then, Krajci moves on to mappings the arrow-operations $\uparrow: L_X^X \rightarrow L_Y^Y\ and\ \downarrow: L_Y^Y \rightarrow L_X^X$ by

$$A^{\uparrow}(y) = \bigvee\{b \in L_Y \mid \forall\, x \in X: A(x) \otimes b \leq I(x,y)\} \tag{21}$$

$$B^{\downarrow}(x) = \bigvee\{a \in L_X \mid \forall\, y \in Y: a \otimes B(y) \leq I(x, y)\} \tag{22}$$

The formal concepts in (X, Y, I) are defined as pairs $(A, B) \in L_X^X \times L_Y^Y$ fulfilling $A^{\uparrow} = B, B^{\downarrow} = A$.

$\mathcal{B} = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$ (the set of all concepts) equipped with the partial order $\leq$ is a complete lattice (i.e., the generalized concept lattice for $\langle X, Y, I, \otimes\rangle$). (Krajci, 2004) provides a key theorem for a generalized concept lattice.

4.2. Issues with fuzzy concept lattices

The potential of vast number of concepts derived from the data can be a problem when utilizing fuzzy concept lattices. Many methods have been proposed to deal with such a situation. In (Belohlavek and Vychodil, 2005d), the authors suggested the first method of "Concept lattices and Galois connections with hedges" as a generalization of the two methods by Pollandt and Belohlávek "fuzzy concept lattices" and " crisply generated fuzzy concepts" , and then became more generalized in (Belohlavek and Vychodil, 2012). The method involves using two unary functions on $\boldsymbol{L}$, known as hedges (hedges are terms like "quite", " extremely", and "highly" concepts (Zadeh, 1972)) , as parameters which may be used in the form of FCA with fuzzy attributes to reduce (control) the amount of generated concepts. A hedge * on $\boldsymbol{L}$ is referred to the truth function of logical connective (very true). It is a unary function $*$ mapping on $\boldsymbol{L}$ fulfilling: (a)$1^* = 1$, (b) $a^* \leq a$, (c) $(a \to b)^* \leq a^* \to b^*$ (d) $a^{**} = a^*$, for each $a, b \in L$.

A triplet (X,Y,I) indicates a formal fuzzy context, where X be a set of objects, and Y is the set of attributes, I is a fuzzy relation between X and Y, $I: X \times Y \to L$ for each $x \in X\ and\ y \in Y$ assigns truth degree (to which objects have attributes). With two truth parameters $*x$, $*y$ (hedges) on $\boldsymbol{L}$. For a fuzzy set $A \in L^X$(set of objects) and $B \in L^Y$(set of attributes), consider $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ as a fuzzy set described by

$$A^{\uparrow}(y) = \bigwedge\nolimits_{x \in X} (A(x)^{*x} \to I(x, y)) \tag{23}$$

$$B^{\downarrow}(x) = \bigwedge\nolimits_{y \in Y} (B(y)^{*y} \to I(x, y)) \tag{24}$$

By putting $\mathcal{B}(X^{*x}, Y^{*y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$ is called a fuzzy concept lattice with hedges. For any pairs $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*x}, Y^{*y}, I)$, put $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle\ iff\ A_1 \subseteq A_2$ or $B_2 \subseteq B_1$. $\langle \mathcal{B}(X^{*x}, Y^{*y}, I), \leq \rangle$, including by (X, Y, I), it's indeed a complete lattice (Belohlavek and Vychodil, 2005d).

Noticed that the method of "fuzzy concept lattice" in (Ben Yahia and Jaoua, 2001) is isomorphic to $\mathcal{B}(X^{*x}, Y^{*y}, I)$ with $*x$ be an identity, $*y$ be a globalization. While, $*x$ be a globalization and $*y$ be an identity, $\mathcal{B}(X^{*x}, Y^{*y}, I)$ is will be isomorphic to the method of "one-sided fuzzy concept lattice" proposed by (Krajci, 2003).

Other methods proposed to reduce (control) the number of generated concepts by using parameters called thresholds, by putting threshold δ (an arbitrary truth degree of L, where we may get a set $\delta_{A\uparrow} = \{y \mid A^{\uparrow}(y) \geq \delta\}$ that only includes attributes that relate to $A^{\uparrow}$ with a degree $\geq \delta$. The methods (Ben Yahia and Jaoua, 2001; Krajci, 2003; Bělohlávek et al., 2005a; Belohlavek and Vychodil, 2005d) are obtained when δ equal to 1. In (Elloumi et al., 2004), the authors applied this form to arbitrary δ, where their (extent and intent) forming operators did not form a Galois connection. Later on this method was figured out in (Shao and Zhang, 2007) in their method the researchers suggested a new operators depending on the concept of (threshold) for general δ. The following is a brief description for their methods. For $A \in L^X, B \in L^Y, C \in 2^X, D \in 2^Y$, new additional three operators ((*, *), (□,□),(◇, ◇)) have been defined by the authors:

- $C^* = \{y \in Y \mid \bigwedge_{x \in X}\big(C(x) \to I(x,y)\big) \geq \delta\} \in 2^Y,\ D^* = \{x \in X \mid \bigwedge_{y \in Y}\big(D(y) \to I(x,y)\big) \geq \delta\} \in 2^X$
- $C\ \ (y) = (\delta \to \bigwedge_{x \in C} I(x,y)) \in L^Y, B\ \ = \{x \in X \mid \bigwedge_{y \in Y}(B(y) \to I(x,y)) \geq \delta\} \in 2^X$
- $A^{\diamond} = \{y \in Y \mid \bigwedge_{x \in X}\big(A(x) \to I(x,y)\big) \geq \delta\} \in 2^Y, D^{\diamond}(x) = \big(\delta \to \bigwedge_{y \in D} I(x,y)\big) \in L^Y$

For each x $\in X$ and y $\in Y$.

According to the authors, these operators form a Galois connection, leading to the concept lattices are shown below:

- $\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \in 2^X \times 2^Y \mid A^* = \mathrm{B}, B^* = \mathrm{A}\}$, it's equivalent to the ordinary concept lattice $\mathcal{B}\big(X,Y,{}^{\delta}I\big)$, ${}^{\delta}I = \{\langle x,y\rangle \in X \times Y \mid I(x,y) \geq \delta\}$.
- $\mathcal{B}(X_{\diamond},Y_{\diamond},I) = \{\langle A,B\rangle \in L^X \times 2^Y \mid A^{\diamond} = \mathrm{B}, B^{\diamond} = \mathrm{A}\}$, included by $(X,Y,\delta \to 1)$, referred to (one-sided fuzzy concept lattice) with fuzzy extents and crisp intents.
- $\mathcal{B}(X\ \ ,Y\ \ ,I) = \{\langle A,B\rangle \in 2^X \times L^Y \mid A\ \ = \mathrm{B}, B^{\diamond} = \mathrm{A}\}$, referred to (one-sided fuzzy concept lattice) with crisp extents and fuzzy intents.

When dealing with data that has fuzzy attributes, note that the fuzzy concept lattice generated by using Zhang's operators (threshold) in (Shao and Zhang, 2007) is isomorphic to the fuzzy concept lattice generated by using hedges as shown in (Bělohlávek et al., 2006).

Another possible issue may arise when using a fuzzy setting. When one deals with collections that are considered fuzzy sets and used a scale of membership degree like the unit interval [0, 1], It's possible that the set of each of these collections is infinite and computationally unsolvable.. The solution for such an issue was proposed in (Belohlavek et al., 2007). The authors discussed the issue of approximating potentially infinite sets of solutions by finite sets of solutions using scales of truth degrees that are discretized. In constraint-based problems like " "find all collections in a given finite universe satisfying constraint C," infinite sets of solutions often arise. The author suggested a solution for this problem by using a finite subset K of $[0,1[$, instead of $[0,1]$, which approximates [0,1] to a satisfactory degree and illustrates the concept on formal concept analysis.

4.3. Applications area

In this part, we will introduce a survey of the papers on using formal concept analysis with fuzzy logic in several domains like information retrieval, knowledge discovery in databases, data mining, and ontology engineering.

4.3.1. The utilization of fuzzy FCA in KDD (Knowledge Discovery in Databases) and DM (Data Mining)

KDD and DM (Information Discovery in Databases and Data Mining) are a multidisciplinary research field that focuses on techniques for extracting valuable knowledge from data. Sklenar et al. (2005) employed formal concept analysis to analyze physical activity data from epidemiological questionnaires to order to identify correlations among demographic data and physical activity levels. Later on, Belohlavek et al. (2007, 2011) expand the work of Sklenar et al. (2005), and Sigmund et al. (2005), where Participants are aggregated, and fuzzy values are used to indicate the relative intensity of attributes in the aggregated items. In (Bertaux et al., 2009), the authors present a framework for defining ecological features of organisms based on biological characteristics analysis. The dataset's complex structure is formally established as a fuzzy many-valued context, which is then converted into a binary context via histogram scaling. The method's structure was built on the creation and interpretation of formal concepts. A hydrobiologist interpreted the concepts, resulting in a set of ecological traits that were added into the initial context. (Fenza et al., 2008), the authors proposed a framework that supports the user in the exploration of semantic web resources by combining FCA with fuzzy attributes. This framework is split into two layers: lower and upper. The semantic representations of web services are converted into fuzzy multisets in the lower layer. This representation is (an OWL-S document) that illustrates the service's capability. The web services are clustered into fuzzy clusters using fuzzy C-Means clustering. Fuzzy matchmaking has been used to find services which are close matches to the input request. A fuzzy formal context has been used in the upper layer to describe prototypes and assigned ontological concepts that are present or not.

4.3.2. The utilization of fuzzy FCA in IR (Information retrieval)

Information retrieval is described as finding content (probably documents) of an unstructured format (probably text) which meets an information need out of large collections (Probably, data saved on machines.) (Manning et al., 2008). Supporting users in searching or filtering document types, as well as sorting a series of collected documents, is used in the field of information retrieval. Clustering is the process of creating a successful grouping of documents depending on its contents. When given a collection of topics, classification is the process of defining which categories each of a collection of documents applies to.

Several articles have been suggested using FCA with fuzzy logic in information retrieval. In a citation-database document retrieval system. (Quan et al., 2004), the authors used formal concept analysis with fuzzy attributes for conceptual clustering. A fuzzy concept lattice is built using fuzzy logic and FCA, on which “a fuzzy conceptual clustering technique “is performed. The retrieval of documents will then be done using fuzzy queries. (Butka et al., 2008), the authors defined a framework for building an ontology using FCA and fuzzy attributes. Using the “Growing Hierarchical Self Organizing Map clustering algorithm”, the initial set of documents is decomposed into smaller groups of related documents. Using “agglomerative clustering”, the models are combined into a hierarchy of concept lattices. To cope with null answers for fuzzy queries, (Chettaoui et al., 2008), the authors used FCA with fuzzy attributes. By presenting the subqueries which are liable for the error, fuzzy querying processing based on Galois lattices allows detecting the explanations for empty results. For query expansion, (Zhang et al., 2008) suggested a system based on FCA and fuzzy attributes.

4.3.3. The utilization of fuzzy FCA in ontology engineering

As a way of formally expressing knowledge, ontologies were created. Their goal is to simulate a similar perception of truth as shown by some individuals to enable knowledge-intensive applications (Liu and Özsu, 2009). In general, the ontology includes individuals or objects, classes, characteristics, relationships among individuals and classes, terms of function, rules, axioms, and activities. Ontologies are usually encoded in ontology languages like ontology (OWL). In comparison, ontologies also use trees to model the universe.

(Quan et al., 2004), the authors employ FCA into fuzzy logic to generate ontologies automatically. The generated ontologies are employed to serve the Scholarly Semantic Web, which is used to share, reuse, and manage scholarly information. Quan et al. (2006) suggest an approach called "Fuzzy Ontology Generation System" based on using FCA into fuzzy logic for automatically creating an ontology. Later on, (Quan et al., 2006) the authors use this approach to construct an ontology that can be used in "web-based help-desk applications". FCA is combined with fuzzy logic by (C De Maio et al., 2009, Carmen De Maio et al., 2012)to construct an ontology for automatically classifying RSS feeds.

4.4. Open research directions

FCA has grown into a collection of reliable methodologies, software tools and algorithms for exploring concepts, arranging them into lattices, scaling them, splitting, and visualizing them as line graphs. still has open issues that must be tackled. Data or context must be given in discrete form for FCA tools and techniques to work. One of the future research directions in this topic would be to integrate interpretation algorithms into FCA instruments so that data given in different formats, such as continuous text, expressions, and so on, could be analyzed for concepts.

Another open topic in FCA is the scalability of the concept lattice. For a limited and tiny context, current techniques can be used to easily visualize and generate the concept lattice. However, as the context grows larger, this becomes a problem. The time and iterations required for concept generation algorithms grow exponentially with each iteration. Another consequence of the concept lattice is that nodes and edges begin to collide as the lattice expands in size. The visualization of formal concepts in their hierarchical order within the concept lattice structure is important for FCA applications. The size of the concept lattice created from "a huge formal context" is a significant concern in this approach. The concept lattice, which is derived from a large context, becomes complicated and impracticable. As a result, in practical FCA implementations, dealing with a large formal context and decreasing the concept lattice are tackled as extreme challenges (Aswani Kumar et al., 2015). We intend to host the references and links to the articles on a public interface in the future, and we hope that this summary will guide both researchers and practitioners to new advanced FCA possibilities.

5. Conclusion

FCA has been a well-known tool in computer science since its release in 1982 as a mathematical model. Many articles on FCA have been presented in recent years, and most of them included case studies explaining the method's effectiveness in real-world situations. Formal concept analysis (FCA) has been expanded to cope with ambiguous and imperfect data, integration of FCA with fuzzy logic has gained a lot of coverage in the literature. In this paper, we attempted to summarize the literature on integrating FCA with fuzzy logic. For integrating FCA with fuzzy logic, the authors were primarily concerned with describing more theoretical models like one-sided fuzzy concept lattice and crisply generated Fuzzy concepts, developing strategies to control or minimize the number of generated concepts, and using FCA with fuzzy logic in disciplines like knowledge discovery in databases, data mining, information retrieval, and ontology engineering.

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