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ON ACCUMULATION POINTS OF GENERALIZED RATIO SETS OF POSITIVE INTEGERS

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Dedicated to the memory of Professor Péter Kiss

Abstract. The paper deals with a generalized ratio set of positive integers defined as

$$R_n(A)=\{a_1a_2\dots a_n/(b_1b_2\dots b_n);a_1,a_2,\dots,a_n,b_1,b_2,\dots,b_n\in A\},\qquad \text{where } A\subset\mathbf{N}.$$

There are characterized the accumulation points of $R_n(A)$. Further it is proved that if $A\subset\mathbf{N}$ has positive lower asymptotic density then for sufficiently large positive integer n the set $R_n(A)$ is dense in $\mathbf{R}^+$.

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1. Introduction

Denote by $\mathbf{R}$ ($\mathbf{R}^+$) the set of all real (positive real) numbers and by $\mathbf{N}$ the set of all positive integer numbers, respectively. The *ratio set* of $A \subset \mathbf{N}$ is denoted by $R(A) = \{\frac{a}{b}; a, b \in A\}$ (see [3], [5]). The symbol X^d will stand for the set of all accumulation points of $X \subset \mathbf{R}^+$. It is easy to see that for any infinite subset A of positive integers $\{0, +\infty\} \subset R(A)^d$. The set $R(A)$ is everywhere dense in $\mathbf{R}^+$ if $R(A)^d = [0, +\infty]$.

It is known that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ for the set $A = \{a_1 < a_2 < \cdots\} \subset \mathbf{N}$ then $R(A)$ is dense in $\mathbf{R}^+$ [5], on the other hand if $\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} = c > 1$ then $R(A)$ is not dense in $\mathbf{R}^+$, moreover $R(A)^d \cap \left(\frac{1}{c}, c\right) = \emptyset$ [6].

The lower and upper asymptotic density of A, denoted by $\underline{d}(A)$ and $\overline{d}(A)$ respectively, are defined as

$$\underline{d}(A) = \underline{\lim}_{x\to\infty} \frac{A(x)}{x}, \qquad \overline{d}(A) = \overline{\lim}_{x\to\infty} \frac{A(x)}{x},$$

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where $A(x) = \#\{a \le x : a \in A\}$. If $\underline{d}(A) = \overline{d}(A) = d(A)$ then the number $d(A)$ is called the asymptotic density of the set A.

We mention some known results on the topics density of ratio sets. Šalát [5] showed that $\underline{d}(A) = \overline{d}(A) > 0$ or $\overline{d}(A) = 1$ implies that $R(A)$ is everywhere dense in $\mathbf{R}^+$ and for every sufficiently small $\varepsilon > 0$ there exists a subset of $A \subset \mathbf{N}$ such that $\overline{d}(A) = 1 - \varepsilon$ and $R(A)$ is not everywhere dense in $\mathbf{R}^+$. He gave an example of $A \subset \mathbf{N}$ for which $\underline{d}(A) = \frac{1}{4}$ and $R(A)$ is not everywhere dense in $\mathbf{R}^+$. Strauch and Tóth [4] proved that $\frac{1}{2}$ is the lower bound of γ's for which $\underline{d}(A) \ge \gamma$ implies that $R(A)$ is everywhere dense in $\mathbf{R}^+$.

We define the *generalized ratio set*

$$R_n(A) = \{\frac{a_1 a_2 \dots a_n}{b_1 b_2 \dots b_n};\ a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A\}.$$

Clearly, $R_1(A) = R(A)$ and $R_n(A) \subset R_m(A)$ for $m \ge n$.

In [2] was asked: For which sets $B \subset \mathbf{R}$ does there exist a set $A \subset \mathbf{N}$ such that $R(A)^d = B$? It is evident that $B \ne \emptyset$ provided A is infinite. On the other hand, $\{0, +\infty\} \subset R(A)^d$ for any infinite $A \subset \mathbf{N}$. Further, if some positive $t \in R(A)^d$, then $\frac{1}{t} \in R(A)^d$, since $\frac{a}{b} \in R(A)$ always implies that $\frac{b}{a} \in R(A)$. Notice also, that the accumulation points of any linear set constitute a closed set in $\mathbf{R}$. Consequently, the nonempty set B must be a closed subset of $[0, +\infty] = \mathbf{R}^+ \cup \{0, +\infty\}$, it must contain 0 and $+\infty$, and if $b \in B$ ($b \in \mathbf{R}^+$) then $\frac{1}{b} \in B$. In [1] was proved that these conditions are also sufficient for the existence of an $A \subset \mathbf{N}$ for that $R(A)^d = B$. We show that the same assertion is valid if we consider the generalized ratio set $R_n(A)$ instead of the ratio set $R(A)$.

2. Theorems and proofs

Theorem 1. *Let $\emptyset \ne B \subset [0, +\infty]$ and n be a positive integer. The followings are equivalent:*

(i) There exists an $A \subset \mathbf{N}$ such that $R_n(A)^d = B$;

(ii) $B \cap \mathbf{R}$ is closed in $\mathbf{R}$, $\{0, +\infty\} \subset B$ and $b \in B$ implies $\frac{1}{b} \in B$.

Proof. As the implication (i) $\Rightarrow$ (ii) is trivial it suffices to prove only (ii) $\Rightarrow$ (i). The case $n = 1$ was considered in [1]. Let us suppose that $n > 1$ and suppose $\emptyset \ne B \subset [0, +\infty]$ satisfies (ii). Let $\mathcal{S}$ stand for the system of intervals $(1 + \frac{i-1}{n}, 1 + \frac{i+1}{n})$ where $n \in \mathbf{N}$ and $i = 1, 2, \dots, n^2$. The length of intervals tends to zero with increasing n and every real number greater than 1 can be covered with infinitely many elements of $\mathcal{S}$. Denote by $((c_k - \delta_k, c_k + \delta_k))_{k=1}^{\infty}$ the sequence of those intervals from $\mathcal{S}$ which meet B (i.e. which contain at least one element from B).

Define the set $A = \{a_0 < a_1 < a_2 < \dots\} \subset \mathbf{N}$ as follows:
Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, further

$$a_{3k} = \lfloor (a_{3k-1})^{n^2}.(c_k+1) \rfloor, a_{3k+1} = (a_{3k})^{n^2}, a_{3k+2} = \left\lceil \frac{c_k.(a_{3k+1})^n}{(a_{3k})^{n-1}} \right\rceil \text{for } k = 1, 2, \dots$$

We will show that $R_n(A)^d = B$.

(1) $B \subset R_n(A)^d$: Let $t \in B$ be a positive real number. We may suppose that $t > 1$. Let $((c_{m_k} - \delta_{m_k}, c_{m_k} + \delta_{m_k}))_{k=1}^{\infty}$ be a sequence of intervals containing t. Then $\lim_{k\to\infty} c_{m_k} = t$ since $\lim_{k\to\infty} \delta_{m_k} = 0$. Accordingly the sequence

$$(1) \qquad \frac{a_{3m_k+2} \cdot (a_{3m_k})^{n-1}}{(a_{3m_k+1})^n} = \left\lceil \frac{c_{m_k} \cdot (a_{3m_k+1})^n}{(a_{3m_k})^{n-1}} \right\rceil \cdot \frac{(a_{3m_k})^{n-1}}{(a_{3m_k+1})^n} \qquad (k = 1, 2, \dots)$$

converges to t; thus, $t \in R(A)^d$.

(2) $R_n(A)^d \subset B$: Let us consider the fraction

$$r = \frac{a_{i_1} a_{i_2} \dots a_{i_m}}{a_{j_1} a_{j_2} \dots a_{j_m}} \in R_n(A),$$

where $m \le n$, $a_{i_1}, \dots, a_{i_m}, a_{j_1}, \dots, a_{j_m} \in A$ further $a_{i_1} \ge a_{i_2} \ge \dots \ge a_{i_m}$, $a_{i_1} > a_{j_1} \ge a_{j_2} \ge \dots \ge a_{j_m}$ and the fraction r cannot be simplified. Our aim is to show that only a sequence like (1) from $R_n(A)$ can have finite limit. To prove this we consider the following possibilities:

(a) $i_1 = 3k$ or $i_1 = 3k+1$. In this case we have

$$r \ge \frac{a_{i_1}}{(a_{i_1-1})^n} \ge (a_{i_1-1})^{n^2-n}.$$

(b) $m < n$, $i_1 = 3k+2$, $j_1, \dots, j_m \le 3k+1$ or $m = n$, $j_1, \dots, j_{n-1} \le 3k+1$, $j_n \le 3k$. Now we have

$$r \ge \frac{a_{3k+2}}{(a_{3k+1})^{n-1} \cdot a_{3k}} = \frac{\left\lceil \frac{c_k \cdot (a_{3k+1})^n}{(a_{3k})^{n-1}} \right\rceil}{(a_{3k+1})^{n-1} \cdot a_{3k}} \ge \frac{a_{3k+1}}{(a_{3k})^n} = (a_{3k})^{n^2-n}.$$

(c) $i_1 = 3k+2$, $i_2, \dots, i_{n-1} \le 3k$, $i_n \le 3k-1$, $j_1 = j_2 = \dots = j_n = 3k+1$. Then we have the following estimation

$$r \le \frac{a_{3k+2} \cdot (a_{3k})^{n-2} \cdot a_{3k-1}}{(a_{3k+1})^n} < \frac{(c_k+1) \cdot (a_{3k+1})^n}{(a_{3k})^{n-1}} \cdot \frac{(a_{3k})^{n-2} \cdot a_{3k-1}}{(a_{3k+1})^n} \cdot \frac{(c_k+1) \cdot a_{3k-1}}{a_{3k}}$$

$$\leq (a_{3k-1})^{1-n^2}.$$

The last case we have to consider is related to (1)

(d) $i_1 = 3k+2$, $i_2 = \cdots = i_n = 3k$, $j_1 = \cdots = j_n = 3k+1$.

Let now $t \in R(A)^d$ and $t > 1$. Then there exist sequences $(s_{i,l})_{l=1}^{\infty}$ and $(r_{i,l})_{l=1}^{\infty}$, $i = 1, 2, \ldots, n$ of positive integers such that

$$\lim_{l\to\infty} \frac{a_{s_{1,l}} \cdot a_{s_{2,l}} \cdots a_{s_{n,l}}}{a_{r_{1,l}} \cdot a_{r_{2,l}} \cdots a_{r_{n,l}}} = t. \tag{2}$$

Observe that if the fractions in (2) are of the form (a) and (b) then their limit is $+\infty$ and if these fractions are of the form (c) then their limit is 0. So (2) can hold only if for sufficiently large numbers we have the case (d). Therefore for some subsequence $(m_k)_{k=1}^{\infty}$ of positive integers we have

$$\lim_{k\to+\infty} c_{m_k} = t.$$

Taking into account that every interval $(c_{m_k} - \delta_{m_k}, c_{m_k} + \delta_{m_k})$ contains some $t_k \in B$, therefore $\lim\limits_{k\to\infty} t_k = t$. Finally, the closedness of $B \cap \mathbf{R}$ in $\mathbf{R}$ ensures that $t \in B$.

Remark. As a consequence of the theorem we immediately have that for each $n \geq 1$ there exists a set $A \subset \mathbf{N}$ such that $R_n(A)$ is not dense in $\mathbf{R}^+$, but R_{n+1} is already dense in $\mathbf{R}^+$. Indeed, there is a set A such that the set of all accumulation points of $R_n(A)$ is equal to $B = \{n, \frac{1}{n}; n = 1, 2, \ldots\}$. Obviously, then $R_{n+1}(A)$ is dense in $\mathbf{R}^+$.

Strauch and Tóth [4] have proved that for any $A \subset \mathbf{N}$ and the interval (α, β), $0 \leq \alpha < \beta \leq 1$ if $(\alpha, \beta) \cap R(A) = \emptyset$ then $\overline{d}(A) \leq 1 - (\beta - \alpha)$. The following lemma generalizes this result and it is basic for the proof of the theorem below.

Lemma. *Let $A \subset \mathbf{N}$ and the pairwise disjoint intervals (α_i, β_i), $0 \leq \alpha_i < \beta_i \leq 1$ are such that $(\alpha_i, \beta_i) \cap R(A) = \emptyset$, $i = 1, 2, \ldots, m$. Then*

$$\overline{d}(A) \leq 1 - \sum_{i=1}^{m} (\beta_i - \alpha_i)$$

Proof. In the cases $\overline{d}(A) = 0$ or $\overline{d}(A) = 1$ the assertion is trivial (it was proved by Šalát [5] that $\overline{d}(A) = 1$ implies that $R(A)$ is everywhere dense in $\mathbf{R}^+$), so we can suppose that the set A is infinite and A has infinite complement in $\mathbf{N}$. Thus A can be expressed as the set of integer points lying in the intervals

$$[b_1, c_1],\ [b_2, c_2], \ldots, [b_n, c_n], \ldots,$$

whose endpoints are ordered as

$$b_1 \le c_1 < b_2 \le c_2 < \cdots < b_n \le c_n < \cdots$$

Obviously,

$$\overline{d}(A) = \overline{\lim}_{n\to+\infty} \frac{1}{c_n} \sum_{i=1}^{n} (c_i - b_i + 1).$$

Let us consider the fractions $\frac{a}{c_n}$, where $a \in A$, $a \le c_n$. All these fractions are contained in the union of the intervals

$$\left[\frac{b_1}{c_n}, \frac{c_1}{c_n}\right], \ \left[\frac{b_2}{c_n}, \frac{c_2}{c_n}\right], \dots, \left[\frac{b_n}{c_n}, \frac{c_n}{c_n}\right]. \tag{3}$$

The distance of any two neighbouring fractions lying in the same interval of (3) is $\frac{1}{b_n} \to 0$ as $n \to +\infty$. Therefore, for sufficiently large n, each interval $(\alpha_i, \beta_i) \subset [0,1]$, $i = 1, 2, \dots, m$ must lie in the complement of

$$\left[\frac{b_k}{c_n}, \frac{c_k}{c_n}\right], \quad k = 1, 2, \dots, n.$$

This complement is formed by the pairwise disjoint intervals

$$\left(\frac{c_k}{c_n}, \frac{b_{k+1}}{c_n}\right), \quad k = 1, 2, \dots, n-1.$$

Hence

$$\cup_{i=1}^{m} (\alpha_i, \beta_i) \subset \cup_{k=1}^{n} \left(\frac{c_k}{c_n}, \frac{b_{k+1}}{c_n}\right)$$

and therefore

$$\sum_{i=1}^{m} (\beta_i - \alpha_i) \le \sum_{k=1}^{n} \frac{b_{k+1} - c_k}{c_n}.$$

The upper asymptotic density of the set A we can write as

$$\overline{d}(A) = \overline{\lim}_{n\to+\infty} \left(\frac{c_n - b_1}{c_n} + \frac{n}{c_n} - \frac{1}{c_n} \left[(b_2 - c_1) + (b_3 - c_2) + \cdots + (b_n - c_{n-1}) \right] \right)$$

whence

$$\overline{d}(A) - \overline{d}(C) \le 1 - \sum_{i=1}^{m} (\beta_i - \alpha_i),$$

where C is the range of c_n. Now, for a positive integer t, transform $[b_n, c_n] \to [tb_n, tc_n + t - 1]$ and denote by A_t the set of all integer points lying in $[tb_n, tc_n + t - 1]$,

$n = 1, 2, \ldots$. Analogously, C_t is the set of all $tc_n + t - 1$. Then we have $\overline{d}(A_t) = \overline{d}(A)$ and $\overline{d}(C_t) = \overline{d}(C)/t$, which gives

$$\overline{d}(A) - \frac{\overline{d}(C)}{t} \le 1 - \sum_{i=1}^{m} (\beta_i - \alpha_i)$$

and the assertion of the lemma follows.

Theorem 2. *For arbitrary $A = \{a_1 < a_2 < \cdots\} \subset \mathbf{N}$ having positive lower asymptotic density ($\underline{d}(A) > 0$) there exists a positive integer n such that the set $R_n(A)$ is dense in $\mathbf{R}^+$.*

Proof. First, we claim that $\underline{d}(A) > 0$ implies that for some interval $[\gamma, \delta]$, $1 \le \gamma < \delta$ the set $R(A)$ is dense in $[\gamma, \delta]$. Indeed, if such interval $\gamma, \delta]$ does not exist, then there exist pairwise disjoint intervals (α_i, β_i), $0 \le \alpha_i < \beta_i \le 1$ such that $(\alpha_i, \beta_i) \cap R(A) = \emptyset$, $i = 1, 2, \ldots, m$ and the sum of the length of these intervals can be arbitrary near to 1, i.e.

$$\sum_{i=1}^{m} (\beta_i - \alpha_i) > 1 - \underline{d}(A)$$

which is a contradiction with the lemma.

From the condition $\underline{d}(A) > 0$ follows that for sufficiently large K we have

$$\frac{a_{k+1}}{a_k} < K, \quad k = 1, 2, \ldots$$

If $R(A)$ is dense in $[\gamma, \delta]$, $(1 \le \gamma < \delta)$ then $R_2(A)$ is dense in $[1, \frac{\delta}{\gamma}]$ and $R_4(A)$ is dense in $[1, (\frac{\delta}{\gamma})^2], \ldots$. To see this, we remark that

$$R_{2^{n+1}}(A) = R(R_{2^n}(A)), \quad n = 1, 2, \ldots$$

Evidently $(\frac{\delta}{\gamma})^n \to +\infty$ for $n \to +\infty$, therefore for sufficiently large n we have that $R_{n-1}(A)$ is dense in $[1, K]$. Using this fact we have that the set

$$\left\{ t \cdot \frac{a_k}{a_1};\ t \in R_{n-1}(A) \right\} \subset R_n(A)$$

is dense in each $[\frac{a_k}{a_1}, \frac{a_{k+1}}{a_1}]$, $k = 1, 2, \ldots$, hence $R_n(A)$ is dense in $\mathbf{R}^+$.

To conclude this paper, let us describe some open problems associated with this topic.

Let $\gamma(n)$ be the least value of γ for which $\underline{d}(A) \ge \gamma$ implies that $R_n(A)$ is dense in $\mathbf{R}^+$, $n = 1, 2, \ldots$. It is known that $\gamma(1) = 1/2$. Determine the exact value of $\gamma(2)$. What can be said about the function $\gamma(n)$?

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