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ON THE L^1 NORM OF THE WEIGHTED MAXIMAL FUNCTION OF THE WALSH–KACZMARZ–DIRICHLET KERNELS

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Dedicated to the memory of Professor Péter Kiss

Abstract. In this paper we investigate the integral of the weighted maximal function of the Walsh–Paley–Dirichlet, and the Walsh–Kaczmarz–Dirichlet kernels. We find necessary and sufficient conditions for the finiteness of the integrals. The conditions are quite different for the two rearrangements of the Walsh system.

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1. Introduction

The Walsh system in the Kaczmarz enumeration was studied by a lot of authors (see [4], [5], [8], [7], [1], [6], [9]). In [2] it has been pointed out that the behavior of the Dirichlet kernel of the Walsh–Kaczmarz system is worse than of the kernel of the Walsh–Paley system considered more often. Namely, it is proved [2]that for the Dirichlet kernel $D_n(x)$ of the Walsh–Kaczmarz system the inequality $\limsup_{n\to\infty} \frac{|D_n(x)|}{\log n} \geq C > 0$ holds a.e. This "spreadness" of this system makes easier to construct examples of divergent Fourier series [1].

A number of pathological properties is due to this "spreadness" property of the kernel. For example, for Fourier series with respect to the Walsh–Kaczmarz system it is impossible to establish any local test for convergence at a point or on an interval, since the principle of localization does not hold for this system.

On the other hand, the global behavior of the Fourier series with respect to this system is similar in many aspects to the case of the Walsh–Paley system. Schipp [5] and Wo–Sang Young [9] proved that the Walsh–Kaczmarz system is a convergence system. Skvorcov [8] verified the everywhere (and uniform) convergence of the Fejér

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means of continuous functions, and Gát proved [3] that the Fejér–Lebesgue theorem also holds for the Walsh–Kaczmarz system.

Beyond the convergence theorems of the Fourier series one can often find some boundedness properties of the Dirichlet kernel functions. For instance, for the Walsh–Paley system we have $\sup_{n\in\mathbb{N}} |D_n(x)| < \infty$ for each $x \neq 0$. This —as we have seen above— is not the case for the Kaczmarz rearrangement. What can be said for the norm of maximal functions? It is easy to have that the L^1 norm of $\sup_{n\in\mathbb{N}} |D_n|$ with respect to both systems is infinite. What happens if we apply some weight function α? That is, on what conditions find we the inequality

$$\left\| \sup_{n\in\mathbb{N}} \left| \frac{D_n}{\alpha(n)} \right| \right\|_1 < \infty$$

valid? The aim of this paper is to find the necessary and sufficient conditions for the both rearrangement of the Walsh system.

Let P denote the set of positive integers, $N := P \cup \{0\}$ the set of nonnegative integers and Z_2 the discrete cyclic group of order 2, respectively. That is, $Z_2 = \{0, 1\}$ the group operation is the mod 2 addition and every subset is open. Haar measure is given in a way that the measure of a singleton is 1/2. Set

$$G := \times_{\infty}^{k=0} Z_2$$

the complete direct product. Thus, every $x \in G$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in \{0, 1\}\,(i \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, (which is the so-called logical addition) the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. Set $e_i := (0, 0, \ldots, 1, 0, 0, \ldots) \in G$ the i-th coordinate of which is 1, the rest are zeros.

A base for the neighborhoods of G can be given as follows

$$I_0(x) := G, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G : y_i = x_i \text{ for } i < n\}$$

for $x \in G, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G$ denote the nullelement of $G, I_n := I_n(0)\,(n \in \mathbb{N})$. Let $\mathcal{I} := \{I_n(x) : x \in G, n \in \mathbb{N}\}$. The elements of $\mathcal{I}$ are called the dyadic intervals on G. Furthermore, let $L^p(G)\,(1 \leq p \leq \infty)$ denote the usual Lebesgue spaces ($|\,.\,|_p$ the corresponding norms) on G, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbb{N})\,(f \in L^1)$.

Let $n \in \mathbb{N}$. Then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}\,(n \in \mathbb{N})$, i.e. n is expressed in the number system based 2. Denote by $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. The Rademacher functions are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in G,\, n \in \mathbb{N}).$$

The Walsh–Paley system is defined as the sequence of the Walsh–Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty}(r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}, \quad (x \in G,\, n \in \mathbb{N}).$$

That is, $\omega := (\omega_n, n \in \mathbb{N})$. The n-th Walsh–Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} \left(r_{|n|-1-k}(x)\right)^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

for $n \in \mathbb{P}$, $\kappa_0(x) := 1, x \in G$. The Walsh–Kaczmarz system $\kappa := (\kappa_n, n \in \mathbb{N})$ can be obtained from the Walsh–Paley system by renumbering the functions within the dyadic "block" with indices from the segment $[2^n, 2^{n+1} - 1]$. That is, $\{\kappa_n \colon 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}$ for all $k \in \mathbb{N}$, $\kappa_0 = \omega_0$.

By means of the transformation $\tau_A \colon G \to G$

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots) \in G,$$

which is clearly measure-preserving and such that $\tau_A(\tau_A(x)) = x$ we have

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)) \quad (n \in \mathbb{N}).$$

Let us consider the Dirichlet kernel functions:

$$D_n^{\phi} := \sum_{k=0}^{n-1} \phi_k,$$

where ϕ is either κ or ω and $n \in \mathbb{P}$.

Let function $\alpha \colon [0, +\infty) \to [1, +\infty)$ be monotone increasing, and define the weighted maximal function of the Dirichlet kernels:

$$D_{\alpha}^{\phi}(x) := \sup_{n \in \mathbb{N}} \frac{|D_n^{\phi}(x)|}{\alpha(\lfloor \log n \rfloor)} \quad (x \in G),$$

where ϕ is either the Walsh–Paley, or the Walsh–Kaczmarz system. If it does not cause confusion the notation ϕ is omitted. First we discuss the Walsh–Paley case.

Proposition 1. *$D_{\alpha}^{\omega} \in L^1$ if and only if $\sum_{A=0}^{\infty} \frac{1}{\alpha(A)} < \infty$. Moreover,*

$$\frac{1}{2} \sum_{A=0}^{\infty} \frac{1}{\alpha(A)} \leq \| D_{\alpha}^{\omega} \|_1 \leq 2 \sum_{A=0}^{\infty} \frac{1}{\alpha(A)}.$$

Proof. In [6] one can read that for arbitrary $x \in I_A \setminus I_{A+1}$, and $A \in \mathbb{N}$ the inequality

$$|D_n(x)| \leq \min\{n, 2^A\}.$$

This immediately follows

$$D_\alpha(x) \leq 2 \sum_{k=0}^{A} \frac{2^k}{\alpha(k)}.$$

That is,

$$\begin{aligned} \| D_\alpha \|_1 &= \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} D_\alpha(x) d\mu(x) \\ &\leq 2 \sum_{A=0}^{\infty} \frac{1}{2^{A+1}} \sum_{k=0}^{A} \frac{2^k}{\alpha(k)} \\ &= \sum_{k=0}^{\infty} \sum_{A=k}^{\infty} \frac{1}{2^A} \frac{2^k}{\alpha(k)} \\ &\leq 2 \sum_{k=0}^{\infty} \frac{1}{\alpha(k).} \end{aligned}$$

That is, we have proved that $(1/\alpha(n)) \in l^1$ implies $D_\alpha \in L^1$. On the other hand, in the same way as above we have

$$\begin{aligned} | D_\alpha |_1 &= \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} D_\alpha(x) d\mu(x) \\ &\geq \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} \frac{D_{2^A}(x)}{\alpha(A)} d\mu(x) \\ &= \sum_{A=0}^{\infty} \frac{1}{2^{A+1}} \frac{2^A}{\alpha(A)}. \end{aligned}$$

In the case of the Walsh–Kaczmarz system the situation changes. Namely, we prove the following two propositions:

Proposition 2. *If* $\sum_{A=1}^{\infty} \frac{A}{\alpha(A)} < \infty$, *then* $D_\alpha^\kappa \in L^1$. *Moreover,* $\| D_\alpha^\kappa \|_1 \leq 4 \sum_{A=1}^{\infty} \frac{A}{\alpha(A)} + C$, *where* C *is some constant, such that may depend on* α *(but anyway it is a finite real).*

Proposition 3. *There exists a positive constant* C *(which may depend on* α*) such that*

$$| D_\alpha^\kappa |_1 \geq \frac{1}{25} \sum_{A=1}^{\infty} \frac{A}{\alpha(A)} - C.$$

These propositions give

Corollary 4. *$D_\alpha^\kappa \in L^1$ if and only if $\sum_{A=1}^\infty \frac{A}{\alpha(A)} < \infty$.*

That is, in the case of the Kaczmarz rearrangement we have to divide by a "greater" weight function α if we want the maximal function D_α^κ to be integrable. Besides, by the method of the proof of Proposition 3 one can prove that if $D_\alpha^\kappa \notin L^1$ (that is, $\sum_{A=1}^\infty \frac{A}{\alpha(A)} = \infty$), then it is not integrable on any dyadic interval. This is quite different in the Walsh–Paley case. Since for this system even the maximal function $\sup_n |D_n^\omega|$ is bounded by 2^A on $G \setminus I_A$. In order to prove Proposition 2 we use to following lemma. Let

$$L_\alpha(x) := \sup\{\frac{D_{2^j}(\tau_A(x))}{\alpha(A)} : j \le A, j, A \in \mathbb{N}\}, \quad x \in G.$$

Lemma 5. *We prove $\| L_\alpha \|_1 \le 2\sum_{A=1}^\infty \frac{A}{\alpha(A)} + C$.*

Proof.

$$| L_\alpha |_1 \le \sum_{A=0}^\infty \sum_{j=0}^A \frac{| D_{2^j} \circ \tau_A |_1}{\alpha(A)} = \sum_{A=1}^\infty \frac{A+1}{\alpha(A)} + C.$$

Proof of Proposition 2. It is known ([6]) that for $1 \le n \in \mathbb{N}$

$$D_n^\kappa = D_{2^{|n|}} + r_{|n|} D^\omega_{n-2^{|n|}} \circ \tau_{|n|}.$$

Since in [6] one can find the inequality

$$|D_n^\omega(x)| \le 2^j = D_{2^j}(x)$$

for any $x \in I_j \setminus I_{j+1}$, then

$$\sup_{|n|=A} |D_n^\kappa(x)| \le D_{2^A}(x) + \sup_{|n|<A} |D_n^\omega(\tau_A(x))| \le D_{2^A}(x) + \sup\{D_{2^j}(\tau_A(x)) : j < A\}.$$

This gives

$$D_\alpha^\kappa(x) = \sup_A \sup_{|n|=A} \frac{|D_n^\kappa(x)|}{\alpha(A)} \le \sup_A \frac{|D_{2^A}(x)|}{\alpha(A)} + L_\alpha(x) \le D_\alpha^\omega(x) + L_\alpha(x).$$

By Proposition 1 we have

$$| D_\alpha^\omega |_1 \le 2\sum_{A=1}^\infty \frac{A}{\alpha(A)} + C,$$

and by Lemma 5, that is, by

$$| L_\alpha |_1 \leq 2 \sum_{A=1}^{\infty} \frac{A}{\alpha(A)} + C$$

the proof of the inequality

$$| D_\alpha^\kappa |_1 \leq 4 \sum_{A=1}^{\infty} \frac{A}{\alpha(A)} + C,$$

that is, the proof of Proposition 2 is complete.

Proof of Proposition 3. Introduce the following notations:

$$L_{\alpha,N} := \sup_{\substack{A \leq N \\ j \leq A}} \left| \frac{D_{2^j} \circ \tau_A}{\alpha(A)} \right|, \quad a_N := | L_{\alpha,N} |_1 \ \ (N \in \mathbb{N}).$$

First, we prove that

$$a_N \leq CN^2. \tag{1}$$

This in equality can be proved in the following way.

$$a_N \leq \sum_{A=0}^{N} \sum_{j=0}^{A} \left\| \frac{D_{2^j} \circ \tau_A}{\alpha(0)} \right\|_1 \leq \sum_{A=0}^{N} \sum_{j=0}^{A} C \leq CN^2.$$

Next, for $N \in \mathbb{N}$, and $k \in \mathbb{N}, 1 \leq k$ denote by $J_{N,k}$ the following subset of G.

$$J_{N,k} := \begin{cases} \{x \in G : x_{N-k} = 1, x_{N-k+1} = \cdots = x_{N-1} = 0\} & \text{if } N \geq k \geq 2, \\ \{x \in G : x_{N-1} = 1\} & \text{if } N \geq k = 1. \end{cases}$$

Since for fixed N the sets$J_{N,k}$, I_N are disjoint, and $\cup_{k=1}^{N} J_{N,k} \cup I_N = G$, then we have

$$a_N = \sum_{k=1}^{N} \int_{J_{N,k}} L_{\alpha,N}\, d\mu + \int_{I_N} L_{\alpha,N}\, d\mu. \tag{2}$$

We give another upper bound for a_N, a different one from the inequality (1). Investigate the function $L_{\alpha,N}$ on the set $J_{N,k}$.
If $A = N$, then for $y = \tau_A(x)$ we have $y_0 = \cdots = y_{k-2} = 0, y_{k-1} = 1$.
Thus, $\sup_{j \leq A} D_{2^j}(\tau_A(x))/\alpha(A) = 2^{k-1}/\alpha(N)$.
For $A = N - 1$ we have $\sup_{j \leq A} D_{2^j}(\tau_A(x))/\alpha(A) = 2^{k-2}/\alpha(N-1)$.

And so on ...
Finally, if $A = N-k+1$ we have $\sup_{j\le A} D_{2^j}(\tau_A(x))/\alpha(A) = 1/\alpha(N-k+1)$.
That is, for $x \in J_{N,k}$

$$\sup_{\substack{N-k<A\le N\\ j\le A}} \left|\frac{D_{2^j}\circ\tau_A(x)}{\alpha(A)}\right| = \max\left\{\frac{2^{k-1}}{\alpha(N)}, \frac{2^{k-2}}{\alpha(N-1)}, \dots, \frac{1}{\alpha(N-k+1)}\right\}.$$

This, and

$$\int_{J_{N,k}} \sup_{\substack{A\le N-k\\ j\le A}} \left|\frac{D_{2^j}\circ\tau_A(x)}{\alpha(A)}\right| d\mu = \frac{1}{2^k} a_{N-k}$$

implies

$$\int_{J_{N,k}} L_{\alpha,N}(x)\, d\mu(x) = \max\left\{\frac{1}{2\alpha(N)}, \frac{1}{2^2\alpha(N-1)}, \dots, \frac{1}{2^k\alpha(N-k+1)}, \frac{1}{2^k}a_{N-k}\right\}.$$

Consequently, by (2) we have

$$\begin{aligned}
a_N &\le \sum_{k=1}^{N} \sup_{l\in[1,\dots,k]} \frac{1}{2^l\alpha(N-l+1)} + \sum_{k=1}^{N}\frac{1}{2^k}a_{N-k} + \int_{I_N} L_{\alpha,N}\, d\mu \\
&\le \sum_{k=1}^{N} \sup_{l\in[1,\dots,k]} \frac{1}{2^l\alpha(N-l+1)} + \sum_{k=1}^{N}\frac{1}{2^k}a_{N-k} + \frac{1}{2^N}L_{\alpha,N}(0) \\
&\le N \sup_{A\in[1,\dots,N]} \frac{1}{2^{N-A+1}\alpha(A)} + \sum_{k=1}^{N}\frac{1}{2^k}a_{N-k} + \sup_{A\le N}\frac{1}{2^{N-A}\alpha(A)} \\
&\le \left(\frac{N}{2}+1\right) \sup_{0<A\le N} \frac{1}{2^{N-A}\alpha(A)} + \sum_{k=1}^{N}\frac{1}{2^k}a_{N-k} + \frac{1}{2^N\alpha(0)}.
\end{aligned} \tag{3}$$

Next, we prove the inequality below (constant C depends on the function α).

$$\sum_{n=1}^{N} \frac{n}{2}\sup\left\{\frac{1}{\alpha(n)}, \frac{1}{2\alpha(n-1)}, \dots, \frac{1}{2^{n-1}\alpha(1)}\right\} \le C + \frac{2}{3}\sum_{n=1}^{N}\frac{n}{\alpha(n)}. \tag{4}$$

If

$$\sup\left\{\frac{1}{\alpha(n)}, \frac{1}{2\alpha(n-1)}, \dots, \frac{1}{2^{n-1}\alpha(1)}\right\} = \frac{1}{2^k\alpha(n-k)}$$

for some $1 \le k < n$, then we have

$$\sup\left\{\frac{1}{\alpha(n)},\frac{1}{2\alpha(n-1)},\dots,\frac{1}{2^{n-1}\alpha(1)}\right\}=\frac{1}{2^k\alpha(n-k)}$$

$$\sup\left\{\frac{1}{\alpha(n-1)},\frac{1}{2\alpha(n-2)},\dots,\frac{1}{2^{n-2}\alpha(1)}\right\}=\frac{1}{2^{k-1}\alpha(n-k)}$$

$$\vdots$$

$$\sup\left\{\frac{1}{\alpha(n-k)},\frac{1}{2\alpha(n-k-1)},\dots,\frac{1}{2^{n-k-1}\alpha(1)}\right\}=\frac{1}{\alpha(n-k)}.$$

Consequently, for the left side of (4) we have the following upper bound.

$$\frac{1}{2}\sum_{i=1}^{K}\left(\frac{n_i}{\alpha(n_i)}+\frac{n_i+1}{2\alpha(n_i)}+\frac{n_i+2}{2^2\alpha(n_i)}+\dots+\frac{n_{i+1}-1}{2^{n_{i+1}-n_i-1}\alpha(n_i)}\right),$$

where for the strictly monotone increasing sequence (n_i) we have $n_1 = 1$, and $K \in \mathbb{N}$ is defined as $n_{K+1} - 1 = N$. If

$$\{i \in \mathbb{N} : n_i + 1 < n_{i+1}\} = \emptyset,$$

then the left side of (4) is bounded by

$$\frac{1}{2}\sum_{n=1}^{K}\frac{n}{\alpha(n)}=\frac{1}{2}\sum_{n=1}^{N}\frac{n}{\alpha(n)}.$$

On the other hand, if

$$\{i \in \mathbb{N} : n_i + 1 < n_{i+1}\} \neq \emptyset,$$

then let ρ denote its minimal element. That is, $n_1 = 1, n_2 = 2, \dots, n_\rho = \rho, n_{\rho+1} \ge \rho + 2$. Consequently for the left side of (4) we have

$$\begin{aligned}
&\sum_{n=1}^{N}\frac{n}{2}\sup\left\{\frac{1}{\alpha(n)},\frac{1}{2\alpha(n-1)},\dots,\frac{1}{2^{n-1}\alpha(1)}\right\}\\
&=\frac{1}{2}\left(\frac{1}{\alpha(1)}+\frac{2}{\alpha(2)}+\dots+\frac{\rho-1}{\alpha(\rho-1)}\right)\\
&+\frac{1}{2}\left(\frac{n_\rho}{\alpha(n_\rho)}+\frac{n_\rho+1}{2\alpha(n_\rho)}+\dots+\frac{n_{\rho+1}-1}{2^{n_{\rho+1}-n_\rho-1}\alpha(n_\rho)}\right)\\
&+\frac{1}{2}\sum_{i=\rho+1}^{K}\left(\frac{n_i}{\alpha(n_i)}+\frac{n_i+1}{2\alpha(n_i)}+\frac{n_i+2}{2^2\alpha(n_i)}+\dots+\frac{n_{i+1}-1}{2^{n_{i+1}-n_i-1}\alpha(n_i)}\right) \qquad (5)\\
&\le C+\frac{1}{2}\sum_{i=\rho+1}^{K}\left(\frac{n_i}{\alpha(n_i)}+\frac{1}{\alpha(n_i)}\left(n_i+\sum_{j=1}^{\infty}\frac{j}{2^j}\right)\right)\\
&\le C+\frac{1}{2}\sum_{i=\rho+1}^{K}\left(\frac{n_i}{\alpha(n_i)}+\frac{n_i+2}{\alpha(n_i)}\right)
\end{aligned}$$

(C depends on α). Since the function $\alpha\colon [0,+\infty) \to [1,+\infty)$ is monotone increasing, then we have

$$\sum_{n=1}^{N} \frac{n}{2} \sup\left\{ \frac{1}{\alpha(n)}, \frac{1}{2\alpha(n-1)}, \dots, \frac{1}{2^{n-1}\alpha(1)} \right\}$$
$$\le C + \frac{1}{2} \sum_{i=\rho+1}^{K} \left(\frac{n_i}{\alpha(n_i-1)} + \frac{n_i+2}{\alpha(n_i)} \right)$$
$$\le C + \frac{1}{2} \sum_{n=n_{\rho+1}-1}^{N} \frac{n+2}{\alpha(n)}$$
$$\le C + \frac{1}{2} \cdot \frac{4}{3} \sum_{n=1}^{N} \frac{n}{\alpha(n)}.$$

That is, the inequality (4) is verified. On the other hand, (2) also implies

$$a_N \ge \sum_{k=1}^{N} \max\left\{ \frac{1}{2\alpha(N)}, \frac{1}{2^k} a_{N-k} \right\} + \sup_{A\le N} \frac{1}{2^{N-A}\alpha(A)}$$
$$\ge \sum_{k=\lfloor N/4\rfloor+1}^{N} \frac{1}{2\alpha(N)} + \sum_{k=1}^{\lfloor N/4\rfloor} \frac{1}{2^k} a_{N-k} + \sup_{A\le N} \frac{1}{2^{N-A}\alpha(A)}$$
$$\ge \frac{3N/8}{\alpha(N)} + \frac{a_{N-1}}{2} + \frac{a_{N-2}}{2^2} + \dots + \frac{a_{\lceil 3N/4\rceil}}{2^{\lfloor N/4\rfloor}} + \sup_{A\le N} \frac{1}{2^{N-A}\alpha(A)}.$$

By this inequality we have

$$2a_N - a_{N-1} \ge \frac{3N/4}{\alpha(N)} + \frac{a_{N-2}}{2} + \frac{a_{N-3}}{2^2} + \dots + \frac{a_{\lceil 3N/4\rceil}}{2^{\lfloor N/4\rfloor-1}} + 2\sup_{A\le N} \frac{1}{2^{N-A}\alpha(A)}.$$

Consequently, (3) gives

$$2a_N - 2a_{N-1} \ge \frac{3N/4}{\alpha(N)} + \frac{a_{N-2}}{2} + \frac{a_{N-3}}{2^2} + \dots + \frac{a_{\lceil 3N/4\rceil}}{2^{\lfloor N/4\rfloor-1}} + 2\sup_{A\le N} \frac{1}{2^{N-A}\alpha(A)}$$
$$- \frac{N-1}{2} \sup_{A\le N-1} \frac{1}{2^{N-1-A}\alpha(A)} - \sum_{k=1}^{N-1} \frac{1}{2^k} a_{N-1-k} - \sup_{A\le N-1} \frac{1}{2^{N-A}\alpha(A)}$$
$$\ge \frac{3N/4}{\alpha(N)} - \frac{N-1}{2} \sup_{A\le N-1} \frac{1}{2^{N-1-A}\alpha(A)} - \sum_{k=\lfloor N/4\rfloor}^{N-1} \frac{1}{2^k} a_{N-1-k}.$$

At last by (1) and (4) we have the following lower bound for a_N.

$$\begin{aligned}
2a_N &= \sum_{n=1}^{N}(2a_n - 2a_{n-1}) \\
&\geq \sum_{n=1}^{N}\frac{3n/4}{\alpha(n)} - \sum_{n=0}^{N-1}\frac{n}{2}\sup_{A\leq n}\frac{1}{2^{n-A}\alpha(A)} - C\sum_{n=1}^{N}\frac{n^2}{2^{n/4}} \\
&\geq \left(\frac{3}{4}-\frac{2}{3}\right)\sum_{n=1}^{N}\frac{n}{\alpha(n)} - C.
\end{aligned}$$

Apply Proposition (1), or more exactly, the method its proof, and the inequality given for a_N above.

$$\begin{aligned}
&\| D_\alpha^\kappa \|_1 = \sup_{N\in\mathbb{N}} \sup\left\{ \left\| \frac{D_n^\kappa}{\alpha(\log(\lfloor n \rfloor))} \right\|_1 : n \leq N \right\} \\
&\geq \sup_{N\in\mathbb{N}} \sup\left\{ \left\| \frac{D_{2^A+2^j}^\kappa}{\alpha(A)} \right\|_1 : j < A \leq N, (j, A, N \in \mathbb{N}) \right\} \\
&\geq \sup_{N\in\mathbb{N}} \sup\left\{ \left\| \frac{D_{2^j}^\omega \circ \tau_A}{\alpha(A)} - \frac{D_{2^A}}{\alpha(A)} \right\|_1 : j < A \leq N, (j, A, N \in \mathbb{N}) \right\} \\
&\geq \sup_{N\in\mathbb{N}} \left(a_N - 2\sum_{A=0}^{N}\frac{1}{\alpha(A)} \right) \\
&\geq \sup_{N\in\mathbb{N}} \left(\frac{1}{24}\sum_{n=1}^{N}\frac{n}{\alpha(n)} - 2\sum_{A=1}^{N}\frac{1}{\alpha(A)} - C \right) \\
&\geq \frac{1}{25}\sum_{n=1}^{\infty}\frac{n}{\alpha(n)} - C.
\end{aligned}$$

This completes the proof of Proposition (3).

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