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RELATIONSHIPS BETWEEN TRANSLATION AND ADDITIVE RELATIONS

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Dedicated to the memory of Professor Péter Kiss

Abstract. According to our former papers, a relation F on a groupoid X is called a translation relation if $x+F(y)\subset F(x+y)$ for all $x,y\in X$. Moreover, a relation F on one groupoid X to another Y is called an additive relation if $F(x)+F(y)\subset F(x+y)$ for all $x,y\in X$.

In particular, a reflexive additive relation on a groupoid is a translation relation. Moreover, translation relations play important roles in the extensions and uniformizations of semigroups and groups, respectively. Therefore, it is of some interest to investigate the relationships between translation and additive relations.

In particular, we show that a normal translation relation on a group is odd (additive) if and only if it is symmetric (transitive). Moreover, if F is an odd additive relation of one group X into another Y and S is a translation relation on Y, then $R=F^{-1}\circ S\circ F$ is a translation relation on X such that $R=F^{-1}\circ S\circ f$ for any selection f of F.

A translation relation F on a group X is called normal if $F(0)+x\subset x+F(0)$ for all $x\in X$. Moreover, a relation F on one group X to another Y is called odd if $-F(x)\subset F(-x)$ for all $x\in X$. In particular, we also show that an additive function on one group to another is odd if and only if its domain is symmetric.

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1. A few basic facts on relations and groupoids

A subset F of a product set $X\times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X=\{(x,x):\ \ x\in X\}$ and $X^2=X\times X$ are called the identity and universal relations on X, respectively.

Namely, if in particular $F\subset X^2$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X\cup Y$. Therefore, it is frequently not a severe restriction to assume that $X=Y$.

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If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively. Whenever $A \in X$ seems unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(y) = \big\{x \in X : \ \ y \in F(x)\big\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G\big(F(x)\big)$ for all $x \in X$. Note that thus we have $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

If F is a relation on X to Y, then the sets $D_F = F^{-1}(X)$ and $R_F = F(X)$ are called the domain and range of F, respectively. If in particular $X = D_F$ (and $Y = R_F$) , then we say that F is a relation of X into (onto) Y.

A relation F on X to Y is called a function if for each $x \in D_F$ there exists $y \in Y$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If F is a relation on X to Y, then a function f of D_F into Y is called a selection of F if $f \subset F$. In terms of selections, the axiom of choice can be briefly reformulated by saying that every relation has a selection.

A relation F on X is called reflexive, symmetric and transitive if $\Delta_F \subset F$, $F^{-1} \subset F$ and $F \circ F \subset F$, respectively. Note that if F is a symmetric relation, then we actually have $F = F^{-1}$.

If X is nonvoid set and $+$ is a function of X^2 into X, then the ordered pair $X(+) = (X, +)$ is called a groupoid. In this case, we may also naturally write $x + y = +(x, y)$ for all $x, y \in X$.

Moreover, if X is a groupoid, then we may also naturally write $A + B = \big\{x + y : x \in A, \ y \in B\big\}$ for all $A, B \subset X$. Thus, the family $\mathcal{P}(\mathcal{X})$ of all subsets of X is also a groupoid.

Note that if X is, in particular, a group, then $\mathcal{P}(\mathcal{X})$ is, in general, only a semigroup with zero element $\{0\}$. However, we can still naturally use the notations $-A = \big\{-x : \ x \in A\big\}$ and $A - B = A + (-B)$.

A subset A of a groupoid X is called additive and normal if $A + A \subset A$ and $A + x \subset x + A$ for all $x \in X$, respectively. Moreover, a subset A of a group X is called symmetric if $-A \subset A$.

Note that if A is a symmetric set, then we also have $A \subset -A$. Moreover, if A is a normal subset of a group X, then we also have $x + A = x + (A - x) + x \subset x + (-x + A) + x = A + x$ for all $x \in X$.

2. A few basic facts on translation relations

Definition 2.1. A relation F on a groupoid X is called a translation relation if

$$x + F(y) \subset F(x+y)$$

for all $x, y \in X$.

Remark 2.2. Note that thus we have $X + D_F \subset D_F$. Therefore, D_F is an ideal of X whenever $F \neq \emptyset$.

Hence, it is clear that $D_F = X + D_F$ whenever X has a zero element. Moreover, $D_F = X$ whenever X is a group and $F \neq \emptyset$.

Remark 2.3. Moreover, it is also worth mentioning that, by using the notation xFy instead of $y \in F(x)$, the inclusion $x + F(y) \subset F(x+y)$ can be expressed by saying that yFz implies $(x+y)F(x+z)$.

Example 2.4. Clearly, the identity function Δ_X of a groupoid X is a translation relation on X.

Moreover, the order relation $\leq$ of a left-ordered group X [5, p. 127] is, in particular, a translation relation on X.

Example 2.5. More generally, we can also note that if Y is a subset of a semigroup X and F is a relation on X such that $F(x) = x + Y$ for all $x \in X$, then F is a translation relation on X.

Moreover, we can also easily establish the following

Theorem 2.6. *If F is a relation on a group X, then the following assertions are equivalent :*

(1) F is a translation; *(2) $F(x) = x + F(0)$ for all $x \in X$.*

Proof. If (1) holds, then $x + F(0) \subset F(x+0) = F(x)$ and $F(x) = x - x + F(x) \subset x + F(-x+x) = x + F(0)$ for all $x \in X$. Therefore, (2) also holds. Moreover, by Example 2.5, the converse implication (2) $\Longrightarrow$ (1) is also true.

Remark 2.7. In this respect, it is also worth mentioning that if F is a relation on a group X such that $F(x+y) \subset x + F(y)$ for all $x, y \in X$, then we also have $F(x) = x + F(0)$ for all $x \in X$. Therefore, F is a translation relation on X.

Example 2.8. If $X = [0, +\infty]$ and F is a relation on X such that $F(x) = [0, x]$ for all $x \in X$, then F is a translation (and a total order) relation on X such that $F(x) \neq x + F(0)$ for all $x \in X \setminus \{0\}$.

Example 2.9. More generally, we can also note that if p is a function on a group X to $[0, +\infty]$, and moreover $r \in [0, +\infty]$ and F is a relation on X such that

$F(x) = \{ y \in X : \quad p(-x+y) \le r \}$ for all $x \in X$, then F is a translation relation on X. Therefore, $F(x) = x + F(0)$ for all $x \in X$.

Concerning translation relations, we shall also need the following theorems which have been mostly proved in [13].

Theorem 2.10. *If F is a relation on a groupoid X, then the following assertions are equivalent :*

(1) F is a translation ; *(2) $\Delta_X + F \subset F$.*

Remark 2.11. If F is a translation relation on a groupoid X with a zero element, then the equality $F = \Delta_X + F$ is also true.

Theorem 2.12. *If F is a translation relation on a groupoid X and $A, B \subset X$, then*

$$A + F(B) \subset F(A+B).$$

Moreover, if X is a group, then the corresponding equality is also true.

Remark 2.13. In this respect, it is also worth noticing that if F is a normal translation relation on a group X in the sense that $F(0)$ is a normal subset of X, then we also have $F(A+B) = F(A) + B$.

Theorem 2.14. *If F is a translation relation on a groupoid X, then F^{-1} is also a translation relation on X.*

Theorem 2.15. *If F is a normal translation relation on a group X and $A \subset X$, then*

$$F^{-1}(A) = -F(-A).$$

Remark 2.16. The equality $F^{-1}(0) = -F(0)$ is true even if the translation relation F is not normal.

Theorem 2.17. *If F and G are translation relations on a groupoid X, then $G \circ F$ is also a translation relation on X.*

Theorem 2.18. *If F is a normal and G is an arbitrary translation relation on a group X and $A, B \subset X$, then*

$$(G \circ F)(A+B) = F(A) + G(B).$$

Remark 2.19. The equality $(G \circ F)(0) = F(0) + G(0)$ is true even if F is an arbitrary relation on X.

Moreover, the equality $(G \circ F)(A) = F(A) + G(0)$ is true even if the translation relation F is not normal.

Therefore, under the conditions of Theorem 2.18, we also have $(G \circ F)(x) = (F \circ G)(x)$ for all $x \in X$, and hence $G \circ F = F \circ G$.

3. A few basic facts on additive relations

Definition 3.1. A relation F on one groupoid X to another Y is called additive if

$$F(x)+F(y)\subset F(x+y)$$

for all $x,y\in X$.

Remark 3.2. Note that thus we have $D_F+D_F\subset D_F$. Therefore, D_F is a subgroupoid of X whenever $F\neq\emptyset$.

Remark 3.3. Moreover, it is also worth noticing that, by using the notation xFy instead of $y\in F(x)$, the inclusion $F(x)+F(y)\subset F(x+y)$ can be expressed by saying that xFz and yFw imply $(x+y)F(z+w)$.

Example 3.4. Note that the order relation $\leq$ of an ordered group X [3, p. 9] is, in particular, an additive relation on X.

Example 3.5. More generally, we can also note that if X is a groupoid and Z is an additive and normal subset of a semigroup Y, and moreover f is an additive function on X to Y and F is a relation on X to Y such that $F(x)=f(x)+Z$ for all $x\in X$, then F is an additive relation on X to Y.

Moreover, we can also easily establish the following

Theorem 3.6. *If F is an additive relation on one group X to another Y and f is a selection for F such that $-f(x)\in F(-x)$ for all $x\in D_F$, then for all $x\in D_F$ we also have*

$$F(x)=f(x)+F(0).$$

Proof. Namely, we evidently have $f(x)+F(0)\subset F(x)+F(0)\subset F(x)$ and

$$F(x)=f(x)-f(x)+F(x)\subset f(x)+F(-x)+F(x)\subset f(x)+F(0)$$

for all $x\in X$. Therefore, the required equality is also true.

Remark 3.7. Quite similarly, we can also prove that $F(x)=F(0)+f(x)$ for all $x\in D_F$.

Example 3.8. If X and F are as in Example 2.8, then F is an additive relation on X such that for any function f of X and $Z\subset X$, with $F(0)=f(0)+Z$, we have $F(x)\neq f(x)+Z$ for all $x\in X\setminus\{0\}$.

Example 3.9. More generally, we can also note that if $X=[0,+\infty]$, p is a subadditive function on a groupoid Y to X, and F is a relation on X to Y such that $F(x)=\big\{y\in Y:\ \ p(y)\leq x\big\}$ for all $x\in X$, then F is an additive relation on X to Y.

Concerning additive relations, we can also easily prove the following counterparts of the corresponding results of [14].

Theorem 3.10. *If F is a relation on one groupoid X to another Y, then the following assertions are equivalent:*

(1) F is additive ; *(2) $F+F \subset F$.*

Theorem 3.11. *If F is an additive relation on one groupoid X to another Y and $A, B \subset X$, then*

$$F(A)+F(B) \subset F(A+B).$$

Theorem 3.12. *If F is an additive relation on one groupoid X to another Y, then F^{-1} is an additive relation on Y to X.*

Theorem 3.13. *If F is an additive relation on one groupoid X to another Y and G is an additive relation on Y to a groupoid Z, then $G \circ F$ is an additive relation on X to Z.*

The relationship between translation and additive relations can be cleared up by the following

Theorem 3.14. *If F is a normal translation relation on a group X, then the following assertions are equivalent :*

(1) F is additive ; *(2) F is transitive.*

Proof. If (1) holds, then by Remark 2.19 we have $(F \circ F)(x) = F(x)+F(0) \subset F(x)$ for all $x \in X$ even if F is not normal. Therefore, $F \circ F \subset F$, and thus (2) also holds.

While, if (2) holds, then by Theorem 2.18 we have $F(x)+F(y) = (F \circ F)(x+y) \subset F(x+y)$ for all $x, y \in X$. Therefore, (1) also holds.

Remark 3.15. Quite similarly, we can also prove that a translation relation F on a group X is transitive if and only if the set $F(0)$ is additive.

Moreover, in addition to Theorem 3.14, it is also worth noticing that a reflexive and additive relation F on a groupoid X is a translation relation.

4. A few basic facts on odd relations

Definition 4.1. A relation F on one group X to another Y is called odd if

$$-F(x) \subset F(-x)$$

for all $x \in X$.

Remark 4.2. Note that thus we have $-D_F \subset D_F$. Therefore, D_F is a symmetric subset of X, and we have $D_F = -D_F$.

Remark 4.3. Moreover, it is also worth noticing that, by using the notation xFy instead of $y \in F(x)$, the inclusion $-F(x) \subset F(-x)$ can be expressed by saying that xFy implies $(-x)F(-y)$.

Example 4.4. Note that the order relation $\leq$ of an ordered group X is odd if and only if $\leq$ coincides with Δ_X.

Concerning odd relations, we can also easily prove the following theorems.

Theorem 4.5. *If F is a relation on one group X to another Y, then the following assertions are equivalent :*

(1) F is odd; *(2) $-F = F$.*

Theorem 4.6. *If F is an odd relation on one group X to another Y and $A \subset X$, then*

$$F(-A) = -F(A).$$

Proof. If $y \in -F(A)$, then $-y \in F(A)$. Therefore, there exists $x \in A$ such that $-y \in F(x)$. Hence, we can already see that $y \in -F(x) \subset F(-x) \subset F(-A)$. Therefore, $-F(A) \subset F(-A)$.

Now, by writing $-A$ in place of A, we can also see that $-F(-A) \subset F(A)$, and hence $F(-A) \subset -F(A)$ is also true.

Theorem 4.7. *If F is an odd relation on one group X to another Y, then F^{-1} is an odd relation on Y to X.*

Theorem 4.8. *If F is odd relation on one group X to another Y and G is an odd relation on Y to a group Z, then $G \circ F$ is an odd relation on X to Z.*

The relationship between odd and translation relations can be cleared up by the following

Theorem 4.9. *If F is a normal translation relation on a group X, then the following assertions are equivalent :*

(1) F is odd; *(2) F is symmetric.*

Proof. If (1) holds, then by Theorems 2.15 and 4.6 we have $F^{-1}(x) = -F(-x) = -\big(-F(x)\big) = F(x)$ for all $x \in X$. Therefore, $F^{-1} = F$, and thus (2) also holds.

While, if (2) holds, then only by Theorem 2.15 we have $-F(x) = -F^{-1}(x) = -\big(-F(-x)\big) = F(-x)$ for all $x \in X$. Therefore, (1) also holds.

Remark 4.10. Quite similarly, we can also prove that a translation relation F on a group X is symmetric if and only if the set $F(0)$ is symmetric.

Concerning the relationship between odd and additive relations, we can only prove the following theorems.

Theorem 4.11. *If F is an additive relation on one group X to another Y and f is an odd selection for F, then for all $x \in D_F$ we have*

$$F(x) = f(x) + F(0).$$

Proof. Namely, $-f(x) = f(-x) \in F(-x)$ for all $x \in D_F$. Therefore, Theorem 3.6 can be applied.

Theorem 4.12. *If f is an additive function on one group X to another Y, then the following assertions are equivalent:*

(1) f is odd; *(2) D_f is symmetric.*

Proof. By Remark 4.2, it is clear that the implication (1)$\Longrightarrow$(2) is true even if f is not additive.

Moreover, if (2) holds, then for any $x \in D_f$, we also have $-x \in D_f$. Hence, by the additivity of f, it follows that $f(x) + f(-x) = f(0)$. Now, by writing 0 in place of x, we can see that $f(0) + f(0) = f(0)$, and thus $f(0) = 0$. Therefore, we actually have $f(x) + f(-x) = 0$, and hence $-f(x) = f(-x)$. That is, (1) also holds.

Example 4.13. If X is a group and Z is an additive and normal subset of a group Y, and moreover f is an additive function on X to Y, with a symmetric domain, and F is a relation on X to Y such that $F(x) = f(x) + Z$ for all $x \in X$, then F is an odd additive relation on X to Y.

5. A few basic facts on odd additive relations

Theorem 5.1. *If F is a nonvoid odd additive relation on one group X to another Y, then $0 \in F(0)$.*

Proof. Since $F \neq \emptyset$, there exist $x \in X$ and $y \in Y$ such that $y \in F(x)$. Hence, by the assumed properties of F, it is clear that $0 = y - y \in F(x) - F(x) = F(x) + F(-x) \subset F(0)$.

Corollary 5.2. *If F is an odd additive relation on one group X to another Y and f is a function of D_F into Y such that $F(x) = f(x) + F(0)$ for all $x \in D_F$, then f is a selection for F.*

Theorem 5.3. *If F is an odd additive relation on one group X to another Y and f is a selection for F, then for all $x \in D_F$ we have*

$$F(x) = f(x) + F(0).$$

Proof. Namely, $-f(x) \in -F(x) = F(-x)$ for all $x \in D_F$. Therefore, Theorem 3.6 can be applied.

Theorem 5.4. *If F is an odd additive relation on one group X to another Y and $A \subset D_F$ and $B \subset X$, then*

$$F(A+B) = F(A) + F(B).$$

Proof. Since $A \subset D_F$, for each $x \in A$ there exists $y \in F(x)$. Hence, by Theorem 3.11, it is clear that

$$\begin{aligned} F(x+B) &= y - y + F(x+B) \subset F(x) - F(x) + F(x+B) \\ &= F(x) + F(-x) + F(x+B) \subset F(A) + F(-x+x+B) = F(A) + F(B). \end{aligned}$$

Therefore, we also have

$$F(A+B) = F\Big(\bigcup_{x \in A} (x+B)\Big) = \bigcup_{x \in A} F(x+B) \subset F(A) + F(B).$$

Thus, by Theorem 3.11, the corresponding equality is also true.

Remark 5.5. If $A \subset X$ and $B \subset D_F$, then we can quite similarly see that the equality $F(A+B) = F(A) + F(B)$ is also true.

Example 5.6. If $F = \Delta_{\Delta_{\mathbf{R}}}$, then F is a linear relation on $\mathbf{R}^2$. Moreover, if $A = \mathbf{R} \times \{0\}$ and $B = \{0\} \times \mathbf{R}$, then

$$F(A+B) = \Delta_{\mathbf{R}}, \quad \text{but} \quad F(A) + F(B) = \{(0,0)\}.$$

Remark 5.7. A relation F on one vector space X to another Y over the same field K is called linear [15] if in addition to the additivity of F we also have $\lambda F(x) \subset F(\lambda x)$ for all $\lambda \in K$ and $x \in X$.

6. Projective generation of translation relations

Theorem 6.1. *If F is an additive relation of one groupoid X into another Y and S is a translation relation on Y, then $R = F^{-1} \circ S \circ F$ is a translation relation on X.*

Proof. If $x, y \in X$ and $z \in R(y)$, then by the corresponding definitions we also have $z \in \big(F^{-1} \circ S \circ F\big)(y) = F^{-1}\big(S\big(F(y)\big)\big)$. Therefore, there exists $w \in S\big(F(y)\big)$ such that $z \in F^{-1}(w)$, and hence $w \in F(z)$. Consequently, we also have $F(z) \cap S\big(F(y)\big) \neq \emptyset$. Hence, since $F(x) \neq \emptyset$, it follows that

$$\big(F(x) + F(z)\big) \cap \big(F(x) + S\big(F(0)\big)\big) \neq \emptyset.$$

Hence, by using that $F(x)+F(z)\subset F(x+z)$ and

$$F(x)+S\big(F(y)\big)\subset S\big(F(x)+F(y)\big)\subset S\big(F(x+y)\big),$$

we can infer that

$$F(x+z)\cap S\big(F(x+y)\big)\neq\emptyset.$$

Therefore, there exists $\omega\in S\big(F(x+y)\big)$ such that $\omega\in F(x+z)$, and hence $x+z\in F^{-1}(\omega)$. Consequently, we also have

$$x+z\in F^{-1}\left(S\big(F(x+y)\big)\right)=\big(F^{-1}\circ S\circ F\big)(x+y)=R(x+y).$$

Therefore, the inclusion $x+R(y)\subset R(x+y)$ is also true.

Theorem 6.2. *If F is a relation on one group X to another Y such that*

(1) $0\in F(0)$, *(2)* $-F(0)\subset F(0)$,

(3) $F(0)+F(x)\subset F(x)$ *for all* $x\in X$,

and S is a translation relation on Y, then

$$F^{-1}\big(S(0)\big)=F^{-1}\left(S\big(F(0)\big)\right).$$

Proof. Because of hypothesis (1), we evidently have $S(0)\subset S\big(F(0)\big)$, and hence $F^{-1}\big(S(0)\big)\subset F^{-1}\left(S\big(F(0)\big)\right)$.

To prove the converse inclusion, note that if $x\in F^{-1}\left(S\big(F(0)\big)\right)$, then there exists $y\in S\big(F(0)\big)$ such that $x\in F^{-1}(y)$, and hence $y\in F(x)$. Moreover, there exists $z\in F(0)$ such that $y\in S(z)$. Hence, by using the translation property of S, we can see that

$$-z+y\subset -z+S(z)\subset S(-z+z)=S(0).$$

Moreover, by using hypotheses (2) and (3), we can see that

$$-z+y\in -F(0)+F(x)\subset F(0)+F(x)\subset F(x),$$

and hence $x\in F^{-1}(-z+y)$. Therefore, we also have $x\in F^{-1}\big(S(0)\big)$.

Corollary 6.3. *If F is an odd additive relation on one group X to another Y and S is a translation relation on Y, then $F^{-1}\big(S(0)\big)=F^{-1}\left(S\big(F(0)\big)\right)$.*

Proof. If $F=\emptyset$, then the required assertion trivially holds. While, if $F\neq\emptyset$, then by Theorem 5.1 we have $0\in F(0)$. Thus, Theorem 6.2, can be applied.

Theorem 6.4. *If F is an odd additive relation on one group X to another Y and S is a translation relation on Y, then for any selection f of F we have*

$$F^{-1}\circ S\circ f=F^{-1}\circ S\circ F.$$

Proof. In this case, by Theorems 4.7 and 3.12, it is clear that F^{-1} is an odd additive relation on Y to X. Moreover, by using Theorems 3.11, 5.3 and 5.4 and Corollary 6.3, we can see that

$$\begin{aligned}
\big(F^{-1}\circ S\circ f\big)(x) &= F^{-1}\left(S\big(f(x)\big)\right) = F^{-1}\big(f(x)+S(0)\big)\\
&= F^{-1}\big(f(x)\big)+F^{-1}\big(S(0)\big) = F^{-1}\big(f(x)\big)+F^{-1}\left(S\big(F(0)\big)\right)\\
&= F^{-1}\left(f(x)+S\big(F(0)\big)\right) = F^{-1}\left(S\big(f(x)+F(0)\big)\right)\\
&= F^{-1}\left(S\big(F(x)\big)\right) = \big(F^{-1}\circ S\circ F\big)(x)
\end{aligned}$$

for all $x \in D_F$. Therefore, the required equality is also true.

Remark 6.5. If F and S are in Theorem 6.4, then as a more direct generalization of Corollary 6.3, we can also prove that $F^{-1}\big(S(y)\big)= F^{-1}\left(S\big(F(x)\big)\right)$ for all $x \in D_F$ and $y \in F(x)$.

However, the latter statement can also be easily derived from Theorem 6.4 since by the axiom of choice for any $x \in D_F$ and $y \in F(x)$ there exists a selection f of F such that $f(x) = y$.

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