Acta Acad. Paed. Agriensis, Sectio Mathematicae **29** *(2002) 15–34*

ON THE CUBE MODEL OF THREE-DIMENSIONAL EUCLIDEAN SPACE

I. Szalay (Szeged, Hungary)

Dedicated to the memory of Professor Péter Kiss

Abstract. In [4] the open interval $\underline{R}=]-1,1[$ with the sub-addition $\oplus$ and sub-multiplication $\odot$ was considered as a compressed model of the field of real numbers $(R,+,\cdot)$. Considering the points of the open cube $\underline{R}^3=\{X=(x_1,x_2,x_3):\ x_1,x_2,x_3\in\underline{R}\}$ we give the concepts of sub-line and sub-plane and construct a bounded model of the three dimensional Euclidean geometry which is isomorphic with the familiar model R^3.

Preliminary

The first exact formulation of classical Euclidean geometry was given by Hilbert. Nowadays, Hilbert's axiom-system is well-known. (For example, see [2], pp. 172, 102, 31, 326, 135–136, 187, 351, 77, 326, 25, 45 and 405.) It is a very comfortable model, the three-dimensional Descartes coordinate-system R^3 is a real vector space with a canonical inner product. It is used in the secondary and higher schools, in general. Another model, given by Fjodoroff (see [2] p. 117), is less-known. Its speciality is that it is able to interpret the points of R^3 in a given basic plane by a point (lying on the basic plane) together with a directed circle. Both mentioned models, are boundless.

Our cube model, being an (open) cube in R^3, is bounded. Its speciality is that it is able to show the "end of line" or the "meeting of parallel lines" and so on. On the other hand, the elements of this model are less spectacular in a traditional sense. "Line" may be a screwed curve which does not lie in any traditional plane. The form "ball" depends on not only its "radius" but the place of its "centre", too.

The importence of the cube model is in the methodology of teacher training. Seeing that the axioms are not trivial helps to understand the role of parallelism in the history of mathematics: Namely, the axiom of parallelism was the only axiom which seemed to be provable by the other axioms.

The cube model is based on the ordered field of compressed real numbers situated on the open interval]-1, 1[denoted by $\underline{R}$. Introducing the sub-addition $\oplus$ and sub-multiplication $\odot$, the ordered field $(\underline{R},\oplus,\odot)$ is isomorphic with the ordered field $(R,+,\cdot)$. The points of open cube $\underline{R}^3=\{X=(x_1,x_2,x_3):\ x_1,x_2,x_3\in\underline{R}\}$ give the points of the cube model.

Introduction

Having the compression function $u \in R \mapsto \operatorname{th} u \in]-1,1[$ ([1], I. 7.54–7.58) we say that the compressed of u is given by the equation

$$\underbracket{u} = \operatorname{th} u, \quad u \in R. \tag{0.1}$$

Hence, we have that the compresseds of real numbers are just on the open interval $\underbracket{R} =]-1,1[$. Considering the compression function as an isomorphism between the fields $(R, +, \cdot)$ and $(\underbracket{R}, \oplus, \odot)$ we define the sub-addition and sub-multiplication by the identities

$$\underbracket{u} \oplus \underbracket{v} := \underbracket{u + v}, \quad u, v \in R \tag{0.2}$$

and

$$\underbracket{u} \odot \underbracket{v} := \underbracket{u \cdot v}, \quad u, v \in R, \tag{0.3}$$

respectively. If $x = \underbracket{u}$ and $y = \underbracket{v}$, then (0,1), (0,2) and (0,3) yield the relations

$$x \oplus y = \frac{x + y}{1 + xy}, \quad x, y \in \underbracket{R} \tag{0.4}$$

and

$$x \odot y = \operatorname{th}((\operatorname{ar\,th} x)(\operatorname{ar\,th} y)), \quad x, y \in \underbracket{R}. \tag{0.5}$$

Moreover, we can use the identities

$$\underbracket{u} \ominus \underbracket{v} := \underbracket{u - v}, \quad u, v \in R \tag{0.6}$$

$$\underbracket{u} \mathbin{\bigcirc\kern-0.75em:\kern0.4em} \underbracket{v} := \underbracket{u : v}, \quad u, v \in R, \quad v \neq 0 \tag{0.7}$$

or

$$x \ominus y = \frac{x - y}{1 - xy}, \quad x, y \in \underbracket{R} \tag{0.8}$$

and

$$x \mathbin{\bigcirc\kern-0.75em:\kern0.4em} y = \operatorname{th}\left(\frac{\operatorname{ar\,th} x}{\operatorname{ar\,th} y}\right), \quad x, y \in \underbracket{R}, y \neq 0, \tag{0.9}$$

where the operations $\ominus$ and $\mathbin{\bigcirc\hspace{-0.72em}:}$ are called sub-subtraction and sub-division, respectively.

The inverse of compression is explosion defined by the equation

$$\overset{\sqcup}{x} = \operatorname{ar\,th} x, \quad x \in \underset{\sqcup}{R} \tag{0.10}$$

and $\overset{\sqcup}{x}$ is called the exploded of x. Clearly, by (0.1) and (0.10) we have the identities

$$x = \underset{\sqcup}{(\overset{\sqcup}{x})}, \quad x \in \underset{\sqcup}{R} \tag{0.11}$$

and

$$u = \overset{\sqcup}{(\underset{\sqcup}{u})}, \quad u \in R. \tag{0.12}$$

1. Operations on $\underset{\sqcup}{R}^3$

Having the familiar three dimensional Euclidean vector-space R^3 with the traditional operations (addition, multiplication by scalar, inner product) as well as the concepts of norm and metric, we give their isomorphic concepts for $\underset{\sqcup}{R}^3$ which is the set of points $X = (x_1, x_2, x_3)$ such that $x_1, x_2, x_3 \in \underset{\sqcup}{R}$. Clearly, $\underset{\sqcup}{R}^3$ forms an open cube in R^3. Considering the vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ from $\underset{\sqcup}{R}^3$ we define sub-addition as

$$X \oplus Y = (x_1 \oplus y_1, x_2 \oplus y_2, x_3 \oplus y_3), \tag{1.1}$$

sub-multiplication by scalar $c \in \underset{\sqcup}{R}$ as

$$c \odot X = (c \odot x_1, c \odot x_2, c \odot x_3) \tag{1.2}$$

and sub-inner product as

$$X \odot Y = (x_1 \odot y_1) \oplus (x_2 \odot y_2) \oplus (x_3 \odot y_3). \tag{1.3}$$

Introducing the exploded of the point $X = (x_1, x_2, x_3)$ as

$$\overset{\sqcup}{X} = (\overset{\sqcup}{x_1}, \overset{\sqcup}{x_2}, \overset{\sqcup}{x_3}), \quad X \in \underset{\sqcup}{R}^3 \tag{1.4}$$

and the compressed of the point $U = (u_1, u_2, u_3)$ as

$$\underbracket{U} = (\underbracket{u_1}, \underbracket{u_2}, \underbracket{u_3}), \quad U \in R^3 \tag{1.5}$$

we have the identities

$$X = \underbracket{(\overbracket{X})}, \quad X \in \underbracket{R}^3 \tag{1.6}$$

and

$$U = \overbracket{(\underbracket{U})}, \quad U \in R^3. \tag{1.7}$$

Using (0.11), (0.2), (1.5) and (1.4), the identity (1.1) yields

$$X \oplus Y = \underbracket{\overbracket{X} + \overbracket{Y}}, \quad X, Y \in \underbracket{R}^3. \tag{1.8}$$

Moreover, by (0.11), (0.3), (1.4) the identity (1.2) yields

$$c \odot X = \underbracket{\overbracket{c} \cdot \overbracket{X}}, \quad c \in \underbracket{R} \quad \text{and} \quad X \in \underbracket{R}^3. \tag{1.9}$$

Considering the operations (1.1) and (1.2) we have the following

Theorem 1.10. $\underbracket{R}^3$ *is a real vector space with the sub-addition* (1.1) *and scalar sub-multiplication* (1.2). *In detail, we have the following identities:*

$$X \oplus Y = Y \oplus X, \quad X, Y \in \underbracket{R}^3, \tag{1.11}$$

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z), \quad X, Y, Z \in \underbracket{R}^3, \tag{1.12}$$

$$X \oplus o = X, \quad \textit{where} \quad X \in \underbracket{R}^3 \quad \textit{arbitrary and} \quad o = (0, 0, 0), \tag{1.13}$$

$$\begin{gathered} X \oplus (-X) = o, \quad \textit{where} \quad -X \\ \textit{is the familiar additive inverse of} \quad x \in \underbracket{R}^3. \end{gathered} \tag{1.14}$$

Moreover, the identities

$$\underbracket{1} \odot X = X, \quad X \in \underbracket{R}^3, \tag{1.15}$$

$$c \odot (X \oplus Y) = (c \odot X) \oplus (c \odot Y), \quad c \in \lfloor R \rfloor, \ X, Y \in \lfloor R \rfloor^3, \tag{1.16}$$

$$(c_1 \oplus c_2) \odot X = (c_1 \odot X) \oplus (c_2 \odot X), \quad c_1, c_2 \in \lfloor R \rfloor, \ x \in \lfloor R \rfloor^3, \tag{1.17}$$

$$(c_1 \odot c_2) \odot X = c_1 \odot (c_2 \odot X), \quad c_1, c_2 \in \lfloor R \rfloor \ X \in \lfloor R \rfloor^3, \tag{1.18}$$

also hold.

Remark 1.19. By Theorem 1.10 we say that $\lfloor R \rfloor^3$ is a sub-linear space with the operations (1.1) and (1.2).

Using (0.11), (0.3), (0.2) and (1.4) the identity (1.3) yields

$$X \odot Y = \lfloor \overset{\sqcup}{X} \cdot \overset{\sqcup}{Y} \rfloor, \quad X, Y \in \lfloor R \rfloor^3, \tag{1.20}$$

where "$\cdot$" means the familiar inner product (of vectors $\overset{\sqcup}{X}$ and $\overset{\sqcup}{Y}$) in R^3.

For the sub-inner product we have

Theorem 1.21. $\lfloor R \rfloor^3$ *in a Euclidean vector space with the sub-inner product defined by* (1.3) *such that the sub-inner product has the following properties*

$$X \odot Y = Y \odot X, \quad X, Y \in \lfloor R \rfloor^3, \tag{1.22}$$

$$(X \oplus Y) \odot Z = (X \odot Z) \oplus (Y \odot Z), \quad X, Y, Z \in \lfloor R \rfloor^3 \tag{1.23}$$

$$(c \odot X) \odot Y = c \odot (X \odot Y), \quad c \in \lfloor R \rfloor, \quad X, Y \in \lfloor R \rfloor^3 \tag{1.24}$$

and for any $X \in \lfloor R \rfloor^3$ *the inequality*

$$X \odot X \geq 0 \quad \textit{holds such that} \quad X \odot X = 0 \quad \textit{if and only if} \quad X = 0. \tag{1.25}$$

Remark 1.26. By Theorem 1.21 we say that $\lfloor R \rfloor^3$ is a sub-euclidean space with the sub-inner product (1.3).

In [4] the concept of sub-function was defined for one variable (see [4], (0.8) and (0.9)). Hence, we have the sub-square root function

$$\text{sub}\,\sqrt{x} = \underline{\sqrt{\overset{\sqcup}{x}}}, \quad x \in [0, 1). \tag{1.27}$$

Having the property (1.25) and using the sub-square root function we can define the sub-norm as follows:

$$\|X\|_{\underline{R}^3} = \text{sub}\,\sqrt{X \odot X}, \quad x \in \underline{R}^3. \tag{1.28}$$

Using (1.27), (1.20) and (0.12) the definition (1.28) yields

$$\|X\|_{\underline{R}^3} = \underline{\| \overset{\sqcup}{X} \|_{R^3}}, \quad x \in \underline{R}^3, \tag{1.29}$$

where $\| \cdot \|_{R^3}$ means the familiar norm of vectors.

Remark 1.30. Applying the familiar Cauchy's inequality by (1.20), (0.1), (0.3) and (1.29) we have the inequality

$$|X \odot Y| \le \|X\|_{\underline{R}^3} \odot \|Y\|_{\underline{R}^3}, \quad X, Y, \in \underline{R}^3.$$

Corollary 1.31. *The sub-norm has the following properties*

$$\|X\|_{\underline{R}^3} \ge 0,\ (X \in \underline{R}^3) \textit{ such that} \|X\|_{\underline{R}^3} = 0 \textit{ if and only if } X = 0, \tag{1.32}$$

$$\|c \odot X\|_{\underline{R}^3} = |c| \odot \|X\|_{\underline{R}^3}, \quad c \in \underline{R}, \quad X \in \underline{R}^3 \tag{1.33}$$

and

$$\|X \oplus Y\|_{\underline{R}^3} \le \|X\|_{\underline{R}^3} \oplus \|Y\|_{\underline{R}^3}, \quad X, Y \in \underline{R}^3. \tag{1.34}$$

Remark 1.35. By Corollary 1.31 we say that $\underline{R}^3$ is a sub-normed space with the sub-norm (1.28).

Finally, we define the sub-distance of elements of $\underline{R}^3$ as follows

$$d_{\underline{R}^3}(X, Y) = \|X \ominus Y\|_{\underline{R}^3}, \quad X, Y \in \underline{R}^3, \tag{1.36}$$

where the sub-subtraction of vectors is defined by

$$X \ominus Y = X \oplus (-Y). \tag{1.37}$$

Using (1.29), (1.37), (1.8), (1.7), (1.4) and (0.10) the definition (1.36) yields

$$d_{\underset{\sqcup}{R}^3}(X,Y) = \underset{\sqcup}{\underbrace{d_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{Y})}}, \quad X,Y \in \underset{\sqcup}{R}^3, \tag{1.38}$$

where d_{R^3} is the familiar distance of the points of R^3.

Corollary 1.39. *The sub-distance has the following properties*

$$d_{\underset{\sqcup}{R}^3}(X,Y) = d_{\underset{\sqcup}{R}^3}(Y,X), \quad X,Y \in \underset{\sqcup}{R}^3, \tag{1.40}$$

$$d_{\underset{\sqcup}{R}^3}(X,Y) \geq 0 \quad \textit{such that} \quad d_{\underset{\sqcup}{R}^3}(X,Y) = 0 \quad \textit{if and only if} \tag{1.41}$$

$$X = Y, \quad X,Y \in \underset{\sqcup}{R}^3$$

and

$$d_{\underset{\sqcup}{R}^3}(X,Y) \leq d_{\underset{\sqcup}{R}^3}(X,Z) \oplus d_{\underset{\sqcup}{R}^3}(Z,Y), \quad X,Y,Z \in \underset{\sqcup}{R}^3. \tag{1.42}$$

Remark 1.43. By Corollary 1.39 we say that $\underset{\sqcup}{R}^3$ is a sub-metrical space with the sub-distance (1.36).

2. On the geometry of $\underset{\sqcup}{R}^3$

Our starting point is the Euclidean geometry of R^3 with its points, lines and planes based on the axioms formulated by Hilbert. Now we construct the cube-model of the classical Euclidean geometry. The points of the model will be the points of R^3. Considering a line ℓ in R^3 its compressed will be the set of compressed points of ℓ denoted by $\underset{\sqcup}{\ell}$. Considering a plane s in R^3 its compressed will be the set of compressed points of s denoted by $\underset{\sqcup}{s}$. The set $\lambda = \underset{\sqcup}{\ell}$ is called *sub-line* and the set $\sigma = \underset{\sqcup}{s}$ is called *sub-plane*. Clearly, $\lambda \subset \underset{\sqcup}{R}^3$ and $\sigma \subset \underset{\sqcup}{R}^3$. Moreover, the exploded of a sub-line is a line and the exploded of a sub-plane is a plane, that is $\overset{\sqcup}{\lambda} = \ell$ and $\overset{\sqcup}{\sigma} = s$.

By the axioms of the euclidean geometry of R^3 we have the properties of the geometry of $\underbracket{R}^{3}$.

Denoting by $\mathbf{L}$ the set of lines of R^3, by $\mathbf{P}$ the set of planes of R^3, $(R^3, \mathbf{L}, \mathbf{P})$ is a so-called incidence geometry (see [3]). Considering $\underbracket{\mathbf{L}} = \{\underbracket{\ell}\colon\ \ell \in \mathbf{L}\}$ and $\underbracket{\mathbf{P}} = \{\underbracket{s}\colon\ s \in \mathbf{P}\}$, $(\underbracket{R}^{3}, \underbracket{\mathbf{L}}, \underbracket{\mathbf{P}})$ is also an incidence geometry. Now we give the properties of "incidence".

Property 2.1. If X and Y are distinct points of $\underbracket{R}^{3}$ then there exists a sub-line λ that contains both X and Y

Property 2.2. There is only one λ such that $X \in \lambda$ and $Y \in \lambda$.

Property 2.3. Any sub-line has at least two points. There exist at least three points not all in one sub-line.

Property 2.4. If X, Y and Z not are in the same sub-line then there exists a sub-plane σ such that X, Y and Z are in σ. Any sub-plane has a point at least.

Property 2.5. If X, Y and Z are different non sub-collinear points, there is exactly one sub-plane containing them.

Property 2.6. If two points lie in a sub-line, then the line containing them lies in the plane.

Property 2.7. If two sub-planes have a joint point then they have another joint point, too.

Property 2.8. There exists at least four points such that they are not on the same sub-plane.

We will say that the *point Z is between the points X and Y on the sub-line λ if $\overset{\sqcup}{Z}$ is between $\overset{\sqcup}{X}$ and $\overset{\sqcup}{Y}$ on the line $\overset{\sqcup}{\lambda}$*. The concept of "between" has the following properties:

Property 2.9. If Z is between X and Y then X, Y and Z are three different points of a sub-line and Z is between Y and X.

Property 2.10. For any arbitrary point X and Y there exists at least one point Z lying on the sub-line determined by X and Y such that Z is between X and Y.

Property 2.11. For any three points of a sub-line there is only one between the other two.

Property 2.12. (Pasch-type property.) If X, Y and Z are not in the same sub-line and λ is a sub-line of the sub-plane determined by the points X, Y and Z such that λ has not points X, Y or Z but it has a joint point with the sub-segment XY of the sub-line determined by X and Y then λ has a joint point with one of the sub-segmentes XZ or YZ of the sub-lines determined by X and Z or Y and Z, respectively.

We will say that *two sets in $\underbracket{R}^{3}$ are sub-congruent if their explodeds are congruent in the familiar sense.* Let two half-lines be given with the same starting point W and let be U and V their inner points. Let us consider the familiar convex angle $\measuredangle\ UWV$. Compressing this angle we obtain the sub-angle sub $\measuredangle\ \llcorner U \lrcorner \llcorner W \lrcorner \llcorner V \lrcorner$ (or sub-angle sub $\measuredangle XZY$ where $X = \llcorner U \lrcorner$, $Y = \llcorner V \lrcorner$ and $Z = \llcorner W \lrcorner$) with the peak-point $\llcorner W \lrcorner$ and bordered by the sub-half-lines determined by the points $\llcorner W \lrcorner$, $\llcorner U \lrcorner$ and $\llcorner W \lrcorner$, $\llcorner V \lrcorner$. The concept of "sub-congruence" and "sub-angle" have the following properties

Property 2.13. On a given sub-half-line there always exists at least one sub-segment such that one of its end-points is the starting point of the sub-half-line and this sub-segment is sub-congruent with an earlier given sub-segment.

Property 2.14. If both sub-segments p_1 and p_2 are sub-congruent with the sub-segments p_3 then p_1 and p_2 are sub-congruent.

Property 2.15. If sub-segment p_1 is sub-congruent with sub-segment q_1 and p_2 is sub-congruent with q_2 then $p_1 \cup p_2$ is sub-congruent with $q_1 \cup q_2$.

Property 2.16. On a given side of a sub-half-lines there exists only one sub-angle which is sub congruent with a given sub-angle. Each sub angle is sub-congruent with itself.

Property 2.17. Let us consider two sub-triangles. If two sides and sub-angles enclosed by these sides are sub-congruent in the sub-triangles mentioned above then they have another sub-congruent sub-angles.

We say that *the sub-lines λ_1 and λ_2 are sub-parallel if their explodeds $\overset{\sqcup}{\lambda_1}$ and $\overset{\sqcup}{\lambda_2}$ are parallel lines in the familiar sense.* Now we have

Property 2.18. If a sub-line λ_1 and a point X are given such that X is off λ_1 then there exists only one sub-line λ_2 through X that is sub-parallel to λ_1.

Finally, we mention two properties for continuity.

Property 2.19. (Archimedes-type property.) If a point X_1 is between the points X and Y on a sub-line then there are points X_2 $X_3, \ldots, X_n$ such that the sub-segments $X_{i-1}X_i$; $(i = 2, 3, \ldots, n)$ are sub-congruent with sub-segment XX_1 and Y is between points X and X_n.

Property 2.20. (Cantor-type property.) If $\{X_nY_n\}_{n=1}^{\infty}$ is a sequence of sub-segments lying on a sub-line λ such that for any $n = 1, 2, 3, \ldots$, $X_{n+1}Y_{n+1} \subset X_nY_n$ then there exists at least one point Z of λ such that Z belongs to each $X_n\ Y_n$.

To measure the sub-segments and sub-angles we can use the principle of isomorphic expressed by the identities (1.8), (1.9) and (1.20). If the sub-segment p has the end-points X and Y then its sub-measure can be defined as follows:

$$\text{sub meas}\, p = \llcorner \text{meas}\ \overset{\sqcup}{p} \lrcorner, \tag{2.21}$$

where meas $\overset{\sqcup}{p}$ is understood in the traditional sense. Considering that $\overset{\sqcup}{p}$ is a segment bordered by $\overset{\sqcup}{X}$ and $\overset{\sqcup}{Y}$ we have that

$$\text{meas}\ \overset{\sqcup}{p} = D_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{Y}).$$

Hence, by (2.21) and (1.38) we have

$$\text{sub meas}\, p = d_{\underset{\sqcup}{R}^3}(X, Y), \tag{2.22}$$

which is the *sub-distance* of X and Y.

Similarly, to measure sub-angles we write

$$\text{sub meas sub} \measuredangle XZY = \underbracket{\text{meas} \measuredangle \overset{\sqcup}{X}\ \overset{\sqcup}{Z}\ \overset{\sqcup}{Y}} \tag{2.23}$$

where meas $\measuredangle \overset{\sqcup}{X}\ \overset{\sqcup}{Z}\ \overset{\sqcup}{Y}$ is understood in the traditional sense. Using the concept of sub-function again, we obtain that

$$\text{sub arc cos}\, x = \underbracket{\text{arc cos}\ \overset{\sqcup}{x}}, \quad x \in [-\underset{\sqcup}{1}, \underset{\sqcup}{1}]. \tag{2.24}$$

Moreover, we have the following

Theorem 2.25. *If X, Y and Z are given points of $\underset{\sqcup}{R}^3$ such that $X \neq Z$ and $Y \neq Z$ then*

$$\begin{aligned} &\text{sub meas sub} \measuredangle XZY \\ &= \text{sub arc cos}(((X \ominus Z) \odot (Y \ominus Z)) \oslash (d_{\underset{\sqcup}{R}^3}(X, Z) \odot d_{\underset{\sqcup}{R}^3}(Y, Z))). \end{aligned} \tag{2.26}$$

3. Examples for special subsets of $\underset{\sqcup}{R}^3$

Example 3.1. First, we show that the equation

$$X = B \oplus (\tau \odot M), \quad \tau \in \underset{\sqcup}{R} \tag{3.2}$$

where B, M are given points of $\underset{\sqcup}{R}^3$ with the condition

$$\|M\|_{\underset{\sqcup}{R}^3} = \underset{\sqcup}{1} \tag{3.3}$$

represents a sub-line. Really, using (1.8), (1.7) and (1.9) the equation (3.2) yields the equation

$$\overset{\sqcup}{X} = \overset{\sqcup}{B} + t\, \overset{\sqcup}{M}, \quad t = \overset{\sqcup}{\tau} \in R \tag{3.4}$$

which represents a line. Moreovoer, by (1.29) and (0.12) the condition (3.3) means that

$$\| \overset{\sqcup}{M} \|_{R^3} = 1 \tag{3.5}$$

holds. Writing that $B = (b_1, b_2, b_3)$ and $M = (m_1, m_2, m_3)$, the equation (3.2) is equivalent to the equation-system

$$\begin{aligned} x_1 &= b_1 \oplus (\tau \odot m_1) \\ x_2 &= b_2 \oplus (\tau \odot m_2)\,, \quad \tau \in \underset{\sqcup}{R} \\ x_3 &= b_3 \oplus (\tau \odot m_3) \end{aligned}$$

which considering (0.4) and (0.5) can be written in the following form

$$\begin{aligned} x_1 &= \frac{b_1 + \operatorname{th}((\operatorname{ar\,th} \tau)(\operatorname{ar\,th} m_1))}{1 + b_2 \operatorname{th}((\operatorname{ar\,th} \tau)(\operatorname{ar\,th} m_1))} \\ x_2 &= \frac{b_2 + \operatorname{th}((\operatorname{ar\,th} \tau)(\operatorname{ar\,th} m_2))}{1 + b_2 \operatorname{th}((\operatorname{ar\,th} \tau)(\operatorname{ar\,th} m_2))}, \quad (-1 < \tau < 1) \\ x_3 &= \frac{b_3 + \operatorname{th}((\operatorname{ar\,th} \tau)(\operatorname{ar\,th} m_3))}{1 + b_3 \operatorname{th}((\operatorname{ar\,th} \tau)(\operatorname{ar\,th} m_3))}. \end{aligned} \tag{3.6}$$

In the special case $B = \left(0, 0, \frac{1}{2}\right)$ and $M = \left(\operatorname{th} \frac{1}{\sqrt{6}}, \operatorname{th} \frac{1}{\sqrt{6}}, \operatorname{th} \frac{2}{\sqrt{6}}\right)$ then (3.6) is

$$\begin{aligned} x_1 &= \operatorname{th}\left(\frac{1}{\sqrt{6}} \operatorname{ar\,th} \tau\right) \\ x_2 &= \operatorname{th}\left(\frac{1}{\sqrt{6}} \operatorname{ar\,th} \tau\right), \quad -1 < \tau < 1 \\ x_3 &= \frac{1 + 2\operatorname{th}\left(\frac{2}{\sqrt{6}} \operatorname{ar\,th} \tau\right)}{2 + \operatorname{th}\left(\frac{2}{\sqrt{6}} \operatorname{ar\,th} \tau\right)} \end{aligned} \tag{3.7}$$

and the sub-line is shown in the following figure:

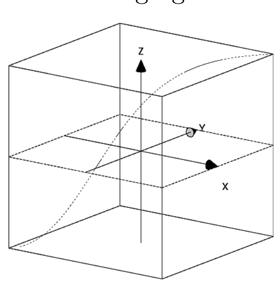

Fig. 3.8

Example 3.9. The sub-line given by the equation-system (3.7) (see Fig. 3.8) and the sub-line given by the equation-system

$$(3.10)\qquad \begin{aligned} x_1 &= \operatorname{th}\left(\frac{1}{\sqrt{6}}\operatorname{ar\,th}\tau\right) \\ x_2 &= \operatorname{th}\left(\frac{1}{\sqrt{6}}\operatorname{ar\,th}\tau\right), \quad -1 < \tau < 1 \\ x_3 &= \operatorname{th}\left(\frac{2}{\sqrt{6}}\operatorname{ar\,th}\tau\right) \end{aligned}$$

are sub-parallel and their graphs are shown in the following figure:

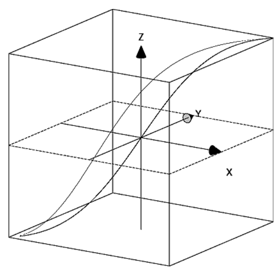

Fig. 3.11

Example 3.12. The sub-line given by the equation system (3.7) (see Fig. 3.8) and the sub-line given by the equation-system

$$(3.13)\qquad \begin{aligned} x_1 &= \operatorname{th}\left(\frac{1}{\sqrt{6}}\operatorname{ar\,th}\tau\right) \\ x_2 &= -\operatorname{th}\left(\frac{1}{\sqrt{6}}\operatorname{ar\,th}\tau\right), \quad -1 < \tau < 1 \\ x_3 &= \frac{1 + 2\operatorname{th}\left(\frac{2}{\sqrt{6}}\operatorname{ar\,th}\tau\right)}{2 + \operatorname{th}\left(\frac{2}{\sqrt{6}}\operatorname{ar\,th}\tau\right)} \end{aligned}$$

has the joint point $B = \left(0, 0, \frac{1}{2}\right)$. Their graphs are shown in the following figure:

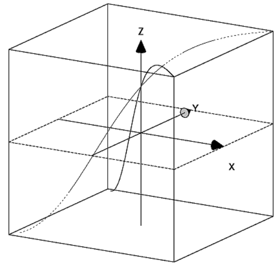

Fig. 3.14

Example 3.15. The sub-lines given by (3.10) (see Fig. 3.11) and (3.13) (see Fig. 3.14) have neither a joint point nor a joint sub-plane. They can be seen in the following figure

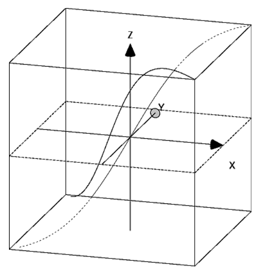

Fig. 3.16

Example 3.17. The equation

$$z = x \oplus y \oplus \frac{1}{2}, \quad x, y \in \underset{\sqcup}{R} \tag{3.18}$$

represents a sub-plane. Really, by (0.11) and (0.2) the computation

$$\begin{aligned} \overset{\sqcup}{z} &= \overset{\sqcup}{x \oplus y \oplus \tfrac{1}{2}} = \overset{\sqcup}{\underset{\sqcup}{(\overset{\sqcup}{x})} \oplus \underset{\sqcup}{(\overset{\sqcup}{y})}} \oplus \underset{\sqcup}{(\overset{\sqcup}{(\tfrac{1}{2})})} = \overset{\sqcup}{\underset{\sqcup}{\overset{\sqcup}{x} + \overset{\sqcup}{y}} \oplus \underset{\sqcup}{(\overset{\sqcup}{(\tfrac{1}{2})})}} \\ &= (\overset{\sqcup}{\underset{\sqcup}{\overset{\sqcup}{x} + \overset{\sqcup}{y} + \overset{\sqcup}{(\tfrac{1}{2})}}}) = \overset{\sqcup}{x} + \overset{\sqcup}{y} + \overset{\sqcup}{(\tfrac{1}{2})} \end{aligned}$$

shows that if (x, y, z) satisfies (3.18) then the points $(\overset{\sqcup}{x}, \overset{\sqcup}{y}, \overset{\sqcup}{z})$ form a plane. By (0.4) the equation (3.18) is equivalent to the

$$z = \frac{xy + 2x + 2y + 1}{2xy + x + y + 2}, \quad -1 < x, y < 1, \tag{3.19}$$

so we have the surface of a sub-plane

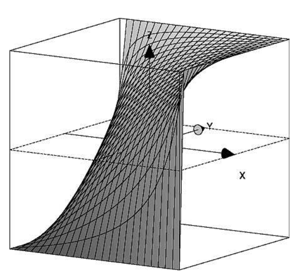

Fig. 3.20

The equation (3.19) shows that the sub-line (3.7) coincides with the sub-plane given by (3.18) . The Fig. 3.20 shows this fact.

The sub-plane determined by the equation

$$z = x \oplus y \left(= \frac{x+y}{1+xy} \right), \quad x, y \in \underline{R} \tag{3.21}$$

is sub-parallel with the sub-plane given by (3.18). Their surfaces are shown in the following figure:

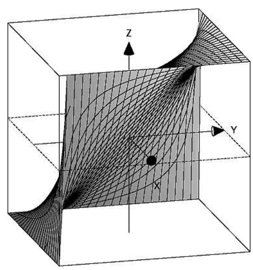

Fig. 3.22

Fig. 3.22 shows that the sub-line (3.10) is on the sub-plane (3.21).

Example 3.23. Considering the set

$$S_{X_0}(\rho) = \{X \in \underline{R}^3 : d_{\underline{R}^3}(X, X_0) = \rho, X_0 \in \underline{R}^3 \quad \text{and} \quad \rho \in \underline{R^+}\} \tag{3.24}$$

by (1.38) and (0.12) we have

$$d_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{X_0}) = \overset{\sqcup}{\rho}, \tag{3.25}$$

that is the points $\overset{\sqcup}{X} \in R^3$ form a sphere with centre $\overset{\sqcup}{X}_0$ and radius $\overset{\sqcup}{\rho}$. Therefore $S_{X_0}(\rho)$ is called a *sub-sphere with centre* X_0 *and radius* ρ. By (1.4), (3.25) and (0.10) we get the equation of sub-sphere

$$\begin{aligned} &(\operatorname{ar\,th} x - \operatorname{ar\,th} x_0)^2 + (\operatorname{ar\,th} y - \operatorname{ar\,th} y_0)^2 + \\ &+ (\operatorname{ar\,th} z - \operatorname{ar\,th} z_0)^2 = (\operatorname{ar\,th} \rho)^2 \end{aligned} \tag{3.26}$$

where $X = (x, y, z)$ and $X_0 = (x_0, y_0, z_0)$ are elements of $\underline{R}^3$.

Although the sub-sphere is determined anambiguously by its centre and radius, its form depends on the place of the centre too. Moreover, it is not symmetrical in a traditional sense for its centre. The following figures show the sub-spheres

$$S_{(\underline{1},\underline{1},\underline{1})}\left(\underline{\left(\tfrac{1}{2}\right)}\right), \; S_{(\underline{1},\underline{1},\underline{1})}(\underline{1}) \quad \text{and} \quad S_{(\underline{1},\underline{1},\underline{1})}\left(\underline{\left(\tfrac{3}{2}\right)}\right)$$

having the equations

$$(\operatorname{ar\,th} x - 1)^2 + (\operatorname{ar\,th} y - 1)^2 + (\operatorname{ar\,th} z - 1)^2 = \frac{1}{4}$$

$$(\operatorname{ar\,th} x - 1)^2 + (\operatorname{ar\,th} y - 1)^2 + (\operatorname{ar\,th} z - 1)^2 = 1$$

and

$$(\operatorname{ar\,th} x - 1)^2 + (\operatorname{ar\,th} y - 1)^2 + (\operatorname{ar\,th} z - 1)^2 = \frac{9}{4},$$

respectively

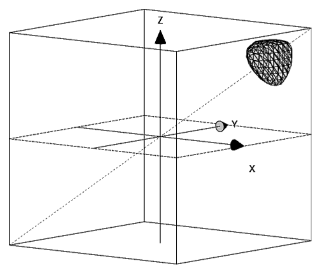

Fig. 3.27

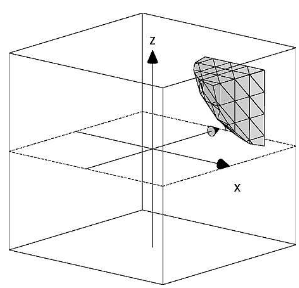

Fig. 3.28

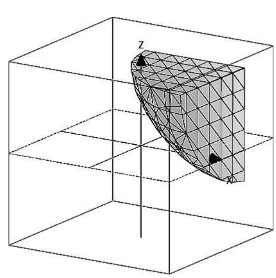

Fig. 3.29

The sub-spheres $S_O(\rho)$ are symmetrical in a traditional sense for their centre o. By (3.26) the sub-sphere $S_0(\rho)$ has the following equation

$$(\operatorname{ar\,th} x)^2 + (\operatorname{ar\,th} y)^2 + (\operatorname{ar\,th} z)^2 = (\operatorname{ar\,th} \rho)^2. \tag{3.30}$$

Considering now the sub-spheres $S_O(\underline{(\frac{1}{2})})$, $S_O(\underline{1})$ and $S_o(\underline{(\frac{3}{2})})$ we obtain their equations by (3.30)

$$(\operatorname{ar\,th} x)^2 + (\operatorname{ar\,th} y)^2 + (\operatorname{ar\,th} z)^2 = \frac{1}{4}$$

$$(\operatorname{ar\,th} x)^2 + (\operatorname{ar\,th} y)^2 + (\operatorname{ar\,th} z)^2 = 1$$

and

$$(\operatorname{ar\,th} x)^2 + (\operatorname{ar\,th} y)^2 + (\operatorname{ar\,th} z)^2 = \frac{9}{4}$$

and they are shown in the following figures:

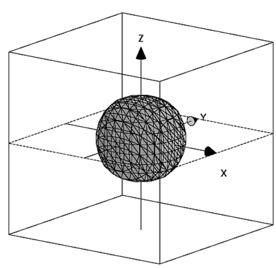

Fig. 3.31

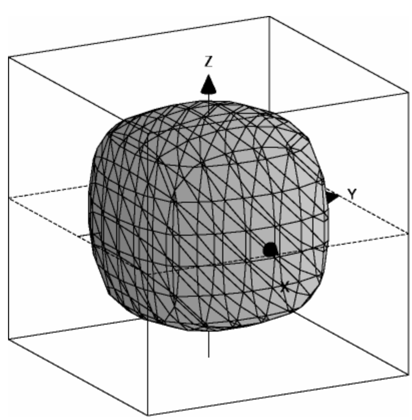

Fig. 3.32

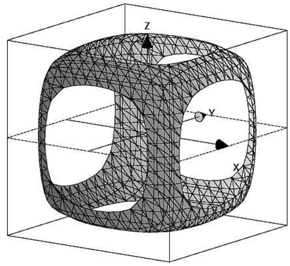

Fig. 3.33

4. Proof of Theorems

4.1. Proof of Theorem 1.10. Considering that the verifications of identitites (1.11)–(1.14) are very similar, we give the proof of identity (1.12), only. After (1.8) and (1.7) we apply the familiar associativity of addition of vectors and using (1.7) and (18) again, we obtain:

$$(X \oplus Y) \oplus Z = \underline{\overset{\sqcup}{X \oplus Y} + \overset{\sqcup}{Z}} = \underline{\overline{(\underline{\overset{\sqcup}{X} + \overset{\sqcup}{Y}})} + \overset{\sqcup}{Z}}$$

$$= \underline{(\overset{\sqcup}{X} + \overset{\sqcup}{Y}) + \overset{\sqcup}{Z}} = \underline{\overset{\sqcup}{X} + (\overset{\sqcup}{Y} + \overset{\sqcup}{Z})} = \underline{\overset{\sqcup}{X} + \overline{(\underline{\overset{\sqcup}{Y} + \overset{\sqcup}{Z}})}}$$

$$= \underline{\overset{\sqcup}{X} + \overset{\sqcup}{Y \oplus Z}} = X \oplus (Y \oplus Z).$$

Considering that the verifications of identities (1.15)–(1.18) are very similar, we give the proof of identity (1.16), only. After (1.19), (1.8), and (1.7) we apply a familiar distributive property of multiplication of vectors by scalar and using (1.9) and (18) again, we get

$$c \odot (X \oplus Y) = \underline{\overset{\sqcup}{c} \oplus \overset{\sqcup}{X \oplus Y}} = \underline{\overset{\sqcup}{c}\,\overline{(\underline{\overset{\sqcup}{X} + \overset{\sqcup}{Y}})}} = \underline{\overset{\sqcup}{c}\,(\overset{\sqcup}{X} + \overset{\sqcup}{Y})}$$

$$= \underline{\overset{\sqcup}{c}\,\overset{\sqcup}{X} + \overset{\sqcup}{c}\,\overset{\sqcup}{Y}} = \underline{\overline{(\underline{\overset{\sqcup}{c}\,\overset{\sqcup}{X}})} + \overline{(\underline{\overset{\sqcup}{c}\,\overset{\sqcup}{Y}})}}$$

$$= \underline{\overset{\sqcup}{c \odot X} + \overset{\sqcup}{c \odot Y}} = (c \odot X) \oplus (c \odot Y).$$

4.2. Proof of Theorem 1.21. The verifications of identities (1.22)–(1.24) are very similar to the verifications mentioned above, so we prove the property (1.25), only. Using (1.20), (1.4) and (0.1)

$$X \odot X = \underline{\overset{\sqcup}{X} \cdot \overset{\sqcup}{X}} = \mathrm{th}((\overset{\sqcup}{x}_1)^2) + (\overset{\sqcup}{x}_2)^2 + (\overset{\sqcup}{x}_3)^2) \geq 0$$

is obtained. Moreover, we have zero if and only if $\overset{\sqcup}{X} = O$ which by (1.4) and (0.10) means that $X = O$.

4.3. Proof of Theorem 1.31. The proof of property (1.32) is very similar to the proof of property (1.25), so we omit it. The identity (1.33) does not need new methods, so we accept it. We prove inequality (1.34). After (1.29), (1.8), (1.7) and (0.1) we apply the Minkowski-inequality and using (0.2) and (1.29), we can write

$$\|X \oplus Y\|_{\underset{\sqcup}{R}^3} = \underset{\sqcup}{\|\overset{\sqcup}{X \oplus Y}\|_{R^3}} = \underset{\sqcup}{\|\overset{\sqcup}{X} + \overset{\sqcup}{Y}\|_{R^3}} \leq$$

$$\leq \underset{\sqcup}{\|\overset{\sqcup}{X}\|_{R^3} + \|\overset{\sqcup}{Y}\|_{R^3}} = \underset{\sqcup}{\|\overset{\sqcup}{X}\|_{R^3}} \oplus \underset{\sqcup}{\|\overset{\sqcup}{Y}\|_{R^3}} = \|X\|_{\underset{\sqcup}{R}^3} \oplus \|Y\|_{\underset{\sqcup}{R}^3}.$$

4.4. Proof of Theorem 1.39. Identity (1.40) is trivial, the verification of property (1.41) is easy, so we omit them. We verify the inequality (1.42), only. After (1.38) we use the triangular inequality and using (0.2) and (1.38) again, we have

$$d_{\underset{\sqcup}{R}^3}(X,Y) = \underset{\sqcup}{d_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{Y})} \leq \underset{\sqcup}{d_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{Z}) + d_{R^3}(\overset{\sqcup}{Z}, \overset{\sqcup}{Y})} =$$

$$= \underset{\sqcup}{d_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{Z})} \oplus \underset{\sqcup}{d_{R^3}(\overset{\sqcup}{Z}, \overset{\sqcup}{Y})} = d_{\underset{\sqcup}{R}^3}(X,Z) \oplus d_{\underset{\sqcup}{R}^3}(Z,Y).$$

4.5. Proof of Theorem 2.25. Our proof is based on the well-known inequality concerning the familiar angles enclosed by vectors. Namely,

$$(U - W)(V - W) = d_{R^3}(U,W) \cdot d_{R^3}(V,W) \cos \measuredangle UWV$$

where $U, V, W \in R^3$ such that $U \neq W$ and $V \neq W$. Hence, denoting by $U = \overset{\sqcup}{X}$, $V = \overset{\sqcup}{Y}$ and $W = \overset{\sqcup}{Z}$, we have

$$\text{meas} \measuredangle \overset{\sqcup}{X}\,\overset{\sqcup}{Z}\,\overset{\sqcup}{Y} = \arccos \frac{(\overset{\sqcup}{X} - \overset{\sqcup}{Z}) \cdot (\overset{\sqcup}{Y} - \overset{\sqcup}{Z})}{d_{R^3}(\overset{\sqcup}{X}, \overset{\sqcup}{Z}) \cdot d_{R^3}(\overset{\sqcup}{Y}, \overset{\sqcup}{Z})}. \tag{4.6}$$

Applying (1.7), (1.8) and (1.37) we have that $\overset{\sqcup}{X} - \overset{\sqcup}{Z} = \overset{\sqcup}{X \ominus Z}$ and $\overset{\sqcup}{Y} - \overset{\sqcup}{Z} = \overset{\sqcup}{Y \ominus Z}$ hold. Hence (1.7) and (1.20) yield

$$(\overset{\sqcup}{X} - \overset{\sqcup}{Z}) \cdot (\overset{\sqcup}{Y} - \overset{\sqcup}{Z}) = \overset{\sqcup}{(X \ominus Z) \odot (Y \ominus Z)}. \tag{4.7}$$

On the other hand, by (0.12), (1.38), (0.12) again, (0.3) and (0.11) we have

$$d_{R^3}(\overset{\sqcup}{X},\overset{\sqcup}{Z})\cdot d_{R^3}(\overset{\sqcup}{Y},\overset{\sqcup}{Z}) = \overline{\underline{(d_{R^3}(\overset{\sqcup}{X},\overset{\sqcup}{Z}))}}\cdot\overline{\underline{(d_{R^3}(\overset{\sqcup}{Y},\overset{\sqcup}{Z}))}}$$

$$= \overline{d_{\underset{\sqcup}{R}^3}(X,Y)}\cdot\overline{d_{\underset{\sqcup}{R}^3}(Y,Z)} = \underline{\overline{(\overline{d_{\underset{\sqcup}{R}^3}(x,Y)}\cdot\overline{d_{\underset{\sqcup}{R}^3}(Y,Z)})}}$$

$$= \overline{\underline{(\overline{d_{\underset{\sqcup}{R}^3}(X,Z)})}\odot\underline{(\overline{d_{\underset{\sqcup}{R}^3}(Y,Z)})}} = \overline{d_{\underset{\sqcup}{R}^3}(X,Z)\odot d_{\underset{\sqcup}{R}^3}(Y,Z)}\,.$$

Hence, by (4.7), (0.12), (0.7) and (0.11) we can write

$$\frac{(\overset{\sqcup}{X}-\overset{\sqcup}{Z})\cdot(\overset{\sqcup}{Y}-\overset{\sqcup}{Z})}{d_{R^3}(\overset{\sqcup}{X},\overset{\sqcup}{Z})\cdot d_{R^3}(\overset{\sqcup}{Y},\overset{\sqcup}{Z})} = \frac{\overline{(X\ominus Z)\odot(Y\ominus Z)}}{\overline{d_{\underset{\sqcup}{R}^3}(X,Y)\odot d_{\underset{\sqcup}{R}^3}(Y,Z)}}$$

$$= \underline{\overline{\left((\frac{\overline{(X\ominus z)\odot(Y\ominus Z)}}{\overline{d_{\underset{\sqcup}{R}^3}(X,Z)\odot d_{\underset{\sqcup}{R}^3}(Y,Z)}})\right)}}$$

$$= \overline{\underline{(\overline{(X\ominus Z)\odot(Y\ominus Z)})}\,\bigodot\,\underline{(\overline{d_{\underset{\sqcup}{R}^3}(X,Z)\odot d_{\underset{\sqcup}{R}^3}(Y,Z)})}}$$

$$= \overline{((X\ominus Z)\odot(Y\ominus Z))\bigodot(d_{\underset{\sqcup}{R}^3}(X,Z)\odot d_{\underset{\sqcup}{R}^3}(Y,Z))}\,.$$

Returning to (4.6) we obtain that

$$\text{meas}\measuredangle\,\overset{\sqcup}{X}\,\overset{\sqcup}{Z}\,\overset{\sqcup}{Y}$$

$$= \arccos\,\overline{((X\ominus Z)\odot(Y\ominus Z))\bigodot(d_{\underset{\sqcup}{R}^3}(X,Z)\odot d_{\underset{\sqcup}{R}^3}(Y,Z))}$$

holds. Hence, (2.23) and (2.24) yield

$$\text{sub meas sub}\measuredangle XZY$$

$$= \underline{\arccos\,\overline{((X\ominus Z)\odot(Y\ominus Z))\bigodot(d_{\underset{\sqcup}{R}^3}(X,Z)\odot d_{\underset{\sqcup}{R}^3}(Y,Z))}}$$

$$= \operatorname{sub}\arccos(((X \ominus Z) \odot (Y \ominus Z)) \odot (d_{\underline{R}^3}(X,Z) \odot d_{\underline{R}^3}(Y,Z))),$$

that is, we have (2.26).

References

[1] Császár Ákos: *Valós analízis.* Tankönyvkiadó, Budapest, 1983.

[2] *Matematikai kislexikon* (Editor in chief M. Farkas), Műszaki Könyvkiadó 3. kiadás, 1979.

[3] Radó Ferenc és Orbán Béla: *A geometriai mai szemmel.* Dacia Kiadó, Kolozsvár, 1981.

[4] Szalay I., On the compressed Descartes-plane and its applications, *AMAPN* **17** (2001), 37–46.

Szalay István
Department for Matehamatics
Juhász Gyula Teacher Training College
University of Szeged
6701 Szeged, P. O. Box 396.
Hungary
E-mail: szalay@jgytf.u-szeged.hu