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NOTE ON RAMANUJAN SUMS

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Dedicated to the memory of Professor Péter Kiss

Abstract. Let $S=\sum_{1\le a\le q}\left|\sum_{\substack{1\le n\le q\\(n,q)=1}} b_n \exp\left(2\pi i\frac{an}{q}\right)\right|^r$, where $r\ge 1$ is a real number, (b_n) is a sequence of complex numbers. Then we obtain a lower and upper bound for S and moreover, we give an application of the Ramanujan sum to produce some identities given in the formulae $(\star\star)$ and (C).

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1. Introduction

Let $f\colon \mathbf{N}\to\mathbf{C}$ be an arithmetic function and let $f^*=\mu * f$ be the Dirichlet convolution of the Möbius function μ and the function f, i.e. $f^*(n)=\sum_{d|n}\mu(d)f\left(\frac{n}{d}\right)$. Moreover, let $c_q(n)=\sum_{\substack{1\le h\le q\\(h,q)=1}}\exp\left(2\pi i\frac{hn}{q}\right)$ be the Ramanujan sum. Then the series of the form: $\sum_q a_q c_q(n)$, where $a_q=\sum_m \frac{f^*(mq)}{mq}$, are called as Ramanujan series. Important result concerning Ramanujan's expansions of certain arithmetic function has been obtained by Delange [3]. Namely, he proved that, if $\sum_n \frac{2^{\omega(n)}}{n}|f^*(n)|<\infty$ then $\sum_q |a_q c_q(n)|<\infty$ for every positive integer n and $\sum_q a_q c_q(n)=f(n)$. In the proof of this result Delange used of the following inequality:

(D)
$$\sum_{d|k}|c_d(n)|\le 2^{\omega(k)}n,$$

where $\omega(n)$ is the number of distinct prime divisors of n. Delange conjectured (see [3]) Lemma, p. 263) that the inequality (D) is the best possible. However, we proved in [4] that for all positive integers k and n the following identity is true

$$(\star) \qquad \sum_{d|k} |c_d(n)| = 2^{\omega\left(\frac{k}{(k,n)}\right)} (k,n)\,,$$

where (k,n) is the greatest common divisor of k and n.

Redmond [10] generalized $(\star)$ for larger class of arithmetic functions and Johnson [7] evaluated the left hand side of $(\star)$ for second variable of the Ramanujan sums. Further investigations connected with $(\star)$ have been given by Johnson [8], Chidambaraswamy and Krishnaiah [2], Redmond [11] and Haukkanen [6]. Some patrial converse to Delange result and an evaluation of the Ramanujan sums defined on the arithmetical semigroups has been given in our paper [5].

In the present note we give further applications of $(\star)$. Namely, we prove the following:

Theorem 1. *Let* $S\,(k,n) = \sum_{d|k} 2^{\omega\left(\frac{d}{(d,n)}\right)} \mu\left(\frac{k}{d}\right)(d,n)$, *then we have*

$$(\star\star) \qquad S\,(k,n) = \frac{\varphi(k)}{\varphi\left(\frac{k}{(k,n)}\right)}, \ \textit{if } p^2 \nmid \frac{k}{(k,n)} \textit{ for a prime } \ p,$$

$$S\,(k,n) = o \quad \textit{otherwise,}$$

where φ is the Euler function.

Now, we denote by S the following sum:

$$(\star\star\star) \qquad S = S_q\,(b_n, r) = \sum_{1\le a\le q} \left| \sum_{\substack{1\le n\le q\\(n,q)=1}} b_n \exp\left(2\pi i \frac{an}{q}\right)\right|^r,$$

where $r \ge 1$ is a real number and (b_n) is a sequence of complex numbers.

In the paper [1] Bachman proved a very interesting inequality for the sum S defined by $(\star\star\star)$, namely

$$(B) \qquad S \ge (\varphi\,(q))^{-r} \left(\left|\sum_n{}' b_n\right|\right)^r \sum_{1\le k\le q} |c_q\,(k)|^r\,,$$

where $\sum_n{}' b_n$ denotes the summation over all natural numbers n such that $1 \le n \le q$ and $(n,q) = 1$. Using (B) and Hölder inequality we prove the following estimation for the sum S.

Theorem 2. *Let $r \geq 1$ be a real number. Then for any sequence (b_n) of complex numbers we have.*

$$\left(2^{\omega(q)}\left|\sum_{\substack{1\leq n\leq q\\(n,q)=1}} b_n\right|\right)^r q^{1-r} \leq S \leq q\,(\varphi(q))^{r-1} \sum_{\substack{1\leq n\leq q\\(n,q)=1}} |b_n|^r\,. \tag{1}$$

Remark. We note that in general the estimation (1) is the best possible. Indeed, putting in (1) $b_n = i$ for all natural number n, $q = 2^\alpha$ with $\alpha = 1$ and $r = 1$, we get $2 \leq S \leq 2$.

Proof of Theorem 1. In the proof of Theorem 1 we use the following well-known Hölder identity (see, [9]):

$$c_k(n) = \frac{\varphi(k)}{\varphi\left(\frac{k}{(k,n)}\right)}\mu\left(\frac{k}{(k,n)}\right), \tag{HI}$$

where $c_k(n)$ is the Ramanujan sum, φ and μ is the Euler and Möbius function, respectively. Let us denote by

$$F(k) = 2^{\omega\left(\frac{k}{(k,n)}\right)}(k,n)\,. \tag{2}$$

Then, if f and F are given multiplicative arithmetical functions then by Möbius inversion formula we have

$$\sum_{d|k} f(d) = F(k) \text{ if and only if } f(k) = \sum_{d|k}\mu\left(\frac{k}{d}\right)F(d)\,. \tag{3}$$

Using (2) we can represent the identity ($\star$) in the form:

$$F(k) = \sum_{d|k} |c_d(n)|\,. \tag{4}$$

Hence, by (4), (3) and ($\star$) we obtain

$$|c_k(n)| = \sum_{d|k} 2^{\omega\left(\frac{d}{(d,n)}\right)}\mu\left(\frac{k}{d}\right)(d,n)\,. \tag{5}$$

On the other hand by (HI) we have

$$|c_k(n)| = \frac{\varphi(k)}{\varphi\left(\frac{k}{(k,n)}\right)}\left|\mu\left(\frac{k}{(k,n)}\right)\right|. \tag{6}$$

Comparing (5) to (6) we get

$$\sum_{d|k} 2^{\omega\left(\frac{d}{(d,n)}\right)} \mu\left(\frac{k}{d}\right)(d,n) = \frac{\varphi(k)}{\varphi\left(\frac{k}{(k,n)}\right)}\left|\mu\left(\frac{k}{(k,n)}\right)\right|. \tag{7}$$

Now, we remark that by the definition of the Möbius function it follows that, with a prime p if $p^2 \mid \frac{k}{(k,n)}$ then $\left|\mu\left(\frac{k}{(k,n)}\right)\right| = 1$ and $\left|\mu\left(\frac{k}{(k,n)}\right)\right| = 0$ if $p^2 \nmid \frac{k}{(k,n)}$. Hence, the proof of Theorem 1 is complete.

From the Theorem 1 immediately follows the following.

Corollary 1. *Let μ denote the Mbius function and let $\omega(d)$ is the number distinct prime divisors of d. Then we have*

$$S(k) = \sum_{d|k} 2^{\omega(d)} \mu\left(\frac{k}{d}\right) = 1, \text{ if } p^2 \nmid k \text{ and } S(k) = 0, \text{if } p^2 \mid k. \tag{C}$$

Proof of Theorem 2. In the proof of Theorem 2 we use of the following

Lemma 1. *Let $a_k \geq 0$ and $r \geq 1$ be real numbers. Then we have*

$$q^{r-1} \sum_{1\leq k\leq q} a_k^r \geq \left(\sum_{1\leq k\leq q} a_k\right)^r. \tag{8}$$

Proof of Lemma 1. Let $r > 1$ and $a_k \geq 0, b_k \geq 0$ be real numbers and $\frac{1}{r} + \frac{1}{s} = 1$. Then by the well-known Hölder's inequality we have

$$\left(\sum_{1\leq k\leq q} a_k^r\right)^{\frac{1}{r}} \left(\sum_{1\leq k\leq q} b_k^s\right)^{\frac{1}{s}} \geq \sum_{1\leq k\leq q} a_k b_k. \tag{H}$$

Putting in the inequality (H) $b_k = 1$ in virtue of $\frac{1}{s} = 1 - \frac{1}{r}$ we obtain (8). For $r = 1$, (8) follows immediately.

Now, we denote by $a_k = |c_q(k)|$, then from (8) we get

$$q^{r-1} \sum_{1\leq k\leq q} |c_q(k)|^r \geq \left(\sum_{1\leq k\leq q} |c_q(k)|\right)^r. \tag{9}$$

On the other hand we can calculate that

$$\sum_{1\le k\le q} |c_q(k)| = 2^{\omega(q)}\varphi(q). \tag{10}$$

Hence, by (10) and Bachman's inequality (B), we obtain

$$S \ge \left(2^{\omega(q)}\left|\sum_{\substack{1\le n\le q\\(n,q)=1}} b_n\right|\right)^r q^{1-r}. \tag{11}$$

It remains to prove the right hand side of (1). In this purpose denote by

$$S_k^{'} = \left|\sum_{\substack{1\le n\le q\\(n,q)=1}} b_n \exp\left(2\pi i\frac{kn}{q}\right)\right|.$$

Then we have

$$S_k^{'} \le \sum_{\substack{1\le n\le q\\(n,q)=1}} |b_n|, \tag{12}$$

and consequently we obtain

$$\left(S_k^{'}\right)^r \le \left(\sum_{\substack{1\le n\le q\\(n,q)=1}} |b_n|\right)^r. \tag{13}$$

In the same way as in Lemma 1 we can deduce the following inequality

$$\left(\sum_{\substack{1\le n\le q\\(n,q)=1}} |b_n|\right)^r \le (\varphi(q))^{r-1}\sum_{\substack{1\le n\le q\\(n,q)=1}} |b_n|^r. \tag{14}$$

From (13), (14) and $(\star\star\star)$ we obtain

$$S = \sum_{1\le k\le q}\left(S_k^{'}\right)^r \le \sum_{1\le k\le q}(\varphi(q))^{r-1}\sum_{\substack{1\le n\le q\\(n,q)=1}} |b_n|^r = q(\varphi(q))^{r-1}\sum_{\substack{1\le n\le q\\(n,q)=1}} |b_n|^r, \tag{15}$$

that is the proof of Theorem 2 is complete.

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