MATHEMATICS COMPETITIONS AND
THEIR ROLE IN EDUCATION

Erika Gyöngyösi (Debrecen, Hungary)

Abstract. The present paper has two messages: mathematics competitions form an important complementary component of mathematical education, at various levels. They should form an important stimulus to mathematical learning, catalysing discussions which pursue the unknown and mysterious, and in general in many cases catalysing an increased love of learning. Secondly, typical errors in the processes of reasoning can be detected after analysing solutions of different problems given by students participating in mathematics competitions.

Introduction

In the first chapter benefits and challenges of mathematics competitons will be discussed. In the second chapter, some examples of actual competitions which have been stunningly successful will be considered to illustrate certain typical errors. It is of considerable use to teachers to be aware of these errors, to determine their reasons, to prevent and correct them. Furthermore, all useful related results should be published in order to contribute to the improvement of the effectivity of the teaching practice. In the final chapter a short summary can be read.

Benefits and challenges of mathematics competitions

Competitions: what are they?

The first thing we wish to do is to extend the concept of competition. Competitions ideed form significant supplement to the mathematics learning process. However there are many further activities related to competitions. There are a number of activities which help students develop their knowledge and increase their interest in learning more. According to Professor Peter Taylor, an Australian mathematics professor who gave a lecture at an international conference titled "Mathematics competitions and related activities and their role in education" between 15 and 17 June in 2001 in Toulouse, France, these activities can include:

— mathematical aspects of problem creation and solution, a dynamic branch of mathematics;

— research in mathematics education related or pertaining to competitions or their types of activities listed here;

— enrichment courses and activities in mathematics;

— mathematics clubs or "circles";

— mathematics camps, including live-in programs in which students solve open-ended or research-style problems over a period of days;

— publication of journals for students and teachers containing problem sections, book review, review articles on historic and contemporary issues in mathematics;

— support for teachers, schools, region and countries who desire to develop local, regional and national competitions.

In the most obvious sense competitions are organized tests with problems which may be new to students and which have to be solved in a given time frame. Whereas the organizers expect students to use normal logical reasoning to solve the problems, sometimes they are multiple choice. Organisations which administer competitions are not normally the schools or their education departments. Usually they are run by independent organizations comprising dedicated teachers. With the help of the participating mathematics professors at the previously mentioned international conference "Mathematics competitions and related activities and their role in education" benefits and challenges have been collected. According to their valuable experiences these objections can be answered as follows.

Benefits and challenges

Advantages

— Competitions are attractive to students of all standard stimulating their interest in mathematics.

— Questions are often set in a real world, rather than pure mathematical environment, with situations to which the sudents can relate.

— These questions automatically test features of mathematics not commonly tested, such as modeling and interaction with language. However underlying mathematical skills are required.

— Competitions promote mathematics in a favourable light and provide teachers with good quality resources.

— A wide numbers of students have access to competitions.

— Competitions fill the gap of the curriculum, providing an opportunity for talented students to appreciate some of the beautiful parts of mathematics, of a more theoretical and structural nature.

— Experience in competitions and related activities will improve the preparation of the student for University study.
— Competitions provide a great wealth of problems. Competition problems usually provoke vigorous discussion and they can often be solved more than one independent way, showing the richness of mathematics.

Challenges
— Some educators argue that competitions are bad because they cause extra pressure on students.
— There are educators who argue that competitions are not egalitarian.
— Are all students treated equally in competition?
— In some competitions there is a penalty for choosing a wrong answer.
— Problems used in competitions are too special and cover difficult topics in mathematics.
— Organizers of multi-choice competitions do not feel totally comfortable with the concept of asking questions in which students do not need to write out an argument.

Answering objections

— Pressure

Why should competitions cause extra pressure on students? We would argue that this is not the case for a number of reasons. First, in most cases, entry is optional. Further, results of competitions do not form part of normal assessment. Therefore competitions should be seen as a positive experience. For most students, competitions are not events in which students compete against other students, either in teams or as individuals (exceptions being in cases of elite students normally). Rather they should be seen as opportunities for individual challenge.
— Egalitarianism

In the past competitions were not egalitarian as they were for the elite only. However, with competitions such as the Hungarian Competitions (Kurschák, Zrínyi, Gordiusz, Arany Dániel, etc.) and the Kangaroo, hopefully this argument no longer applies. These competitions, which are known as "inclusive", are clearly aimed at students of all standards, and certainly students of average ability.
— Equality

This is a more interesting question. The same rules apply to all participants and judges correcting tests do not know the name of students whose tests they are correcting. In the case of multi-choice tests, answers are unambiguous.
— Risk

In the multi-choice tests there is a penalty for choosing a wrong answer. This is rather historical, and difficult to change, but for our formal reason is to discourage guessing. I am not sure that guessing is necessarily bad. This is a difficult argument. It does however reduce the possibility of many perfect scores, not always for the right reasons, and so makes it easier for the judges to determine the best students.

— Problems with topics

There have been changes in the curriculum and the manner in which mathematics is taught, over the past 20 years. The main result for the changes has been that the syllabus has become less theoretical. The new syllabus is based on "discovery", with students armed at all stages with the latest calculator to assist them make their discoveries. The impact of these changes is meant to broaden the appeal of mathematics to a wider group of students. But there is certainly a vacuum developing in the availability of rigourous mathematics for talented students to learn in the schools. Competition organizers, who are generally volunteers working independently of formal education departments, continue to prepare pedagogical material to fill this gap. The material of competitions, sometimes inspired by material in mathematical Olympiads, is evolving and sometimes takes different forms than material in traditional syllabi.

— Multi-choice competitions

Such competitions can be easily marked, thus freeing themselves up to be more accessible problems to the average student. It is true that students do not need to write out an argument. However it is possible to design such questions so that students reall y need to solve the problem directly, rather than guessing.

Illustration of typical errors in the processes of reasoning

In this chapter we consider some problems revealing typical errors of students participating in a Hungarian Competition.[1]

Observing students' results we may conclude the dominant reason of their errors and thus we can classify typical errors as follows.

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[1] The Zrínyi Ilona Mathematics Competition served as a model for the multi-choice mathematics competition organized in February, 1996. Altogether 600 students of secondary education participated in this competition in Baranya county. After its favourable reception a two-round (regional competition and national final) contest called Gordiusz Mathematics Competition was organized in 1997. 2646 competing students from 10 counties solved the problems on the testpapers of the first round. 115 students were invited to the National Final Competition organized in Pécs. In Gordiusz Mathematical Competition each student has to solve 30 problems in 90 minutes. On the testpapers five solutions (A,B,C,D,E) can be found after each problem and only one of them is correct. During the competition, except for a calculat or no other aid (book, ruler, compasses, etc.) can be used. Writing the processes of reasoning is not required. The considerable number of problems makes it possible to realize an overall and complex measure of students' educational standards.

Examples

1. Which formula is equal to

$$(x^{-1} + y^{-1})^{-1}?$$

(A) $x + y$ (B) $\frac{xy}{x+y}$ (C) $x - y$ (D) $(x + y)^{-1}$ (E) $\frac{x+y}{xy}$

Only 29.5% of the participating students chose the right solution (B) to this problem.[2] 39.5% of the students gave the wrong answer (A). 16.4% of them did not solve the problem and the others chose from the other wrong solutions.

The large proportion of students choosing the wrong answer (A) may show that many of them made an **error based on formalism**. Concepts, problems and results of mathematics are often expressed by mathematics signs and formulas. These procedures performed with signs — for example operation mechanisms — will become skills. In such a case content behind mathematics signs is forgotten, the form dominates and the sense of words gets lost. Those who think formalistically recall their memories instead of understanding the essence of problems, try to find a stereotype and consider mathematics as a collection of rules. In this study we confine the concept of formalism to the cases where mathematics forms dominate over the content of it. Certainly, taking a narrow or wide sense of this concept does not change the fact that reasons for errors influence jointly. The method of replacing variables by concrete values proved to be the best way of correcting this type of errors.

2. How many solutions has the following inequality in the set of integers

$$\frac{x^2 - 5x + 6}{x^2 - 8x + 15} < 0?$$

(A) infinitely many (B) 3 (C) 2 (D) 1 (E) 0

20.6% of participating students gave the right answer (D) to this problem.[3] 22% of the students chose (E). 24% of them did not give any answer and the others chose from the other wrong solutions.

This type of **error** may be made out **of habit**. Teachers often experience that students observe things and phenomena which are not taught and in such a case students consider things of no importance as being important and think certain false conclusions — on which they do not doubt and so they do never

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[2] 14th problem posed to students of age 15–16 in the first round of the Gordiusz Mathematics Competition organized in 1997.

[3] 20th problem posed to students of age 16–17 in the Mathematics Competition organized in 1996.

state them during lessons and thus they remain hidden before their teacher —
to be true. These errors are difficult to prevent and correct because it is rather
difficult to give up wrong habits. Lots of patience and long training are needed.
Therefore it is very important to be aware of this type of errors. Another similar
experience is that students consider concepts previously taught as closed systems
and they separate them from new concepts without integrating new concepts into
the system of the previous ones. Certain concepts are defined at various levels of
development of students again and again as their age changes. When teaching a
certain concept it is not necessarily important to define it in the widest domain of
validity because it can be too complex and difficult for students to understand it for
the first sight. So it is more useful to introduce a concept in a simplier way without
giving a definitive definition of it. Although, later on when the same concept is to
be taught at a higher level in its definitive form teachers should give enough time to
understand the enlargement of the concept and to digest the discontinuity between
the restricted and enlarged concepts which finally represent the same concept. It
is very useful to give more time to practise and to show the application of the
concept and to wait until students are able to understand the difference between
different concepts and if so, the teacher can go on with their formation. The case of
expanding the domain of validity implies the following process: we have a property
which is valid for a certain domain A of objects (a statement of the form: $\forall x \in A$,
$p(x)$). We expand the domain A to B $(A \subset B)$ in such a way that $p(x)$ should have
a sense for every x in B. Then we will study if this particular property continues
to be true for each object of the domain B (is it true that $\forall x \in B$, $p(x)$?). Briefly,
we study if the property contin use to be true in a situation for which it was not
established. In the teaching practice this situation occurs very often. In each case
when one expands the domain of validity of a concept he or she should study if
the properties can be defined in the same way as before the expansion. In this
particular case many students made the following error revealing this type of error.
They multiplied both sides of the inequality by the expression in the denominator
without considering that the order of the inequation changes if one multiplies by
a negative number (however in the set of natural numbers this problem does not
arise). Thus these students search the solution of the expression in the numerator
and consider under which conditions it is negative. In these cases illustration is
of considerable importance. Without illustrating figures students will not believe
the correct result which seems an absurdity to them. After the illustration and the
ensuing understanding, long training and lots of practice may be needed in order
to integrate new conceptions in a higher unit.

3. Find the set of roots of the following function:

$$f(x) = \frac{x^2 - 2|x| + 1}{|x| - 1}.$$

(A) 1 (B) -1 (C) 1; -1 (D) 0 (E) $\emptyset$

16.3% of participating students gave the correct answer (E) to this problem.[4]
20% of the students chose (C). Moreover, 50.2% of them did not give any answer
and the others cho se from the other wrong solutions.

This type of errors are made by students **lacking certain preliminary
knowledge**. We do not mean to classify errors owing to learning or teaching
negligence into this class of errors. In such a case the solution is obvious: it
is necessary to fill the gap. But we intend to consider other errors—which are
not the fault of the student or the teacher—made owing to the lack of some
preliminary knowledge. In mathematics lots of things are regulated by "agreements".
It is compulsory to keep them. Mathematicians are familiar with these agreements
however others—such as students—do not know them because even if they learnt
them they were not emphasized enough thus students may easily forget them. In
our particular example one can not divide by zero which means that the function is
not defined for the variables when the denominator is zero. Thus the above problem
has no solution. In order to prevent these errors it is necessary to emphasize these
agreements which may be trivial and obvious to us but not to students and to train
students to keep them whenever it is required.

4. The greatest divisor of a number is itself. How many integers have the
number 91 as its second greatest divisor?

(A) infinitely many (B) 6 (C) 5 (D) 4 (E) 3

Only 5.1% of participating students gave the right answer (D) to this problem.[5]
23.9% of the students chose (A). 53.2% of them did not give any answer and the
others chose from the other wrong solutions.

In our particular case students giving the wrong solution (A) have **the wrong
intuition** that infinitely many integers have the number 91 as its second greatest
divisor. They are surprised to learn that the solution is not "infinity" in this case.
The solution is as follows: $91 = 13 \times 7$. The integers n we want to find are of the
form $91k$. They have $13k$ as divisors. We wish to have $13k \leq 91$. Thus, on the one
hand it is necessary that $k \leq 7$. On the other hand k must be a prime number
since if k is not a prime then k' is the divisor of k and hence $91k'$ will be a "real"
divisor of n which is greate r than 91. Thus, k can only be 7, 5, 3, or 2.

Certain concepts when we study them make the impression that a result should
be true. Sometimes, however, the result turns out to be false. If this conviction
arises the intuitive concept may mislead us. This type of situation has a subjective
characteris tics since on the one hand intuitions vary individually in each person

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[4] 27th problem posed to students of age 15–16 in the first round of Gordiusz Mathematics
Competition organized in 1997.

[5] 25th problem posed to students of age 16–18 in the "Kangourou Étudiants 2000" French
Mathematics Competition oganized in 2000.

on the other hand they depend on how much one is familiar with a concept in question. Giving counter-examples for these preconvinced ideas allows us to rectify this wrong intuition. When teachers correct testpapers of students they often meet this type of situation. Being aware of these false intuition of students teachers can improve the mathematical teaching and learning process if they train students to avoid deliberately these wrong intuitions.

5. Let two numbers x and y be positive integers which has only one common divisor, 1 and $xy = 300$. Which is the smallest possible value for the sum $x + y$?

(A) 30 (B) 35 (C) 37 (D) 56 (E) 79

32.1% of participating students gave the correct answer **(C)** to this problem.[6] 20% and 10.2% of the students chose (A) and (D), respectively and 27.5% of them did not give any answer and the others chose (B).

Students giving wrong answers confused the concept of "common divisor" and that of "one number is divisible by another".

There are **differences between terms of mathematics and those of the everyday language** which may cause difficulties. Mathematical terms may become reasons for errors if their content does not equal to their meaning in the everyday language, or if the term has not other signification apart from a mathematical conception and it is not enough lifelike. Teachers can correct this type of errors by illustrating controversial cases and by giving counter-examples.

6. We know that certain plants are green, red or blue and they may have from 2 to 5 leaves and they may have from 3 to 20 flowers. Minimum how many plants should be selected by accidently so that we could make a bouquet of 11 perfectly identical plants?

(A) 216 (B) 2376 (C) 2160 (D) 2161 (E) 2375

Only 8.9% of participating students gave the correct answer (D) to this problem.[7] 23.3% the students chose (B) and 47.1% of them did not give any answer.

This error reveals that students do not consider the minimum condition of the pigeonhole principle. During lessons when considering different theorems having more conditions, teachers can train students to examine if these **conditions are really necessary conditions**. Thus students should decide if the new theorem derived from the original by suppressing one of the conditions is true or false. In this

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[6] 8th problem posed to students of age 16–18 in the "Kangourou Étudiants 2000" French Mathematics Competition oganized in 2000.

[7] 26th problem posed to students of age 16–18 in the "Kangourou Étudiants 2000" French Mathematics Competition oganized in 2000.

way teachers can train students to learn theorems accurately and to understand them.

Remarks

— Formulas tend to hide the real content of concepts and to create stereotypes. Certainly we need to learn and apply formulas however it is inadmissible to let the real content of concepts sink into insignificance. Errors based on formalism can be considerably reduced by performing a good teaching practice.

— If the dominant reason of an error is a wrong habit then it is easier to prevent these wrong habits than to correct them.

— Errors caused by lacking preliminary knowledge — which are not issued from teaching or learning negligence — as well as those caused by the different usage of mathematical and everyday language cannot be prevented. Nor can be prevented errors whose dominant reason is wrong intuition. In these cases we have to concentrate to correct these errors.

— According to the work of researchers if the teaching practice is of high standard and the teacher trains students to think for themselves then the occurance of typical errors in the processes of reasoning can be considerably reduced. It is necessary to emphasize how important it is to perform the formation and enlargement of concepts thoroughly. If teachers find enough time to do so they will save lots of troubles and dissapointments. Thus, new concepts should be integrated into the previous ones forming one unit without separating different "sets".

— Detecting and correcting an error are two different procedures. The former can be made by a striking observation or by a counter-example and the latter usually demands long and patient work. Generally, the most useful method of correcting errors is illustration. Although it takes relatively long time. The previous estimation of certain results often proves to be better in some cases because it takes less time and it teaches students to think for themselves. We should be careful if we wish to correct an error by two different methods because they may confuse students' thinking.

Summary

The main role of mathematics competitions is to enrich the study of mathematics. These competitions can be especially inspiring for the able student, but materials of all standards can be developed, even allowing the average student to realise that mathematics can be done in a relaxing atmosphere and even be fun and useful in everyday life. Students who can face unexpected situations and solve new problems should be and will be in great demand. This is the strength that we have. Correctly used, competitions train students to meet these challenges. Competitions provide a valuable supplement to classroom teaching by providing

new ideas on mathematics in an environment with less pressure. Competitions are enjoying and have a greater role in mathematical teaching, as the growth of popular competitions in the past few decades shows, and we expect that one of the next phases in their development will be a clearer understanding and articulation of the values they have for the classroom. On the other hand multi-choice tests are easy to correct and enable us to assess results quickly and impartially. Typical errors in the processes of reasoning can be detected, too. Comparative statistics can be made from results of students and significant statistical data can be obtained. These tests can be used in a class in order to assess the frequency of certain typical errors of students taught by one particular teacher. Moreover, groups of students from different schools can be tested. Thus teachers can detect deficiencies of their own teaching practice which enables them to improve it. Furthermore teachers working in different schools of secondary education can share their experiences and work out projects together in order to prevent and correct typical errors. Hence the effectivity of teaching and learning can be improved.

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