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ON PRODUCTS AND SUMS OF THE TERMS OF LINEAR RECURRENCES

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Abstract. For a fixed integer $m\geq 2$, let $\left\{G_n^{(i)}\right\}_{n=0}^{\infty}$ $(1\leq i\leq m)$ be linear recursive sequences of integers, $\Pi_{x_1,x_2,\dots,x_m}=G_{x_1}^{(1)}G_{x_2}^{(2)}\cdots G_{x_m}^{(m)}$ and let $x=\max\limits_{1\leq i\leq m}(x_i)$. In the paper it is proved, under some restrictions, that there are effectively computable constants c and n_0 such that $\left|s-\Pi_{x_1,x_2,\dots,x_m}\right|> e^{c\cdot x}$ if s is an integer having fixed prime factors only, $x>n_0$ and $x_j>\gamma\cdot x$ for any $1\leq j\leq m$ with a fixed real number $0<\gamma<1$. Similar result can be obtained if we replace the pruduct of the terms by their sum.

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1. Introduction

Let the linear recurrences $G^{(i)}=\left\{G_n^{(i)}\right\}_{n=0}^{\infty}$ $(i=1,2,\dots,m;\ m\geq 2)$ of order k_i be defined by the recursion

$$G_n^{(i)}=A_1^{(i)}G_{n-1}^{(i)}+A_2^{(i)}G_{n-2}^{(i)}+\ldots+A_{k_i}^{(i)}G_{n-k_i}^{(i)}\quad (n\geq k_i\geq 2), \tag{1}$$

where the initial values $G_j^{(i)}$ and the coefficients $A_{j+1}^{(i)}$ $(j=0,1,\dots,k_i-1)$ are rational integers. Denote the distinct roots of the characteristic polynomial

$$g^{(i)}(x)=x^{k_i}-A_1^{(i)}x^{k_i-1}-\ldots-A_{k_i}^{(i)} \tag{2}$$

of the sequence $G^{(i)}$ defined in (1) by $\alpha_1^{(i)},\alpha_2^{(i)},\dots,\alpha_{t_i}^{(i)}$ $(t_i\geq 2)$, and suppose that

$$A_{k_i}^{(i)}\left(\left|G_0^{(i)}\right|+\left|G_1^{(i)}\right|+\ldots+\left|G_{k_i-1}^{(i)}\right|\right)\neq 0$$

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for any $i \quad (1 \leq i \leq m)$. It is known that there exist uniquely determined polynomials $p_j^{(i)}(x) \in \mathbf{Q}(\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{t_i}^{(i)})[x] \quad (j = 1, 2, \ldots, t_i)$ of degree less than the multiplicity $m_j^{(i)}$ of the roots $\alpha_j^{(i)}$ such that for $n \geq 0$

$$G_n^{(i)} = p_1^{(i)}(n)\left(\alpha_1^{(i)}\right)^n + p_2^{(i)}(n)\left(\alpha_2^{(i)}\right)^n + \ldots + p_{t_i}^{(i)}(n)\left(\alpha_{t_i}^{(i)}\right)^n . \tag{3}$$

In that special case when $g^{(i)}(x)$ has a dominant root, say $\alpha_i = \alpha_1^{(i)}$, that is, when the multiplicity of α_i is 1 and $|\alpha_i| > \left|\alpha_j^{(i)}\right|$ for $j = 2, 3, \ldots, t_i$, then $|\alpha_i| > 1$, since $\left|A_{k_i}^{(i)}\right| \geq 1$, and $p_1^{(i)}(n)$ in (3) is a constant which will be denoted by a_i. In this case

$$G_n^{(i)} = a_i\left(\alpha_i\right)^n + p_2^{(i)}(n)\left(\alpha_2^{(i)}\right)^n + \ldots + p_{t_i}^{(i)}(n)\left(\alpha_{t_i}^{(i)}\right)^n , \tag{4}$$

where we suppose that $a_i \neq 0$.

We say $G^{(1)}$ to be the dominant sequence among the sequences $G^{(i)}$ $(1 \leq i \leq m)$ if $g^{(1)}(x)$ has a dominant root α_1 and the inequalities $|\alpha_1| > \left|\alpha_j^{(i)}\right|$ hold for any $(i, j) \neq (1, 1)$, where $1 \leq i \leq m$ and $1 \leq j \leq t_i$.

T. N. Shorey and C. L. Stewart [13] investigated the connection between the sequences defined by (4) and perfect powers, then A. Pethő [11], [12] and P. Kiss [6] proved important results in this field. Recently, some similar multiplicative and additive problems have been solved by B. Brindza, K. Liptai and L. Szalay [3], L. Szalay [14], P. Kiss and F. Mátyás [8-9] and F. Mátyás [10]. All of the authors show, under some restrictions, that if a term (product or sum of terms) of linear recurrences is a perfect power then the exponent of the power is bounded above.

The problem is similar when we want to consider those sequences $G^{(i)}$ where the terms of $G^{(i)}$ have given prime factors only. Let $p_1, p_2, \ldots, p_r$ be given distinct rational primes and let

$$S = \left\{s \in \mathbf{Z} \;:\; s = \pm p_1^{e_1} \ldots p_r^{e_r}, \quad 0 \leq e_i \in \mathbf{N}\right\} . \tag{5}$$

K. Győry, P. Kiss and A. Schinzel [4] showed that if G_x is a term of Lucas or Lehmer (special second order) recurrences then

$$G_x \in S \tag{6}$$

holds only for finitely many sequences and finitely many integers x. K. Győry [5] improved this result.

P. Kiss [6] proved that if $G^{(1)}$ is defined by (4) then, under some conditions, $\left|G_x^{(1)} - s\right| > e^{c'x}$ for all integers $s \in S$ and $x > n'$, where c' and n' are effectively computable positive constants depending only on the pimes $p_1, p_2, \dots, p_r$ and $G^{(1)}$.

P. Kiss gave a summation of the results concerning this topic in [7], where among others there were cited two theorems (Theorem 3 and Theorem 6) without proofs hoping that the paper containing the proofs had already appeared. Unfortunately, because of some technical reasons, these proofs can appear only in this paper. So the purpose of this paper is to restate the above theorems and to present their proofs. These theorems generalize and extend the result of P. Kiss [6] for the products (and the sums) of terms of linear recurrences defined by (3) and (4).

2. Results

For brevity we introduce the following abbreviations:

$$\Pi_{x_1,x_2,\dots,x_m} = \prod_{i=1}^{m} G_{x_i}^{(i)} \tag{7}$$

and

$$\Sigma_{x_1,x_2,\dots,x_m} = \sum_{i=1}^{m} G_{x_i}^{(i)}, \tag{8}$$

where $x_1, x_2, \dots, x_m$ are positive integers. The following two theorems will be proved.

Theorem 1. *Let γ be a real number with $0 < \gamma < 1$ and let S be the set of integers defined by (5). Suppose that for any $1 \le i \le m$ the polynomial $g^{(i)}(x)$ defined by (2) has a dominant root $\alpha_i = \alpha_1^{(i)}$ and the sequence $G^{(i)}$ is defined by (4). Then there exist positive real numbers c_0 and n_0 such that if $x = \max_{1 \le i \le m}(x_i) > n_0$,*

$$\prod_{i=1}^{m} a_i \alpha_i^{x_i} \notin S \quad \text{and} \quad x_i > \gamma x \quad \text{for } 1 \le i \le m, \tag{9a and 9b}$$

then

$$|s - \Pi_{x_1,x_2,\dots,x_m}| > e^{c_0 x} \tag{10}$$

for any $s \in S$ and positive integers $x_1, x_2, \dots, x_m$. The constants c_0 and n_0 are effectively computable positive numbers depending only on γ, the primes $p_1, p_2, \dots, p_r$ and the parameters of the sequences $G^{(i)}$ $(1 \le i \le m)$.

Corollary. *Under the conditions of Theorem 1,* $\Pi_{x_1,x_2,\ldots,x_m} \notin S$ *if* $x = \max_{1\le i\le m}(x_i) > n_0$.

Theorem 2. *Let* $G^{(i)}$ $(1 \le i \le m,\ 2 \le m)$ *be sequences defined by (3) if* $2 \le i \le m$ *and by (4) if* $i = 1$ *and let* S *be the set of integers defined by (5). Suppose that* $G^{(1)}$ *is the dominant sequence (with the dominant root* $\alpha_1 = \alpha_1^{(1)}$*) among the sequences* $G^{(i)}$ $(1 \le i \le m)$*. Then there exist positive real numbers* c_1 *and* n_1 *such that if*

$$a_1\alpha_1^{x_1} \notin S \quad \text{and} \quad x_1 > \max_{2\le i\le m}(x_i), \tag{11a and 11b}$$

then

$$|s - \Sigma_{x_1,x_2,\ldots,x_m}| > e^{c_1 x_1}$$

for any $s \in S$ *and positive integers* $x_1, x_2, \ldots, x_m$ *satisfying the condition* $x_1 > n_1$*. The constants* c_1 *and* n_1 *are effectively computable positive numbers depending only on the primes* $p_1, p_2, \ldots, p_r$ *and the parameters of the sequences* $G^{(i)}$ $(1 \le i \le m)$*.*

Corollary. *Under the conditions of Theorem 2,* $\Sigma_{x_1,x_2,\ldots,x_m} \notin S$ *if* $x_1 > n_1$*.*

3. Lemmas and Proofs

To prove the theorems we need the following auxiliary results.

Lemma 1. *Let*

$$\Lambda = \gamma_0 + \gamma_1 \cdot log\omega_1 + \gamma_2 \cdot log\omega_2 + \cdots + \gamma_n \cdot log\omega_n,$$

where the $\gamma's$ *and* $\omega's$ *denote algebraic numbers* ($\omega_i \ne 0$ *or* 1)*. We assume that not all the* $\gamma's$ *are zero and that the logarithms mean their principal values. Suppose that* ω_i *and* γ_i *have heights at most* $M_i(\ge 4)$ *and* $B(\ge 4)$*, respectively, and that the field generated by the* $\omega's$ *and* $\gamma's$ *over the rational numbers has degree at most* d*. If* $\Lambda \ne 0$*, then*

$$|\Lambda| > (B\Omega)^{-C\Omega\cdot\log\Omega'},$$

where

$$\Omega = \log M_1 \cdot \log M_2 \cdots \log M_n, \quad \Omega' = \Omega/\log M_n$$

and $C = (16nd)^{200n}$*. If* $\gamma_0 = 0$ *and* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *are rational integers, then*

$$|\Lambda| > B^{-C\Omega\log\Omega'}.$$

Proof. It is a result of A. Baker [1]. (We mention that this result was improved by A. Baker and G. Wüstholz [2], but we do not calculate the exact values of the constants thus we use only the result of Lemma 1.)

Lemma 2. *Let γ be a real number with $0 < \gamma < 1$, $\Pi_{x_1,x_2,\dots,x_m}$ be an integer defined by (7) and $G^{(i)}$ $(1 \le i \le m,\ 2 \le m)$ be sequences defined by (4), that is, for any $1 \le i \le m$ the polynomial $g^{(i)}(x)$ has a dominant root $\alpha_i = \alpha_1^{(i)}$. If $x_i > \gamma\cdot\max(x_1, x_2, \dots, x_m)$, then there are effectively computable positive constans c_2 and n_2 depending only on the sequences $G^{(i)}$ and γ, such that*

$$\Pi_{x_1,x_2,\dots,x_m} = \left(\prod_{i=1}^{m} a_i\alpha_i^{x_i}\right)(1+\varepsilon),$$

where $|\varepsilon| < e^{-c_2 x}$ for any $x = \max(x_1, x_2, \dots, x_m) > n_2$.

Proof. For the proof see Lemma 2 in [9].

After these lemmas we present the proofs of the theorems. We mention that the constants c_i and n_i $(i \ge 2)$ shall always denote effectively computable positive real numbers depending on γ, the primes $p_1, p_2, \dots, p_r$ and the parameters of the recurrences. One can compute their explicit values similarly as in [9-10].

Proof of Theorem 1. Suppose that the conditions of the theorem are fulfiled and

$$|s - \Pi_{x_1,x_2,\dots,x_m}| \le e^{c_0' x} \tag{12}$$

with a suitable constant $c_0' > 0$ and sufficiently large x. By Lemma 2,

$$\Pi_{x_1,x_2,\dots,x_m} = \left(\prod_{i=1}^{m} a_i\alpha_i^{x_i}\right)(1+\varepsilon), \tag{13}$$

where $|\varepsilon| < e^{-c_2 x}$ if $x > n_2$. On the other hand, by (9b),

$$\left|\prod_{i=1}^{m} a_i\alpha_i^{x_i}\right| = e^{\sum_{i=1}^{m}\log|a_i| + \sum_{i=1}^{m} x_i \log|\alpha_i|} > e^{\sum_{i=1}^{m}\log|a_i| + \gamma x \sum_{i=1}^{m}\log|\alpha_i|} > e^{c_3 x} \tag{14}$$

if $x > n_3$. Using (13) and (14), from (12) we can get the inequalities

$$\left|\frac{s}{\prod_{i=1}^{m} a_i\alpha_i^{x_i}} - (1+\varepsilon)\right| \le \frac{e^{c_0'}}{\prod_{i=1}^{m} |a_i\alpha_i^{x_i}|} < e^{(c_0' - c_3)x} = e^{-c_4 x}$$

if $c_0' < c_3$. From this estimation, with $|\varepsilon| < e^{-c_2 x}$,

$$\left| \frac{s}{\prod_{i=1}^{m} a_i \alpha_i^{x_i}} \right| < 1 + |\varepsilon| + e^{-c_4 x} < 1 + e^{-c_5 x} \tag{15}$$

follows if $x > n_4$, which implies that

$$|s| < \left(1 + e^{-c_5 x}\right) \prod_{i=1}^{m} |a_i \alpha_i^{x_i}| < \left(1 + e^{-c_5 x}\right) e^{\sum_{i=1}^{m} \log |a_i| + x \sum_{i=1}^{m} \log |\alpha_i|} < e^{c_6 x},$$

if $x > n_5$. Since by (5), using the notation $y = \max_{1 \le i \le r} (e_i)$,

$$e^{c_6 x} > |s| = \prod_{i=1}^{r} p_i^{e_i} \ge \prod_{i=1}^{r} 2^{e_i} \ge 2^y = e^{y \cdot \log 2},$$

therefore

$$y = \max_{1 \le i \le r} (e_i) < c_7 x. \tag{16}$$

Let $\lambda = \left| \log \left| \frac{s}{\prod_{i=1}^{m} a_i \alpha_i^{x_i}} \right| \right|$. It is clear by (9a) that $\lambda \neq 0$, while by (15) and the properties of the logarithm function,

$$0 < \lambda < \log\left(1 + e^{-c_5 x}\right) < e^{-c_5 x}. \tag{17}$$

Now we give a lower estimation for

$$\lambda = \left| \sum_{i=1}^{r} e_i \log p_i - \sum_{i=1}^{m} \log |a_i| - \sum_{i=1}^{m} x_i \log |\alpha_i| \right|.$$

Since the numbers p_i, $|a_i|$ and $|\alpha_i|$ are algebraic ones with bounded heights, further on the numbers e_i and x_i are bounded above by $c_7 x$ (see (16)) and x, respectively, thus by Lemma 1

$$\lambda > e^{-c_8 \log x}. \tag{18}$$

(17) and (18) imply that

$$c_5 x < c_8 \log x,$$

that is,

$$\frac{x}{\log x} < c_9,$$

but this is a contradiction if $x > n_6$. Therefore the inequality (10) of the theorem holds with $0 < c_0 < c_3$ and $n_0 = \max_{2\le i\le 6}(n_i)$.

Proof of Theorem 2. Using the estimation

$$|a_1\alpha_1^{x_1}| = e^{\log|a_1| + x_1 \log|\alpha_1|} > e^{c_{10}x_1} \tag{19}$$

if $x_1 > n_7$, suppose that

$$|s - \Sigma_{x_1,x_2,\ldots,x_m}| \le e^{c_1' x_1} \tag{20}$$

with a suitable constant $0 < c_1' < c_{10}$ and sufficiently large x_1.

Using (4) for $G^{(1)}$ and (3) for $G^{(i)}$ $(2 \le i \le m)$, then

$$\Sigma_{x_1,x_2,\ldots,x_m} = a_1\alpha_1^{x_1}\left(1 + \sum_{j=2}^{t_1}\frac{p_j^{(1)}(x_1)}{a_1}\left(\frac{\alpha_j^{(1)}}{\alpha_1}\right)^{x_1} + \sum_{i=2}^{m}\sum_{j=1}^{t_i}\frac{p_j^{(i)}(x_i)}{a_i}\frac{\left(\alpha_j^{(i)}\right)^{x_i}}{\alpha_1^{x_1}}\right) = a_1\alpha_1^{x_1}(1+\varepsilon_1) \tag{21}$$

for any $x_1 > n_8$, where $|\varepsilon_1| < e^{-c_{11}x_1}$, since $x_1 > \max_{2\le i\le m}(x_i)$ and $|\alpha_1| > \left|\alpha_j^{(i)}\right|$ for any $(i,j) \ne (1,1)$. From (20), by (19) and (21), we get that

$$\left|\frac{s}{a_1\alpha_1^{x_1}} - (1+\varepsilon_1)\right| \le \frac{e^{c_1' x_1}}{|a_1\alpha_1^{x_1}|} < \frac{e^{c_1' x_1}}{e^{c_{10}x_1}} = e^{\left(c_1' - c_{10}\right)x_1} = e^{-c_{12}x_1},$$

if $x_1 > n_9$. This implies the inequalities

$$\left|\frac{s}{a_1\alpha_1^{x_1}}\right| < 1 + |\varepsilon_1| + e^{-c_{12}x_1} < 1 + e^{-c_{13}x_1} \tag{22}$$

if $x_1 > n_{10}$. From this we can get

$$|s| < |a_1\alpha_1^{x_1}|\left(1 + e^{-c_{13}x_1}\right) < e^{c_{14}x_1},$$

if $x_1 > n_{11}$. According to (5) and the notation $y = \max_{1\le i\le r}(e_i)$,

$$e^{c_{14}x_1} > |s| = \prod_{i=1}^{r} p_i^{e_i} \ge 2^y = e^{y\log 2},$$

that is,

$$y = \max_{1\le i\le r}(e_i) < c_{15}x_1. \tag{23}$$

Let $\lambda = \left|\log\left|\frac{s}{a_1\alpha_1^{x_1}}\right|\right|$. By (11a), $\lambda \ne 0$. From (22) we can obtain an upper estimation for λ, as follows:

$$0 < \lambda < \log\left(1 + e^{-c_{13}x_1}\right) < e^{-c_{13}x_1}. \tag{24}$$

To construct a lower estimation for λ we apply Lemma 1 for

$$\lambda = \left|\sum_{i=1}^{r} e_i \log p_i - \log|a_1| - x_1\log|\alpha_1|\right|.$$

We can similarly get, as in the proof of Theorem 1, that

$$\lambda > e^{-c_{16}\log x_1}. \tag{25}$$

Making a comparison between (24) and (25), we get

$$c_{13}x_1 < c_{16}\log x_1,$$

from which

$$\frac{x_1}{\log x_1} < c_{17} \tag{26}$$

follows. This proves the theorem, since (26) is a contradiction if $x_1 > n_{12}$, that is, the theorem holds with $0 < c_1 < c_{10}$ and $n_1 = \max_{7\le i\le 12}(n_i)$.

The statements of the corollaries are obvious by the theorems.

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