Acta Acad. Paed. Agriensis, Sectio Mathematicae **28** *(2001) 21–26*

APPROXIMATION BY QUOTIENTS OF TERMS OF SECOND ORDER LINEAR RECURSIVE SEQUENCES OF INTEGERS

Sándor H.-Molnár (Budapest, Hungary)

Abstract. In the paper real quadratic algebraic numbers are approximated by the quotients of terms of appropriate second order recurrences of integers.

AMS Classification Number: 11J68, 11B39

Keywords: Linear recurrences, approximation, quality of approximation.

1. Introduction

Let $G = G(A, B, G_0, G_1) = \{G_n\}_{n=0}^{\infty}$ be a second order linear recursive sequence of rational integers defined by recursion

$$G_n = AG_{n-1} + BG_{n-2} \qquad (n > 1)$$

where A, B and the initial terms G_0, G_1 are fixed integers with restrictions $AB \neq 0$, $D = A^2 + 4B \neq 0$ and not both G_0 and G_1 are zero. It is well-known that the terms of G can be written in form

$$G_n = a\alpha^n - b\beta^n, \tag{1}$$

where α and β are the roots of the characteristic polynomial $x^2 - Ax - B$ of the sequence G and $a = \frac{G_1 - G_0\beta}{\alpha-\beta}$, $b = \frac{G_1 - G_0\alpha}{\alpha-\beta}$ (see e. g. [7], p. 91).

Throughout this paper we assume $|\alpha| \geq |\beta|$ and the sequence is non-degenerate, i. e. α/β is not a root of unity and $ab \neq 0$. We may also suppose that $G_n \neq 0$ for $n > 0$ since in [1] it was proved that a non-degenerate sequence G has at most one zero term and after a movement of indices this condition can be fulfilled.

In the case $D = A^2 + 4B > 0$ the roots of the characteristic polynomial are real, $|\alpha| > |\beta|, (\beta/\alpha)^n \to 0$ as $n \to \infty$ and so by (1) $\lim\limits_{n\to\infty} \frac{G_{n+1}}{G_n} = \alpha$ follows [6]. In [2] and [3] the quality of the approximation of α by quotients G_{n+1}/G_n was considered. In [3] it was proved that if G is a non-degenerate second order linear recurrence with $D > 0$, and c and k are positive real numbers, then

$$\left|\alpha - \frac{G_{n+1}}{G_n}\right| < \frac{1}{c|G_n|^k}$$

holds for infinitely many integer n if and only if
(i) $k < k_0$ and c is arbitrary,
(ii) or $k = k_0$ and $c < c_0$,
(iii) or $k = k_0$, $c = c_0$ and $B > 0$,
(iv) or $k = k_0, c = c_0, B < 0$ and $b/a > 0$,
where $k_0 = 2 - \frac{\log|B|}{\log|\alpha|}$ and $c_0 = \frac{\sqrt{D}^{k_0-3}}{|a|^{k_0-1}|b|}$.

If $D < 0$ then α and β are non real complex numbers with $|\alpha| = |\beta|$ and by (1) we have $\frac{G_{n+1}}{G_n} = \frac{1-(b/a)(\beta/\alpha)^{n+1}}{1-(b/a)(\beta/\alpha)^n}$. But $|\beta/\alpha| = 1$, thus $\lim\limits_{n\to\infty} \frac{G_{n+1}}{G_n}$ does not even exist. The approximation of $|\alpha|$ by rationals of the form $|G_{n+1}/G_n|$ was considered e.g. in [3], [4] and [5]. In [3] it was proved that if G is a non-degenerate second order linear recurrence with $D < 0$ and initial values $G_0 = 0, G_1 = 1$, then there exists a constant $c_1 > 0$, depending only on the sequence G, such that $\left||\alpha| - \left|\frac{G_{n+1}}{G_n}\right|\right| < \frac{c_1}{n}$ for infinitely many n.

In this paper the root α of the characteristic polynom of the sequence G will not be approximated by the quotients G_{n+1}/G_n, but by G_{n+1}/H_n, where H is an appropriately chosen second order linear recursive sequence. We can always give a better approximation for $|\alpha|$ if $D < 0$, and for α in the most cases if $D > 0$ as it was given by the authors in [3]. This can be achieved by the approximation of the numbers of the quadratic number field $\mathbf{Q}(\alpha)$ when $D > 0$. The theorems in [3] can only approximate quadratic algebraic integers. Since at least one real quadratic algebraic integer α can be found for any real quadratic algebraic number γ, such that $\gamma \in \mathbf{Q}(\alpha)$, our theorem can adequately approximate any irrational quadratic algebraic number, independently whether it is an algebraic integer or not. We are going to illustrate the above statement and its applicability to non-real complex quadratic algebraic numbers.

2. Result

We prove the following theorem:

Theorem. *Let A and B be rational integers with the restrictions $AB \neq 0$ and $D = A^2 + 4B > 0$ is not a perfect square. Denote by α and β the roots of equation $x^2 - Ax - B = 0$, where $|\alpha| > |\beta|$. Let $t = \frac{r}{s} + \frac{p}{q}\alpha \in \mathbf{Q}(\alpha)$ with integers $s, q > 0, p \neq 0$ and r. Define the numbers k_0 and c_0 by*

$$k_0 = 2 - \frac{\log|B|}{\log|\alpha|} \quad \textit{and} \quad c_0 = \left|\frac{\sqrt{D}}{qsB}\right|^{k_0-1} \cdot \frac{1}{|psB|}$$

and let k and c be positive real numbers. Then with linear recurrences $G(A, B, qr, psB)$ and $H(A, B, 0, qsB)$ the inequality

$$\left| t - \frac{G_{n+1}}{H_n} \right| < \frac{1}{c|H_n|^k}$$

holds for infinitely many integer n if and only if

(i) $k < k_0$ and c is arbitrary,

(ii) or $k = k_0$ and $c \le c_0$.

(Note that $k_0 > 0$ since $|B| = |\alpha\beta| < \alpha^2$.)

Corollary. *Since $t = \frac{r}{s} + \frac{p}{q}\alpha$ is an irrational number, then*

$$\left| t - \frac{G_{n+1}}{H_n} \right| < \frac{1}{cH_n^2}$$

holds with some $c > 0$ for infinitely many n if and only if $|B| = 1$.

3. Examples

1^{st} **Example.** $t = \alpha$ is a real quadratic algebraic integer. Let $G(4, 19, G_0, G_1)$, where $G_0, G_1 \in Z$ not both G_0 and G_1 are zero. The characteristic equation is $x^2 - 4x - 19 = 0$ and $\alpha = (4 + \sqrt{92})/2$. If approximation is done according to [3], the quality of approximation $k_0 = 2 - \frac{\log 19}{\log \alpha} = 0.4634845713\ldots$

The equation $92 = A^2 + 4B$ can be written in an infinite variety forms: $\ldots, 2^2 + 4 \cdot 22, 4^2 + 4 \cdot 19, 6^2 + 4 \cdot 14, 8^2 + 4 \cdot 7, 10^2 - 4 \cdot 2, 12^2 - 4 \cdot 13, \ldots$. Using $|B|$ of minimum value $\alpha_1 = \frac{10+\sqrt{100-8}}{2} \Rightarrow A = 10, B = -2$, $\alpha \in \mathbf{Q}(\alpha_1), \alpha = \alpha_1 - 3$. $G(10, -2, -3, -2)$, $H(10, -2, 0, -2)$ and thus $k_0 = 2 - \frac{\log|-2|}{\log|\alpha_1|} = 1,696248791\ldots$.

2^{nd} **Example.** t is a real quadratic non-algebraic integer. Let t be the root of larger absolute value of the equation $36x^2 - 894x + 1399 = 0$. The roots of $x^2 - 894x + 36 \cdot 1399 = x^2 - 894x + 50364 = 0$ are α_1 and β_1. Since $t = \frac{1}{36}\alpha_1$, i.e. $t \in \mathbf{Q}(\alpha_1)$, we can approximate t. $k_0 = 2 - \frac{\log|B|}{\log|\alpha_1|} = 0,3902074312\ldots$, $c_0 = 0,002251014\ldots$.

Since $D = 894^2 - 4\cdot 36\cdot 1399 = 2^2\cdot 3^4\cdot(43^2-4)$, $\sqrt{D} = 2\cdot 3^2\cdot\sqrt{(43^2-4)}$, it follows that $t \in \mathbf{Q}(\alpha)$ is also true for the root α of $x^2 - 43x + 1 = 0$. Indeed, $t = \frac{5}{3} + \frac{1}{2}\alpha$ and thus $G(43, -1, 10, -3)$, $H(43, -1, 0, -6)$. If we approximate α by the quotients G_{n+1}/H_n, we get $k_0 = 2, c_0 = 2,386303511\ldots$, and thus $\left| t - \frac{G_{n+1}}{H_n} \right| < \frac{1}{cH_n^2} < \frac{1}{\sqrt{5}H_n^2}$ holds for infinitely many n.

3^{rd} **Example.** t' is a non-real quadratic algebraic integer.
Let $t' = \alpha_1$, where α_1 is the root of $x^2 + 3x + 10 = 0$, i. e.$|\alpha_1| = \left|\frac{-3+i\sqrt{31}}{2}\right| = \sqrt{10}$. Since $|\alpha_1| = \sqrt{10} = \frac{6+\sqrt{36+4}}{2} - 3 \Rightarrow |\alpha_1| \in \mathbf{Q}(\alpha)$, where α is a root of $x^2 - 6x - 1 = 0$ and $|\alpha_1| = \alpha - 3$. Calculating with the sequences $G(6, 1, -3, 1)$ and $H(6, 1, 0, 1)$, $k_0 = 2$ and $c_0 = \sqrt{40}$ and thus $\left||\alpha_1| - \left|\frac{G_{n+1}}{H_n}\right|\right| < \frac{1}{2\sqrt{10}H_n^2}$. This approximation is the best.

4^{th} **Example.** α is a complex, non-algebraic quadratic integer. $4x^2 + 5x + 6 = 0, |\alpha_1| = \left|\frac{-5-\sqrt{25-96}}{8}\right|, |\alpha_1| = \frac{\sqrt{24}}{4} = \frac{1}{2}\frac{4+\sqrt{4^2+4\cdot 2}}{2} - 1 = \frac{1}{2}\alpha - 1$, where α is root of the equation $x^2 - 4x - 2 = 0$. $A = 4, B = 2, G(4, 2, -2, 2)$ and $H(4, 2, 0, 4)$, $k_0 = 2 - \frac{\log|2|}{\log|\alpha|} = 1,535669821\ldots, c_0 = 0,5573569115\ldots$. Calculating with the sequences $G^\star(4, 2, -1, 1)$ and $H^\star(4, 2, 0, 2), k_0^\star = k_0, c_0^\star = 2^{k_0} \cdot c_0 = 1,615905915\ldots$.

Proof of Theorem. By (1) we can write $G_{n+1} = a_1\alpha^{n+1} - b_1\beta^{n+1}$ and $H_n = a\alpha^n - b\beta^n$ for any $n \geq 0$, where

$$a_1 = \frac{G_1 - G_0\beta}{\alpha - \beta} = \frac{psB - qr\beta}{\alpha - \beta}, \qquad b_1 = \frac{psB - qr\alpha}{\alpha - \beta},$$

$$a = \frac{qsB - 0\beta}{\alpha - \beta} = \frac{qsB}{\alpha - \beta}, \qquad b = \frac{qsB}{\alpha - \beta}.$$

Suppose that for an integer $n > 0$ and the positive real numbers c and k we have

$$\left|t - \frac{G_{n+1}}{H_n}\right| < \frac{1}{c|H_n|^k}. \tag{2}$$

Substituting the explicit values of the terms of the sequences and using the equality

$$at - a_1\alpha = \frac{qsB}{\alpha - \beta}\left(\frac{r}{s} + \frac{p}{q}\alpha\right) - \frac{psB - qr\beta}{\alpha - \beta} = 0, \tag{3}$$

$$\left|t - \frac{G_{n+1}}{H_n}\right| = \left|t - \frac{a_1\alpha^{n+1} - b_1\beta^{n+1}}{a\alpha^n - b\beta^n}\right| = \left|\frac{(at - a_1\alpha)\alpha^n - (bt - b_1\beta)\beta^n}{a\alpha^n - b\beta^n}\right|$$
$$= \left|\frac{(bt - b_1\beta)\beta^n}{H_n}\right|$$

follows.

Therefore using the equality $a = b$, inequality (2) can be written in the form

$$1 > c\,|H_n|^k \cdot \left| t - \frac{G_{n+1}}{H_n} \right| = c\,|H_n|^{k-1}\,|(bt - b_1\beta)\beta^n| \tag{4}$$

$$= c\,|a\alpha^n|^{k-1} \left(1 - \frac{b}{a}\left(\frac{\beta}{\alpha}\right)^n\right)^{k-1} \cdot |\beta^n|\,|bt - b_1\beta|$$

$$= c|a|^{k-1}\left(|\alpha|^{k-1}|\beta|\right)^n |bt - b_1\beta| \left|1 - \left(\frac{\beta}{\alpha}\right)^n\right|^{k-1}.$$

Since $\left|\frac{\beta}{\alpha}\right| < 1$ and $\alpha \cdot \beta = -B$, this inequality holds for infinitely many n only if $|\beta||\alpha|^{k-1} = |B||\alpha|^{k-2} < 1$, that is if $k \le 2 - \frac{\log|B|}{\log|\alpha|} = k_0$ and in the case $k = k_0$ we need

$$c \le \frac{1}{|a|^{k_0-1}|bt - b_1\beta|}.$$

By (3) and by $a = b$ it follows that $|bt - b_1\beta| = \left|b\frac{a_1\alpha}{a} - b_1\beta\right| = |a_1\alpha - b_1\beta| = |G_1| = |psB|$.

Therefore using the fact that $\alpha - \beta = \sqrt{D}$

$$c \le \left|\frac{\sqrt{D}}{qsB}\right|^{k_0-1} \cdot \frac{1}{|psB|} = c_0$$

Thus by (4) we obtain that (2) holds for infinitely many n if $k < k_0$ or $k = k_0$ and $c \le c_0$. (If $\frac{\beta}{\alpha} > 0$ then for any sufficiently large n, else for any sufficiently large even n.)

References

[1] Kiss, P., Zero terms in second order linear recurrences, *Math. Sem. Notes (Kobe Univ.)*, **7** (1979), 145-152.

[2] Kiss, P., A Diophantine approximative property of the second order linear recurrences, *Period. Math. Hungar.*, **11** (1980), 281-287.

[3] Kiss, P. and Sinka, Zs., On the ratios of the terms of second order linear recurrences, *Period. Math. Hungar.*, **23** (1991), 139-143.

[4] Kiss, P. and Tichy, R. F., A discrepancy problem with applications to linear recurrences I., *Proc. Japan Acad.* **65**, No. 5 (1989) 135-138.

[5] Kiss, P. and Tichy, R. F., A discrepancy problem with applications to linear recurrences I., *Proc. Japan Acad.* **65**, No. 6 (1989), 191-194.

[6] Mátyás, F., On the quotients of the elements of linear recursive sequences of second order, *Mat. Lapok* **27** (1976/79), 379-389. (In Hungarian)

[7] Niven, I. and Zuckerman, H. S., An introduction to the theory of numbers, Wiley, New York, 1960.

Sándor H.-Molnár
BGF. PSZFK.
Department of Mathematics
Buzogány str. 10.
1149 Budapest, Hungary