# ON VERY POROSITY AND SPACES OF GENERALIZED UNIFORMLY DISTRIBUTED SEQUENCES

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**Abstract.** In the paper the porosity structure of sets of generalized uniformly distributed sequences is investigated in the Baire's space.

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## 1. Introduction and definitions

In [4] the concept of uniformly distributed sequences of positive integers mod $m$ ($m \geq 2$) and uniformly distributed sequences of positive integers in **Z** is introduced (see also [1], p. 305).

We recall the notion of Baire's space $S$ of all sequences of positive integers. This means the metric space $S$ endowed with the metric $d$ defined on $S \times S$ in the following way.

Let $x = (x_n)_1^\infty \in S$, $y = (y_n)_1^\infty \in S$. If $x = y$, then $d(x, y) = 0$ and if $x \neq y$, then

$$d(x,y) = \frac{1}{\min\{n:\ x_n \neq y_n\}}.$$

In [2] is proved that the set of all uniformly distributed sequences of positive integers is a set of the first Baire category in $(S, d)$. In the present paper we shall generalize this result to the space of all real sequences.

Denote by $(s, d)$ the metric space of all sequences of real numbers with $d$ Baire's metric.

In the sequel we use the following well-known result of $H$. Weyl:

**Theorem A.** *The sequence $x = (x_n)_1^\infty \in s$ is uniformly distributed* (mod 1) *if and only if for each integer $h \neq 0$ the equality*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$$

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*holds (cf. [3], p. 7).*

Denote
$$\mathcal{U} = \{x = (x_n)_1^\infty \in s;\ (x_n)_1^\infty \text{ is u. d. mod } 1\},$$
hence from Theorem $A$ we have
$$\mathcal{U} = \left\{ x = (x_n)_1^\infty \in s;\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \text{ for each integers } h \neq 0 \right\}.$$

We now give definitions and notation from the theory of porosity of sets (cf. [5]-[7]). Let $(Y, \varrho)$ be a metric space. If $y \in Y$ and $r > 0$, then denote by $B(y, r)$ the ball with center $y$ and radius $r$, i.e.
$$B(y, r) = \{x \in Y : \varrho(x, y) < r\}.$$
Let $M \subseteq Y$. Put
$$\gamma(y, r, M) = \sup\{t > 0 : \quad \exists_{z\in Y} \quad [B(z, t) \subseteq B(y, r)] \wedge [B(z, t) \cap M = \emptyset]\}.$$
Define the numbers:
$$\bar{p}(y, M) = \lim_{r\to 0_+} \sup \frac{\gamma(y, r, M)}{r}, \quad \underline{p}(y, M) = \lim_{r\to 0_+} \inf \frac{\gamma(y, r, M)}{r}.$$
Obviously the numbers $\bar{p}(y, M)$, $\underline{p}(y, M)$ belong to the interval [0, 1].

A set $M \subseteq Y$ is said to be porous (c-porous) at $y \in Y$ provided that $\bar{p}(y, M) > 0$ ($\bar{p}(y, M) \geq c > 0$). A set $M \subseteq Y$ is said to be $\sigma$-porous ($\sigma$-c-porous) at $y \in Y$ if $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets $M_n$ $(n = 1, 2, \ldots)$ is porous (c-porous) at $y$.

Let $Y_0 \subseteq Y$. A set $M \subseteq Y$ is said to be porous, c-porous, $\sigma$-porous and $\sigma$-c-porous in $Y_0$ if it is porous, c-porous, $\sigma$-porous and $\sigma$-c-porous at each point $y \in Y_0$, respectively.

If $M$ is c-porous and $\sigma$-c-porous at $y$, then it is porous and $\sigma$-porous at $y$, respectively.

Every set $M \subseteq Y$ which is porous in $Y$ is non-dense in $Y$. Therefore every set $M \subseteq Y$ which is $\sigma$-porous in $Y$, is a set of the first category in $Y$. The converse is not true even in $\mathbf{R}$ (cf. [6]).

A set $M \subseteq Y$ is said to be very porous at $y \in Y$ if $\underline{p}(y, M) > 0$ and very strongly porous at $y \in Y$ if $\underline{p}(y, M) = 1$ (cf. [7] p. 327). A set $M$ is said to be very (strongly) porous in $Y_0 \subseteq Y$ if it is very (strongly) porous at each $y \in Y$.

Obviously, if $M$ is very porous at $y$, it is porous at $y$, as well. Analogously, if $M$ is very strongly porous at $y$, it is 1-porous at $y$.

Further, a set $M \subseteq Y$ is said to be uniformly very porous in $Y_0 \subseteq Y$ provided that there is a $c > 0$ such that for each $y \in Y_0$ we have $\underline{p}(y, M) \geq c$ (cf. [7], p. 327). In agreement with the previous terminology and in analogy with the notion of $\sigma$–porosity, we introduce the following notions. A set $M \subseteq Y$ is said to be uniformly $\sigma$–very porous in $Y_0 \subseteq Y$ provided that $M = \bigcup_{n=1}^{\infty} M_n$ and there is a $c > 0$ such that for each $y \in Y_0$ and each $n = 1, 2, \ldots$ we have $\underline{p}(y, M_n) \geq c$.

## 2. Main Result

In this part of the paper we shall study the set of all uniformly distributed (mod 1) sequences in the space $(s, d)$.

Evidently for an integer $h > 0$ we have

$$\mathcal{U} \subset S^{(h)} = \left\{ x = (x_n)_1^\infty \in s; \ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \right\} \subseteq \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} F(k, n)$$

for every $k = 1, 2, \ldots$, where

$$F(k, n) = \left\{ x = (x_n)_1^\infty \in s; \ \left| \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h x_j} \right| \leq \frac{1}{k} \right\}.$$

Denote

$$F^*(k, r) = \bigcap_{n=r}^{\infty} F(k, n) \text{ for } k = 1, 2, \ldots, r = 1, 2, \ldots$$

First, for $f : \mathbf{R} \to \mathbf{R}$ let us denote

$$S^{(h)}(f) = \left\{ x = (x_n)_1^\infty \in s; \ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i h f(x_n)} = 0 \right\}$$

and similarly

$$\mathcal{U}(f) = \left\{ x = (x_n)_1^\infty \in s; \ \ (f(x_n))_1^\infty \text{ is u. d. mod } 1 \right\}.$$

The next theorem implies, that the set $S^{(h)}$ is $\sigma$–very porous in $(s, d)$. (Hence, it follows that $\sigma$–very porous in $\mathcal{U}$ too, see Corollary 2.)

**Theorem.** *Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Then the set $S^{(h)}(f)$ is uniformly $\sigma$–very porous in $(s, d)$.*

**Proof.** For $f:\mathbf{R}\to\mathbf{R}$ and $k=1,2,\dots$ denote by

$$F(f,k,n)=\left\{x=(x_n)_1^\infty\in s;\quad \left|\frac{1}{n}\sum_{j=1}^{n}e^{2\pi ihf(x_j)}\right|\le\frac{1}{k}\right\}.$$

Then we have

$$S^{(h)}(f)\subset\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}F(f,k,n).$$

Let

$$F^*(f,k,r)=\bigcap_{n=r}^{\infty}F(f,k,n).$$

Choose $r\in\mathbf{N}$ fixed. Let $\varepsilon>0$ and $x\in s$. Further let $\delta>0$ be such that $\delta<\frac{1}{r}$. Then there exists a positive integer $l$ such that $\frac{1}{l}\le\delta<\frac{1}{l-1}$, (consequently $l>r$). Obviously $S^{(h)}(f)\subseteq\bigcup\limits_{r=1}^{\infty}F^*\left(f,2+\left[\frac{3}{\varepsilon}\right],r\right)$. Therefore it suffices to prove

$$\underline{p}\left(x,F^*\left(f,2+\left[\frac{3}{\varepsilon}\right],r\right)\right)\ge\frac{1}{2}.$$

Choose a sequence $y\in s$ as follows:

$$y_j=\begin{cases}x_j, & \text{for } j=1,2,\dots,l,\\ b, & \text{for } j>l,\end{cases}$$

where $b$ is constant. Evidently $y\in B\left(x,\frac{1}{l}\right)$ and $B\left(y,\frac{1}{[(2+\varepsilon)l]+1}\right)\subset B\left(x,\frac{1}{l}\right)$. We will show

$$B\left(y,\frac{1}{[(2+\varepsilon)l]+1}\right)\cap F^*\left(f,2+\left[\frac{3}{\varepsilon}\right],r\right)=\emptyset.$$

Let $z\in B\left(y,\frac{1}{[(2+\varepsilon)l]+1}\right)$. Then we have

$$\left|\frac{1}{[(2+\varepsilon)l]+1}\sum_{j=1}^{[(2+\varepsilon)l]+1}e^{2\pi ihf(z_j)}\right|\ge\left|\frac{1}{[(2+\varepsilon)l]+1}\sum_{j=l+1}^{[(2+\varepsilon)l]+1}e^{2\pi ihf(z_j)}\right|$$

$$-\left|\frac{1}{[(2+\varepsilon)l]+1}\sum_{j=1}^{l}e^{2\pi ihf(z_j)}\right|\ge\frac{[(2+\varepsilon)l]+1-l}{[(2+\varepsilon)l]+1}-\frac{1}{[(2+\varepsilon)l]+1}$$

$$>\frac{(2+\varepsilon)l-2l}{[(2+\varepsilon)l]+1}\ge\frac{\varepsilon l}{(2+\varepsilon)l+1}=\frac{\varepsilon}{2+\varepsilon+\frac{1}{l}}>\frac{\varepsilon}{\varepsilon+3}=\frac{1}{\frac{3}{\varepsilon}+1}>\frac{1}{\left[\frac{3}{\varepsilon}\right]+2},$$

thus $z \notin F^*(f, 2 + \left[\frac{3}{\varepsilon}\right], r)$. Then

$$\frac{\gamma(x, \delta, F^*(f, 2 + \left[\frac{3}{\varepsilon}\right], r))}{\delta} \geq \frac{\frac{1}{[(2+\varepsilon)l]+1}}{\frac{1}{l-1}} \geq \frac{l-1}{(2+\varepsilon)l+1},$$

(i.e.)

$$\underline{p}\left(x, F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)\right) \geq \frac{1}{2+\varepsilon}$$

and letting $\varepsilon \to 0$ we obtain the required inequality.

**Remark.** Since the set $F^*(f, 2 + \left[\frac{3}{\varepsilon}\right], r)$ is closed in $s$, for each $x \in s \backslash F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)$ holds

$$p\left(x, F^*\left(f, 2 + \left[\frac{3}{\varepsilon}\right], r\right)\right) = 1.$$

**Corollary 1.** *Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Then the set $U(f)$ is uniformly $\sigma$–very porous in $(s, d)$.*

**Corollary 2.** *The set $S^{(h)}$ is uniformly $\sigma$–very porous in $(s, d)$ for every $h$ positive integers.*

**Proof.** It follows from the fact that the function $f(x) = x$, $x \in \mathbf{R}$.

## References


[1] Kuipers – Niederreiter, H., Uniform distribution of sequences, Wiley, New York, (1974).

[2] László, V. – Šalát, T., Uniformly distributed sequences of positive integers in Baire's space, *Math. Slovaca* **41. no 3** (1991), 277 – 281.

[3] László, V. – Šalát, T., The structure of some sequences spaces, and uniform distribution (mod 1), *Periodica Math. Hung.*, **Vol. 10(1)** (1979), 89 – 98.

[4] Niven, I., Uniform distribution of sequences of integers, *Trans. Amer. Math. Soc.*, **98** (1961), 52 – 61.

[5] Tkadlec, J., Construction of some non–$\sigma$–porous sets of real line, *Real Anal. Exch.*, **9** (1983 – 84), 473 – 482.

[6] Zajíček, L., Sets of $\sigma$–porosity and sets of $\sigma$–porosity $(q)$, *Čas. pěst. mat.*, **101** (1976), 350 – 359.

[7] Zajíček, L., Porosity and $\sigma$–porosity, *Real Anal. Exch.*, **13** (1987 – 88), 314 – 350.

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