*Acta Acad. Paed. Agriensis, Sectio Mathematicae* **28** *(2001) 69–77*

# A SHAPE MODIFICATION OF B-SPLINE CURVES BY SYMMETRIC TRANSLATION OF TWO KNOTS

**Imre Juhász (Miskolc, Hungary)**

**Abstract.** We study the effect of the symmetric translation of knots $u_i$ and $u_{i+2k-3}$ on the shape of B-spline curves. We examine when the points of the $i+k-2^{th}$ arc of the curve move along straight line segment. Quadric and cubic special cases are studied in detail along with the rational case.

## 1. Introduction

B-spline curve and especially its rational counterpart has become a de facto standard for the description of curves in nowadays CAD systems. A rational B-spline curve is uniquely determined by its degree, control points, weights and knot values. The shape of the curve can continuously be modified by means of its control points, weights and knots. The effect of control points and weights along with their shape modification opportunities are well known. There are numerous publications dealing with different aspects of the topic, cf. [1], [2], [5], [7-9]. For the time being, knot-based shape modifications of B-spline curves are not elaborated, in the related comprehensive books only uniform parametrization is emphasized due to its easy to evaluate formulae, but neither the theoretical nor the application issues of knot-based modifications are described.

In this paper we use the usual definition of normalized B-spline functions and curves as follows:

**Definition 1.** The recursive function $N_j^k(u)$ given by the equations

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}) \\ 0 & \text{otherwise} \end{cases},$$
$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u)$$

is called normalized B-spline basis function of order $k$ (degree $k-1$). The numbers $u_j \le u_{j+1} \in \mathbf{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

**Definition 2.** The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\, \mathbf{d}_l \,, u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order $k$ (degree $k-1$), where $N_l^k(u)$ is the $l^{\text{th}}$ normalized B-spline basis function of order $k$, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. Points $\mathbf{d}_i$ are called control points or de Boor points, while the polygon formed by these points is called control polygon.

The $j^{\text{th}}$ arc of this curve has the form

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l,\; u \in [u_j, u_{j+1}),\; (j = k-1, \ldots, n).$$

The modification of knot $u_i$ effects the arcs $\mathbf{s}_j(u)$, $(j = i-k+1,\ i-k+2, \ldots, i+k-2)$. An arbitrarily chosen point of such an arc, corresponding to the parameter value $\tilde{u} \in [u_j, u_{j+1})$ describes the curve

$$\mathbf{s}_j(\tilde{u}, u_i) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(\tilde{u}, u_i),\; u_i \in [u_{i-1}, u_{i+1}]$$

which we refer to as the *path* of the point $\mathbf{s}_j(\tilde{u})$. In [6] we have proved the following theorem.

**Theorem 1.** *Paths $\mathbf{s}_{i-z-1}(\tilde{u}, u_i)$ and $\mathbf{s}_{i+z}(\tilde{u}, u_i)$ are rational curves in $u_i$ of degree $k-z-1$, $(z = 0, 1, \ldots, k-2)$.*

The derivative of these paths for $k = 4$ is studied in [4]. An important corollary of the above theorem, that will be used in this paper as well, is :

**Corollary 1.** *Paths of the arcs $\mathbf{s}_{i-k+1}(\tilde{u}, u_i)$ and $\mathbf{s}_{i+k-2}(\tilde{u}, u_i)$ are straight line segments that are parallel to the sides $\mathbf{d}_{i-k}, \mathbf{d}_{i-k+1}$ and $\mathbf{d}_{i-1}, \mathbf{d}_i$, respectively.*

In this paper we show some interesting properties of B-spline curve modifications obtained by the symmetric translation of knots $u_i$ and $u_{i+2k-3}$.

## 2. Symmetric translation of knots $u_i$ and $u_{i+2k-3}$

We study how points of a B-spline curve move, i.e. what will the paths be like, when knots $u_i$ and $u_{i+2k-3}$ are symmetrically translated. By symmetric translation of knots $u_i$ and $u_j$, $(i<j)$ we mean the $u_i+\lambda$, $u_j-\lambda$, $\lambda\in\mathbf{R}$ type modification. In order to preserve the monotony of knot values $\lambda$ can not take any value but it has to be within the range $[-c,c]$, $c=\min\{u_i-u_{i-1},u_{i+1}-u_i,u_j-u_{j-1},u_{j+1}-u_j\}$. Under the circumstances, the $i+k-2^{\text{th}}$ arc of the B-spline curve, that is effected by both $u_i$ and $u_{i+2k-3}$, has the form

$$
\begin{aligned}
\mathbf{s}_{i+k-2}(u) &= \sum_{l=i-1}^{i+k-2}\mathbf{d}_l N_l^k(u) \\
&= \sum_{l=i+1}^{i+k-4}\mathbf{d}_l N_l^k(u) + \left(N_i^{k-1}(u)+\frac{u_{i+k}-u}{u_{i+k}-u_{i+1}}N_{i+1}^{k-1}(u)\right)\mathbf{d}_i \\
&+ \left(\frac{u-u_{i+k-3}}{u_{i+2k-4}-u_{i+k-3}}N_{i+k-3}^{k-1}(u)+N_{i+k-2}^{k-1}(u)\right)\mathbf{d}_{i+k-3} \\
&+ \frac{u_{i+k-1}-u}{u_{i+k-1}-u_i}N_i^{k-1}(u)\left(\mathbf{d}_{i-1}-\mathbf{d}_i\right) \\
&+ \frac{u-u_{i+k-2}}{u_{i+2k-3}-u_{i+k-2}}N_{i+k-2}^{k-1}(u)\left(\mathbf{d}_{i+k-2}-\mathbf{d}_{i+k-3}\right).
\end{aligned}
\tag{1}
$$

Further on we examine the B-spline curves $\mathbf{s}_{i+k-2}(u,\lambda)$ and their paths obtained by the substitution $u_i=u_i+\lambda$ and $u_{i+2k-3}=u_{i+2k-3}-\lambda$.

**Theorem 2.** *Paths* $\mathbf{s}_{i+k-2}(u,\lambda),\lambda\in[-c,c]$ *are straight line segments, if and only if, the equality* $u_{i+k-1}-u_i=u_{i+2k-3}-u_{i+k-2}$ *is satisfied.*

**Proof.** In expression (1) only the coefficients of the terms $(\mathbf{d}_{i-1}-\mathbf{d}_i)$ and $(\mathbf{d}_{i+k-2}-\mathbf{d}_{i+k-3})$ depend on $\lambda$, the rest of the sum can be considered as a constant translation vector which we denote by $\mathbf{p}$.

(i) If $\delta=u_{i+k-1}-u_i=u_{i+2k-3}-u_{i+k-2}$ then $1/(\delta-\lambda)$ can be factored out, thus we obtain a straight line of the form

$$
\begin{aligned}
\mathbf{s}_{i+k-2}(u,\lambda) = \mathbf{p}+\frac{1}{\delta-\lambda}\Big(&(u_{i+k-1}-u)N_i^{k-1}(u)\left(\mathbf{d}_{i-1}-\mathbf{d}_i\right) \\
&+(u-u_{i+k-2})N_{i+k-2}^{k-1}(u)\left(\mathbf{d}_{i+k-2}-\mathbf{d}_{i+k-3}\right)\Big).
\end{aligned}
$$

(ii) If $u_{i+k-1}-u_i\neq u_{i+2k-3}-u_{i+k-2}$ then the rational curve (1) (in $\lambda$) has two points at infinity, one at $\lambda=u_{i+k-1}-u_i$, and another at $\lambda=u_{i+2k-3}-u_{i+k-2}$, therefore the curve can not be a straight line.

It is worth having a closer look at two special cases of the above theorem.

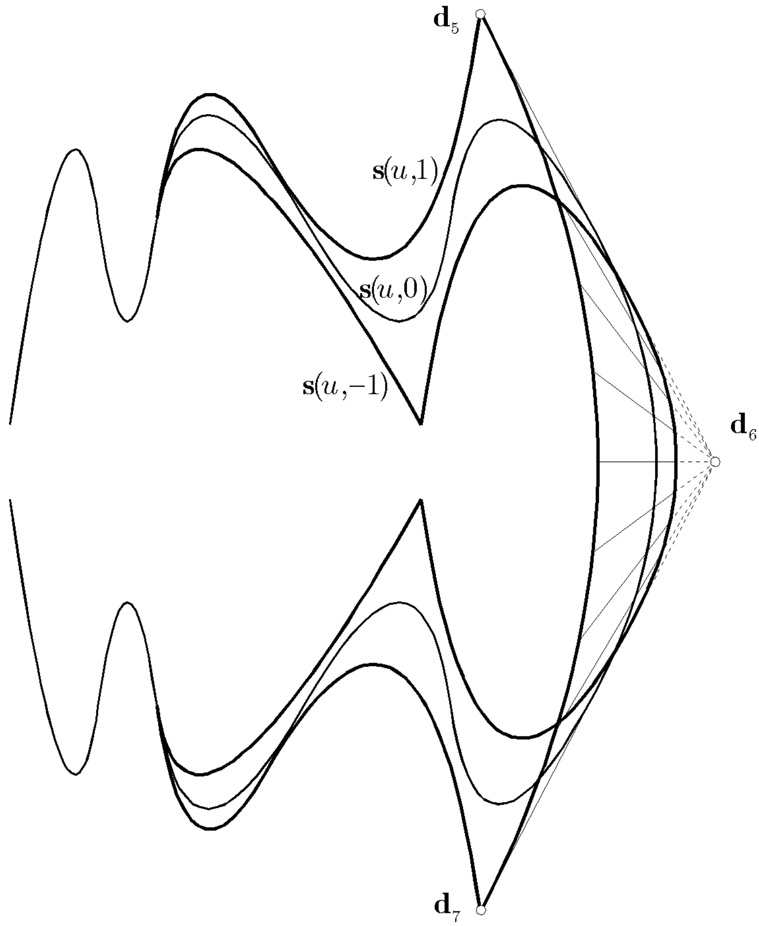


Figure 1: The effect of the symmetric modification of knots $u_i$ and $u_{i+3}$ on the shape of a quadratic B-spline curve, with the indication of paths of the arc $\mathbf{s}_{i+1}(u)$ $(n = 12, k = 3, i = 6, \lambda \in [-1, 1])$.

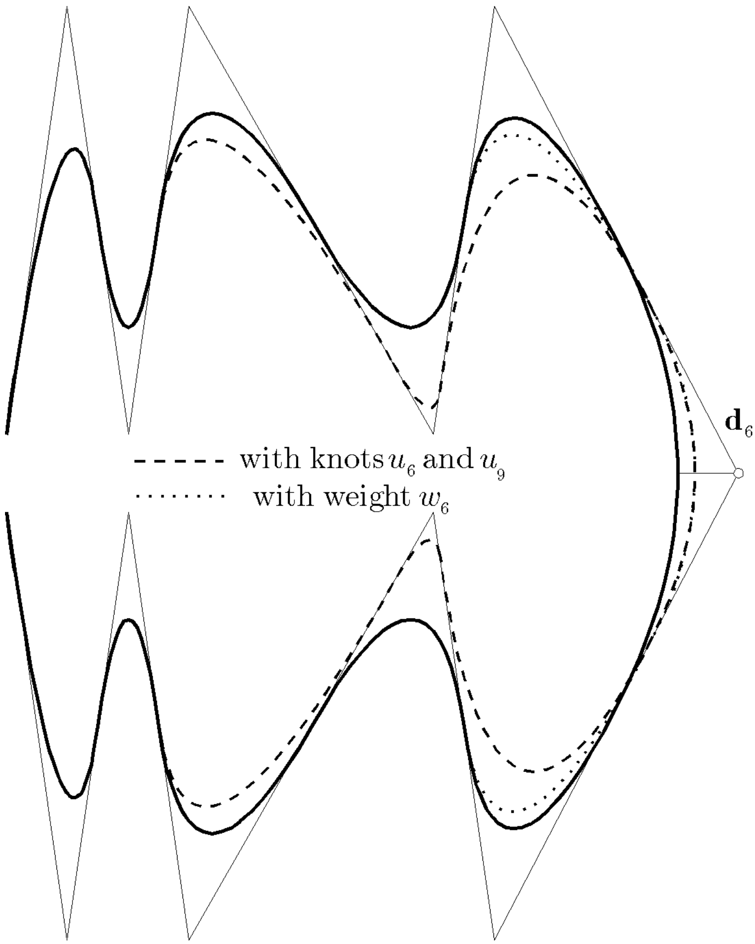


Figure 2: A comparison of two constrained shape modifications gained by the modification of two knots, and by the alteration of a weight ($n = 12, k = 3, i = 6$).

### 2.1. Case $k = 3$

In this case the arc $\mathbf{s}_{i+1}(u, \lambda)$ can be written in the form

$$\mathbf{s}_{i+1}(u, \lambda) = \mathbf{d}_i + \frac{u_{i+2} - u}{u_{i+2} - u_i - \lambda} N_i^2(u)(\mathbf{d}_{i-1} - \mathbf{d}_i) + \frac{u - u_{i+1}}{u_{i+3} - \lambda - u_{i+1}} N_{i+1}^2(u)(\mathbf{d}_{i+1} - \mathbf{d}_i).$$

Obviously, paths of its points are elements of the pencil of lines the base point of which is $\mathbf{d}_i$, provided $u_{i+2} - u_i = u_{i+3} - u_{i+1}$, cf. Fig. 1. (This special case can be found in [3] as well, along with other knot–based shape modifications.)

This means that the symmetric translation of knots $u_i$ and $u_{i+2k-3}$ pulls from / pushes toward the control point $\mathbf{d}_i$ points of the arc $\mathbf{s}_{i+1}(u)$ along straight lines. Therefore this shape modification effect is similar to the one obtained by the alteration of the weigh $w_i$ (the weight of the control point $\mathbf{d}_i$). However, these two shape modification methods are not substitutes of each other. The main difference is that only shape of arcs $\mathbf{s}_i(u), \mathbf{s}_{i+1}(u), \ldots, \mathbf{s}_{i+k-1}(u)$ are modified when $w_i$ is changed, whereas with the symmetric translation of knots $u_i$ and $u_{i+2k-3}$ arcs $\mathbf{s}_{i-k+1}(u), \mathbf{s}_{i-k+2}(u), \ldots, \mathbf{s}_{i+3k-5}(u)$ are modified, i.e. in the latter case a much larger portion of the curve is altered. The difference between these two shape modification methods are illustrated in Fig. 2.

### 2.2. Case $k = 4$

In this case the arc $\mathbf{s}_{i+2}(u, \lambda)$ is of interest which has the form

$$\begin{aligned}
\mathbf{s}_{i+2}(u,\lambda) = & \left( \frac{u_{i+4} - u}{u_{i+4} - u_{i+1}} N_{i+1}^3(u) + N_i^3(u) \right) \mathbf{d}_i \\
& + \left( \frac{u - u_{i+1}}{u_{i+4} - u_{i+1}} N_{i+1}^3(u) + N_{i+2}^3(u) \right) \mathbf{d}_{i+1} \\
& + \frac{u_{i+3} - u}{u_{i+3} - u_i - \lambda} N_i^3(u) (\mathbf{d}_{i-1} - \mathbf{d}_i) \\
& + \frac{u - u_{i+2}}{u_{i+5} - \lambda - u_{i+2}} N_{i+2}^3(u) (\mathbf{d}_{i+2} - \mathbf{d}_{i+1}).
\end{aligned}$$

The coefficients of $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$ are non-negative and sum to 1, i.e. the constant part of the sum is a convex linear combination of the control points $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$. Therefore paths of the arc are straight line segments the extension of which intersect the side $\mathbf{d}_i$, $\mathbf{d}_{i+1}$ at its inner points moreover, they are parallel to the plane determined by the directions $\mathbf{d}_{i-1} - \mathbf{d}_i$ és $\mathbf{d}_{i+2} - \mathbf{d}_{i+1}$, provided $u_{i+3} - u_i = u_{i+5} - u_{i+2}$ holds, cf. Fig. 3. If the directions $\mathbf{d}_{i-1} - \mathbf{d}_i$ and $\mathbf{d}_{i+2} - \mathbf{d}_{i+1}$ are parallel then the paths form a pencil of parallel lines.

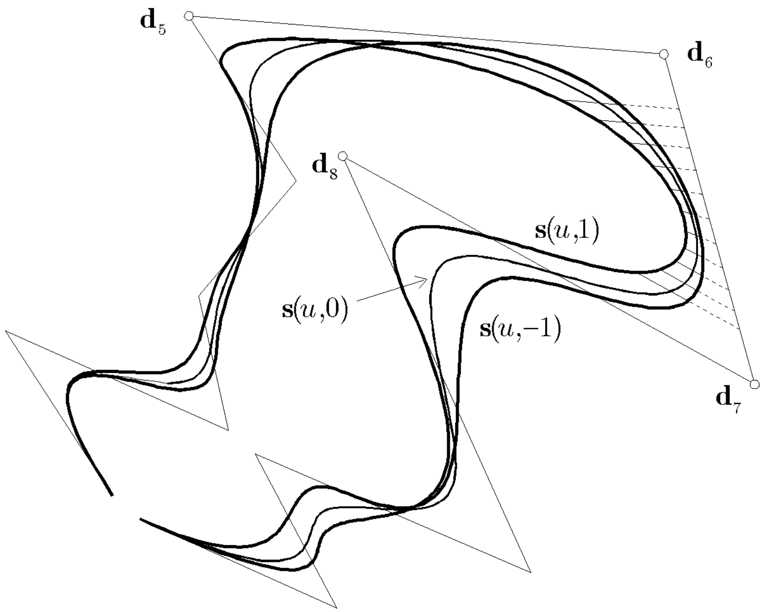


Figure 3: Shape modification of a cubic B-spline curve by means of a symmetric translation of knots $u_i$ and $u_{i+5}$. Paths of points of the arc $\mathbf{s}_{i+2}(u)$ are also shown along with their extensions which intersect the side $\mathbf{d}_6$, $\mathbf{d}_7$ of the control polygon $(n = 12, k = 4, i = 6, \lambda \in [-1, 1])$.

## 3. Rational B-spline curves

The rational equivalents of the results of Section 2 can easily be found utilizing the fact that any rational B-spline (NURBS) curve can be obtained as a central projection of an integral (non-rational) B-spline curve. More precisely, the rational B-spline curve in $\mathbf{R}^d$

$$\mathbf{s}(u) = \sum_{l=0}^{n} w_l \mathbf{d}_l \frac{N_l^k(u)}{\sum_{j=0}^{n} w_j N_j^k(u)}, u \in [u_{k-1}, u_{n+1}]$$

specified by control points $\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_n$, weights $w_0, w_1, \ldots, w_n$ $(w_i \geq 0)$ and knots $(u_0, u_1, \ldots, u_{n+k})$, $(d = 2, 3)$, can be produced by projecting the integral B-spline curve, determined by the same knots and the control points

$$\begin{bmatrix} w_0\mathbf{d}_0 \\ w_0 \end{bmatrix}, \begin{bmatrix} w_1\mathbf{d}_1 \\ w_1 \end{bmatrix}, \ldots, \begin{bmatrix} w_n\mathbf{d}_n \\ w_n \end{bmatrix}$$

in $\mathbf{R}^{d+1}$, from the origin onto the hyperplane $w = 1$.

Thus rational B-spline curves inherit those properties of integral B-spline curves that are invariant under central projection. Therefore Theorem 2 is valid for rational B-spline curves either. Properties of the case $k = 3$ remains valid, since central projection preserves incidence and straight lines. The case $k = 4$ changes in part, since central projection does not preserve parallelism. For this reason, points of the arc $\mathbf{s}_{i+2}(u)$ move along straight line segments, the extension of which intersect the side $\mathbf{d}_i$, $\mathbf{d}_{i+1}$ at its inner points moreover, intersect the line determined by the point of homogeneous coordinates $(w_{i-1}\mathbf{d}_{i-1} - w_i\mathbf{d}_i, w_{i-1} - w_i)$ and $(w_{i+2}\mathbf{d}_{i+2} - w_{i+1}\mathbf{d}_{i+1}, w_{i+2} - w_{i+1})$. (This line is the vanishing line of the plane direction of Subsection 2.2 in this central projection.)

## 4. Conclusions

In this paper we have examined the shape modification effect of the symmetric translation of knots $u_i$ and $u_{i+2k-3}$. We proved that this symmetric translation moves points of the arc $\mathbf{s}_{i+k-2}(u)$ along straight line segments, if and only if, $u_{i+k-1} - u_i = u_{i+2k-3} - u_{i+k-2}$ holds. We studied the $k = 3$ and $k = 4$ special cases in detail, and carried over the results to rational B-spline curves as well. Further research is needed on the simultaneous modification of knots to reveal their shape modification possibilities.

## References

[1] Au, C. K., Yuen, M. M. F., Unified approach to NURBS curve shape modification, *Computer–Aided Design*, **27** (1995), 85–93.

[2] Fowler, B., Bartels, R., Constraint–based curve manipulation, *IEEE Computer Graphics and Applications*, **13** (1993), 43–49.

[3] Hoffmann, M., Juhász, I., Shape Control of Cubic B-spline and NURBS Curves by Knot Modifications, in Banissi, E., Khosrowshahi, F., Srafraz, M., Ursyn, A. (Eds.) Proceedings of the Fifth International Conference on Information Visualisation, 25-27 July 2001, London, England, IEEE Computer Society, 63-68.

[4] Hoffmann, M., On the derivatives of a special family of B-spline curves, *Acta Acad. Paed. Agriensis*, **27**, (2001) (to appear).

[5] Juhász, I., Weight–based shape modification of NURBS curves, *Computer Aided Geometric Design*, **16** (1999), 377–383.

[6] Juhász, I., Hoffmann, M., The effect of knot modifications on the shape of B-spline curves, *Journal for Geometry and Graphics*, (2001) (to appear).

[7] Piegl, L., Modifying the shape of rational B-splines. Part 1: curves, *Computer-Aided Design*, **21** (1989), 509–518.

[8] Piegl, L., Tiller, W., The NURBS book, Springer-Verlag, 1995.

[9] Sánchez-Reyes, J., A simple technique for NURBS shape modification, *IEEE Computer Graphics and Applications*, **17** (1997), 52–59.

**Imre Juhász**
Department of Descriptive Geometry
University of Miskolc
Egyetemváros
H-3515 Miskolc, Hungary
e-mail: agtji@gold.uni-miskolc.hu