THE LIE AUGMENTATION TERMINALS OF GROUPS

Bertalan Király (Eger, Hungary)

Abstract. In this paper we give necessary and sufficient conditions for groups which have finite Lie terminals with respect to commutative ring of non-zero characteristic m, where m is a composite number.

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1. Introduction

Let R be a commutative ring with identity, G a group and RG its group ring and let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the ring homomorphism $\phi : RG \to R$ which maps the group elements to 1. It is easy to see that as R-module $A(RG)$ is a free module with the elements $g-1\,(g \in G)$ as a basis. It is clear that $A(RG)$ is the ideal generated by all elements of the form $g-1$ $(g \in G)$.

The Lie powers $A^{[\lambda]}(RG)$ of $A(RG)$ are defined inductively: $A(RG) = A^{[1]}(RG)$, $A^{[\lambda+1]}(RG) = [A^{[\lambda]}(RG), A(RG)]\cdot RG$, if λ is not a limit ordinal, and $A^{[\lambda]}(RG) = \bigcap_{\nu<\lambda} A^{[\nu]}(RG)$ otherwise, where $[K,M]$ denotes the $R-$submodule of RG generated by $[k,m] = km - mk, k \in K, m \in M$, and for $K \subseteq RG, K\cdot RG$ denotes the right ideal generated by K in RG (similarly $RG\cdot K$ will denote the left ideal generated by K). It is easy to see that the right ideal $A^{[\lambda]}(RG)$ is a two-sided ideal of RG for all ordinals $\lambda \geq 1$. We have the following sequence

$$A(RG) \supseteq A^2(RG) \supseteq \dots$$

of ideals of RG. Evidently there exists the least ordinal $\tau = \tau_R[G]$ such that $A^{[\tau]}(RG) = A^{[\tau+1]}(RG)$ which is called the *Lie augmentation terminal* (or *Lie terminal* for simple) of G with respect to R.

In this paper we give necessary and sufficient conditions for groups which have finite Lie terminal with respect to a commutative ring of non-zero characteristic.

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2. Notations and some known facts

If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short) denotes the ideal of RG generated by all elements of the form $h-1$ $(h \in H)$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi} : RG \to RG/H$ induced by the group homomorphism ϕ of G onto G/H. It is clear that $I(RG) = A(RG)$.

Let F be a free group on the free generators $x_i (i \in I)$, and ZF be its integral group ring (Z denotes the ring of rational integers). Then every homomorphism $\phi : F \to G$ induces a ring homomorphism $\overline{\phi} : ZF \to RG$ by letting $\overline{\phi}(\sum n_y y) = \sum n_y \phi(y)$, where $y \in F$ and the sum runs over the finite set of $n_y y \in ZF$. If $f \in ZF$, we denote by $A_f(RG)$ the two-sided ideal of RG generated by the elements $\overline{\phi}(f)$, $\phi \in \mathrm{Hom}(F, G)$, the set of homomorphism from F to G. In other words $A_f(RG)$ is the ideal generated by the values of f in RG as the elements of G are substituted for the free generators x_i-s.

An ideal J of RG is called a *polynomial ideal* if $J = A_f(RG)$ for some $f \in ZF$, F a free group.

It is easy to see that the augmentation ideal $A(RG)$ is a polynomial ideal. Really, $A(RG)$ is generated as an $R-$module by the elements $g-1$ $(g \in G)$, i.e. by the values of the polynomial $x-1$.

Lemma 2.1. ([2], Corollary 1.9, page 6.) *The Lie powers $A^{[n]}(RG)\,(n \geq 1)$ are polynomial ideals in RG.*

We use the following lemma, too.

Lemma 2.2. ([2] Proposition 1.4, page 2.) *If $f \in ZF$, then f defines a polynomial ideal $A_f(RG)$ in every group ring RG. Further, if $\theta : RG \to KH$ is a ring homomorphism induced by a group homomorphism $\phi : G \to H$ and a ring homomorphism $\psi : R \to K$, then*

$$\theta(A_f(RG)) \subseteq A_f(KH).$$

(It is assumed that $\psi(1_R) = 1_K$, where 1_R and 1_K are identity of the rings R and K, respectively.)

Let $\overline{\theta} : RG \to R/LG$ be an epimorphism induced by the ring homomorphism θ of R onto R/L. By Lemma 2.1 $A^{[n]}(RG) (n \geq 1)$ are polynomial ideal and from Lemma 2.2 it follows that

$$\overline{\theta}(A^{[n]}(RG)) = A^{[n]}(R/LG). \tag{1}$$

Let p be a prime and n a natural number. In this case let's denote by G^{p^n} the subgroup generated by all elements of the form g^{p^n} $(g \in G)$.

If K, L are two subgroups of G, then we denote by (K, L) the subgroup generated by all commutators $(g, h) = g^{-1}h^{-1}gh$, $g \in K$, $h \in L$.

The n^{th} term of the lower central series of G is defined inductively: $\gamma_1(G) = G, \gamma_2(G) = G'$ is the derived group (G, G) of G, and $\gamma_n(G) = (\gamma_{n-1}(G), G)$. The normal subgroups $G_{p,k}$ $(k = 1, 2, \dots)$ is defined by

$$G_{p,k} = \bigcap_{n=1}^{\infty} (G')^{p^n} \gamma_k(G).$$

We have the following sequence of normal subgroups $G_{p,k}$ of a group G

$$G = G_{p,1} \supseteq G_{p,2} \supseteq \ldots \supseteq G_p,$$

where $G_p = \bigcap\limits_{k=1}^{\infty} G_{p,k}$.

In [1] the following theorem was proved.

Theorem 2.1. *Let R be a commutative ring with identity of characteristic p^n, where p a prime number. Then*

1. $\tau_R[G] = 1$ *if and only if* $G = G_p$,
2. $\tau_R[G] = 2$ *if and only if* $G \neq G' = G_p$,
3. $\tau_R[G] > 2$ *if and only if* G/G_p *is a nilpotent group whose derived group is a finite p-group.*

3. The Lie augmentation terminal

It is clear, that if G is an Abelian group, then $A^{[2]}(RG) = 0$. Therefore we may assume that the derived group $G' = \gamma_2(G)$ of G is non-trivial.

We considere the case $\operatorname{char} R = m = p_1^{n_1} p_2^{n_2} \ldots p_s^{n_s} (s \geq 1)$. Let $\Pi(m) = \{p_1, p_2, \ldots, p_s\}$ and $R_{p_i} = R/p_i^{n_i} R$ $(p_i \in \Pi(m))$. If $\overline{\theta}$ is the homomorphism of RG onto $R_{p_i}G$, then by (1)

$$\overline{\theta}(A^{[n]}(RG)) = A^{[n]}(R_{p_i}G) \tag{2}$$

and

$$A^{[n]}(R_{p_i}G) \cong (A^{[n]}(RG) + p_i^{n_i} RG)/p_i^{n_i} RG. \tag{3}$$

Theorem 3.1. *Let G be a non-Abelian group and R be a commutative ring with identity of non-zero characteristic $m = p_1^{n_1} p_2^{n_2} \ldots p_s^{n_s} (s \geq 1)$ Then the Lie augmentation terminal of G with respect to R is finite if and onli if for every $p_i \in \Pi(m)$ one of the following conditions holds:*

1. $G = G_{p_i}$
2. $G \neq G' = G_{p_i}$
3. G/G_{p_i} is a nilpotent group whose derived group is a finite p_i-group.

Proof. Let $p_i \in \Pi(m)$ and let one of the conditions hold: $G = G_{p_i}$ or $G \neq G' = G_{p_i}$ or G/G_{p_i} is a nilpotent group whose derived group is a finite p_i-group. From (2),(3) and Theorem 2.1 it follows, that for every $p_i \in \Pi(m)$ there exists $k_i \geq 1$ such that

$$A^{[k_i]}(R_{p_i}G) = A^{[k_i+1]}(R_{p_i}G) = \dots,$$

where $R_{p_i} = R/p_i^{n_i}R$. If

$$k = \max_{i=1}^{s}\{k_i\},$$

then

$$A^{[k]}(R_{p_i}G) = A^{[k+1]}(R_{p_i}G) = \dots$$

for all $p_i \in \Pi(m)$.

Since $A^{[n]}(R_{p_i}G) \cong (A^{[n]}(RG) + p_i^{n_i}RG)/p_i^{n_i}RG$ for all n and every $p_i \in \Pi(m)$, then from the previous isomorphism it follows, that an arbitrary element $x \in A^{[k]}(RG)$ can be written as

$$x = x_i + p_i^{n_i}a_i,$$

where $x_i \in A^{[k+1]}(RG), a_i \in RG$. If $m_i = m/p_i^{n_i}$, then $m_i x = m_i x_i$ since $m_i p_i^{n_i}$ is zero in R. We have

$$\left(\sum_{p_i \in \Pi(m)} m_i\right) x = \sum_{p_i \in \Pi(m)} m_i x_i.$$

Obviously m_i and $p_i^{n_i}$ are coprime numbers and for all $p_i \in \Pi(m)$ $p_i^{n_i}$ divides m_j for $j \neq i$. Therefore $\sum_{p_i \in \Pi(m)} m_i$ and the characteristic m of the ring R are coprime numbers. Consequently $\sum_{p_i \in \Pi(m)} m_i$ is invertible in R. So

$$x = a \sum_{p_i \in \Pi(m)} m_i x_i,$$

where $a \sum_{p_i \in \Pi(m)} m_i = 1$. Hence $x \in A^{[k+1]}(RG)$ and $x \in A^{[k]}(RG) = A^{[k+1]}(RG)$.

Conversely. Let $\tau_R(G) = n \geq 1$, i.e. $A^{n-1}(RG) \neq A^n(RG) = A^{n+1}(RG) = \dots$. Then for every prime $p_i \in \Pi(m)$

$$A^{k-1} \neq A^{[k]}(R_{p_i}G) = A^{[k+1]}(R_{p_i}G) = \dots$$

holds for a suitable $k \leq n$ and Theorem 2.1 completes the proof.

References

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Bertalan Király
Károly Eszterházy College
Department of Mathematics
H-3300 Eger, Hungary
Leányka str. 4.
e-mail: kiraly@ektf.hu