A NOTE ON THE LOCATION OF ZEROS OF POLYNOMIALS DEFINED BY LINEAR RECURSIONS

Ferenc Mátyás (Eger, Hungary)

Abstract. In this paper it is proved that some earlier results on the location of zeros of polynomials defined by special linear recursions can be improved if the Brauer's theorem is applied instead of the Gershgorin's theorem.

AMS Classification Number: 11B39, 12D10

1. Introduction

Let $n \geq 2$ an integer and define the polynomials $G_n(x)$ by the recursive formula

$$G_n(x) = P(x)G_{n-1}(x) + Q(x)G_{n-2}(x), \tag{1}$$

where the polynomials $P(x), Q(x), G_0(x)$ and $G_1(x)$ are fixed polynomials from $\mathbf{C}[x]$ and at most $G_0(x)$ is the zeropolynomial. If it is needed then we use the notation

$$G_n(P(x), Q(x), G_0(x), G_1(x)) \tag{2}$$

instead of $G_n(x)$. Thus, for example the wellknown Fibonacci $(F_n(x))$ and Chebyshev $(U_n(x))$ polynomials of the second kind can be obtained as

$$F_n(x) = G_n(x, 1, 0, 1) \text{ and } U_n(x) = G_n(2x, -1, 0, 1),$$

respectively.

Recently, we have delt with the location of zeros of polynomials defined by (1), where the polynomials $P(x), Q(x), G_0(x)$ and $G_1(x)$ are special ones (see [3], [4], [5]). If the explicit values of the zeros of polynomials $G_n(x)$ are unknown then one can try to determine such a subset of $\mathbf{C}$ that contains the zeros of $G_n(x)$ for all $n \geq 1$. For example, P. E. Ricci [7] proved that if a complex number z is a zero of the polynomial $G_n(x, 1, 1, x+1)$ for some $n \geq 1$ then $|z| < 2$. In [3] we investigated

Research was sponsored by the Hungarian OTKA Foundation No. T 032898.

the location of zeros of polynomials $G_n(x, 1, c, x + e)$ if $c \neq 0$, and proved that if $z \in \mathbf{C}$ is a zero of these polynomials for some $n \geq 1$ then

$$|z| \leq \max\left(|e| + |c|, 2\right). \tag{3}$$

Similar result was obtained in [5] and for special recursions of order $k \geq 2$ in [4].

To give the location of the zeros of the abovementioned polynomials we applied the wellknown Gershgorin's theorem. But, some papers written by J. Gilewicz and E. Leopold ([1], [2]) suggest that it would be better to apply the Brauer's theorem, since the results are sharper ones. First, see these theorems.

Let $A = (a_{ij})$ be a quadratic matrix of order $n \geq 2$ and $a_{ij} \in \mathbf{C}$. For $1 \leq i \leq n$ let

$$\mathcal{G}_i = \left\{ \omega \in \mathbf{C} \;:\; |\omega - a_{ii}| \leq \sum_{\substack{t=1 \\ t \neq i}}^{n} |a_{it}| \right\} \tag{4}$$

and for $1 \leq i < j \leq n$

$$\mathcal{B}_{ij} = \left\{ \omega \in \mathbf{C} \;:\; |\omega - a_{ii}| \cdot |\omega - a_{jj}| \leq \left(\sum_{\substack{t=1 \\ t \neq i}}^{n} |a_{it}| \right) \left(\sum_{\substack{t=1 \\ t \neq j}}^{n} |a_{jt}| \right) \right\}. \tag{5}$$

Gershgorin's theorem. *All the eigenvalues of A are contained in the set*

$$\mathcal{G} = \bigcup_{i=1}^{n} \mathcal{G}_i.$$

Brauer's theorem. *All the eigenvalues of A are contained in the set*

$$\mathcal{B} = \bigcup_{1 \leq i < j \leq n} \mathcal{B}_{ij}$$

(see [6]).

The purpose of this paper is to obtain a general theorem for the location of the zeros of polynomials defined by (1). Applying the Brauer's theorem we improve the result given in (3).

2. Results

First we need the following lemma.

Lemma. *For every* $n \geq 1$

$$G_n(P(x), Q(x), G_0(x), G_1(x)) = \det(A_n),$$

where A_n *is the following tridiagonal Jacobi matrix of order* n*:*

$$A_n = \begin{pmatrix} G_1(x) & i\sqrt{Q(x)}G_0(x) & 0 & 0 & \dots & 0 & 0 \\ i\sqrt{Q(x)} & P(x) & i\sqrt{Q(x)} & 0 & \dots & 0 & 0 \\ 0 & i\sqrt{Q(x)} & P(x) & i\sqrt{Q(x)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & i\sqrt{Q(x)} & P(x) \end{pmatrix}.$$

Proof. The statement of the Lemma can be obtained by induction on n.

Theorem. *For* $n \geq 2$ *all the zeros of the polynomials*

$$G_n(P(x), Q(x), G_0(x), G_1(x))$$

are located in the sets defined by

$$\text{(6)} \qquad \left\{z \in \mathbf{C} : \, |G_1(z)| \leq \left|\sqrt{Q(z)}G_0(z)\right|\right\} \cup \left\{z \in \mathbf{C} : \, |P(z)| \leq 2\left|\sqrt{Q(z)}\right|\right\}$$

or

$$\text{(7)} \qquad \{z \in \mathbf{C} : \, |G_1(z)P(z)| \leq 2|Q(z)G_0(z)|\} \cup \left\{z \in \mathbf{C} : \, |P(z)| \leq 2\left|\sqrt{Q(z)}\right|\right\}.$$

Proof. It is known that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A_n are the roots of the equation $\det(\lambda I_n - A_n) = 0$, which can be rewritten as

$$\lambda^n + a_{n-1}(x)\lambda^{n-1} + a_{n-2}(x)\lambda^{n-2} + \ldots + a_1(x)\lambda + \det(A_n) = 0,$$

where I_n is the unit matrix of order n and the coefficients $a_i(x)$ of λ^i-s depend on x. Thus for $n \geq 2$, by our Lemma, a complex number z is a zero of the polynomial

$$G_n(P(x), Q(x), G_0(x), G_1(x))$$

iff 0 is an eigenvalue of the tridiagonal matrix A_n. Applying the Gershgorin's theorem and (4) we get that

$$|G_1(z)| \leq \left|\sqrt{Q(z)}G_0(z)\right| \quad \text{or} \quad |P(z)| \leq 2\left|\sqrt{Q(z)}\right|,$$

while according to the Brauer's theorem and (5)

$$|G_1(z)P(z)| \leq 2|Q(z)G_0(z)| \quad \text{or} \quad |P(z)| \leq 2\left|\sqrt{Q(z)}\right|.$$

These prove the theorem.

We note that for $n \geq 1$

$$G_n(P(x), Q(x), 0, G_1(x)) = G_1(x) \cdot G_n(P(x), Q(x), 0, 1) =$$

$$G_1(x) \cdot G_{n-1}(P(x), Q(x), 1, P(x)),$$

thus if z is a zero of the polynomial $G_n(P(x), Q(x), 0, G_1(x))$ then either $G_1(z) = 0$ or z is a zero of the polynomial $G_{n-1}(P(x), Q(x), 1, P(x))$. In the latter case, by our theorem, z satisfies the inequality

$$|P(z)| \leq 2\left|\sqrt{Q(z)}\right|,$$

which matches with a direct consequence of Theorem 1 in [5].

3. Application

In the following part of this paper we shall apply our theorem to give the location of zeros of polynomials $G_n(x, 1, c, x + e)$, where $c, e \in \mathbf{C}$ and $c \neq 0$, since a large class of polynomials $G_n(x)$ can be traced back to this form (see [5]). We have already mentioned that the result (3) can be obtained by the Gershgorin's theorem, thus we demonstrate that the Brauer's theorem (or (7)) gives in general a better estimation for the location of the zeros.

For $n \geq 2$, according to (7), the zeros z of $G_n(x, 1, c, x + e)$ belong to the set

$$\{z \in \mathbf{C} :\ |z + e| \cdot |z| \leq 2|c|\} \cup \{z \in \mathbf{C} :\ |z| \leq 2\}, \tag{8}$$

while by (6), they belong to the set

$$\{z \in \mathbf{C} :\ |z + e| \leq |c|\} \cup \{z \in \mathbf{C} :\ |z| \leq 2\}, \tag{9}$$

from which (3) immediately follows. It can be seen that the zero of $G_1(x) = x + e$ also belongs to the sets (8) and (9), further if a complex number z satisfies (8) then

z also satisfies (9). Therefore, the set defined by (8) can be a narrower one than the set defined by (9).

Let $|e| + |c| \leq 2$. In this case the sets (8) and (9) are equals. Thus (3) cannot be improved, that is, $|z| \leq 2$.

Let $|e| + |c| > 2$. Applying the mapping $\mathbf{C} \longrightarrow \mathbf{C}$ defined by

$$z = |z|(\cos\varphi + i\sin\varphi) \mapsto z' = |z|(\cos(\varphi - \arg(-e)) + i\sin(\varphi - \arg(-e))),$$

the sets (8) and (9) are transformed into the sets

$$\{z' \in \mathbf{C} : \ |z' - |e|| \cdot |z'| \leq 2|c|\} \cup \{z' \in \mathbf{C} : \ |z'| \leq 2\} \tag{10}$$

and

$$\{z' \in \mathbf{C}: |z' - |e|| \leq |c|\} \cup \{z' \in \mathbf{C}: |z'| \leq 2\}, \tag{11}$$

respectively. Without loss of generality it is sufficient to deal with only (10) and (11) since we want to estimate $|z| = |z'|$. Let $z' = x + iy$, where $x, y \in \mathbf{R}$. Then (10) and (11) can be rewritten as

$$\left((x - |e|)^2 + y^2\right)\left(x^2 + y^2\right) \leq 4|c|^2 \ \text{ or } \ x^2 + y^2 \leq 4 \tag{12}$$

and

$$(x - |e|)^2 + y^2 \leq |c|^2 \ \text{ or } \ x^2 + y^2 \leq 4,$$

respectively. Investigating the graph of the implicit function

$$\left((x - |e|)^2 + y^2\right)\left(x^2 + y^2\right) - 4|c|^2 = 0,$$

one can calculate that the graph always intersects the axis x in

$$x_1 = \frac{|e| - \sqrt{|e|^2 + 8|c|}}{2} \ \text{ and } \ x_2 = \frac{|e| + \sqrt{|e|^2 + 8|c|}}{2}, \tag{13}$$

while in the case $|e|^2 \geq 8|c|$ the points

$$x_3 = \frac{|e| - \sqrt{|e|^2 - 8|c|}}{2} \ \text{ and } \ x_4 = \frac{|e| + \sqrt{|e|^2 - 8|c|}}{2} \tag{14}$$

are also intersecting points, and the inequalities

$$0 < -x_1 < x_3 \leq x_4 < x_2$$

hold. Further, if $z' = x + iy$ satisfies (12) and $|e^2| \leq 8|c|$ then

$$|z| = |z'| \leq x_2 < \max(|e| + |c|, 2) = |e| + |c|, \tag{15}$$

while in the case $|e|^2 > 8|c|$

$$|z| = |z^{'}| \le \max(2, x_3) \quad \text{or} \quad x_4 \le |z| = |z^{'}| \le x_2 < |e| + |c|, \tag{16}$$

where x_1, x_2, x_3 and x_4 are defined by (13) and (14). It can be seen that (15) and (16) realy improve (3). (The numerical calculations are omitted in (13)–(16).)

References

[1] Gilewicz, J. & Leopold, E., Location of the zeros of polynomials satisfying three-terms recurrence relation with complex coefficients, *Integral Transforms and Special Functions*, **2** (1994), 267–278.

[2] Gilewicz, J. & Leopold, E., Zeros of polynomials and recurrence relation with periodic coefficients, *Journal of Computational and Applied Math.*, **107** (1999), 241–255.

[3] Mátyás, F., Bound for the zeros of Fibonacci type polynomials, *Acta Acad. Paed. Agriensis Sectio Math.*, **25** (1998), 17–23.

[4] Mátyás, F., On a bound of the zeros of polynomials defined by special linear recurrences of order k, *Rivista di Mat. Univ. Parma*, **6/1** (1998), 173–180.

[5] Mátyás, F., On the location of the zeros of polynomials defined by linear recursions, *Publ. Math. Debrecen*, **55/3–4** (1999), 453–464.

[6] Parodi, M., La localisation des valeurs caractéristiques des matrices et ses applications, Gauthier Villars, Paris, 1959.

[7] Ricci, P. E., Generalized Lucas polynomials and Fibonacci polynomials, *Rivista di Mat. Univ. Parma*, **4/5** (1995), 137–146.

Ferenc Mátyás
Institute of Mathematics and Informatics
Eszterházy Károly College
Leányka str. 4.
H-3300 Eger, Hungary
e-mail: matyas@ektf.hu