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ON POLYNOMIAL VALUES OF THE PRODUCT OF THE TERMS OF LINEAR RECURRENCE SEQUENCES

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Abstract: Let G and H be linear recurrence sequences and let $F(x) = dx^q + d_p x^p + d_{p-1}x^{p-1} + \cdots + d_0$, where d and d_i's are rational integers, be a polynomial. In this paper we showed that for the equation $G_n H_m = F(x)$, with some restriction, there are no solutions in n, m and x if $q > q_0$, where q_0 is an effectively computable positive constant.

1. Introduction

Let $G = \{G_n\}_{n=0}^{\infty}$ be a linear recursive sequence of order k (≥ 2) defined by

$$G_n = A_1 G_{n-1} + \cdots + A_k G_{n-k} \quad (n \geq k),$$

where $G_0, G_1, \ldots, G_{k-1}, A_1, A_2, \ldots, A_k$ are rational integer constants. We need an other sequence, too. Let $H = \{H_n\}_{n=0}^{\infty}$ be another linear recurrence of order l defined by

$$H_n = B_1 H_{n-1} + \cdots + B_l H_{n-l} \quad (n \geq l),$$

where the initial terms $H_0, H_1, \ldots, H_{l-1}$ and the B_i's are given rational integers. We suppose that $A_k \neq 0$, $B_l \neq 0$, and that the initial values of both sequences are not all zero.

Denote the distinct zeros of the characteristic polinomial

$$g(x) = x^k - A_1 x^{k-1} - \cdots - A_k$$

by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$, and similarly let $\beta = \beta_1, \beta_2, \ldots, \beta_t$ be the distinct zeros of the polinomial

$$h(x) = x^l - B_1 x^{l-1} - \cdots - B_l.$$

We suppose that $s > 1$, $t > 1$ and $|\alpha| = |\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq \cdots \geq |\alpha_s|$ and $|\beta| = |\beta_1| > |\beta_2| \geq |\beta_3| \geq \cdots \geq |\beta_t|$. Consequently, we have $|\alpha| > 1, |\beta| > 1$. Assume that α and β have multiplicity 1 in the characteristic polynomials. As it is known the terms of the sequences G and H can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \cdots + r_s(n)\alpha_s^n \quad (n \geq 0), \tag{1}$$

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and

$$H_n = b\beta^n + q_2(n)\beta_2^n + \cdots + q_t(n)\beta_t^n \quad (n \geq 0), \tag{2}$$

where r_i's, q_j's are polynomials and the coefficients of the polynomials, a and b are elements of the algebraic number field $\mathbf{Q}(\alpha, \alpha_2, \dots, \alpha_s, \beta, \beta_2, \dots, \beta_t)$. In the following we assume that $ab \neq 0$ and

$$F(x) = dx^q + d_p x^p + d_{p-1}x^{p-1} + \cdots + d_0, \tag{3}$$

is a polynomial with rational integer coefficients, where $d \neq 0$, $q \geq 2$ and $q > p$.

The Diophantine equation

$$G_n = F(x) \tag{4}$$

with positive integer variables n and x was investigated by several authors. It is known that if G is a nondegenerate second order linear recurrence, with some restrictions, and $F(x) = dx^q$ then the equation (4) have finitely many integer solutions in variables $n \geq 0, x$ and $q \geq 2$.

For general linear recurrences we know a similar result (see [4]). A more general result was proved by I. Nemes and A. Pethő [3]. They proved the following theorem: let G_n be a linear recurrence sequence defined by (1) and let $F(x)$ be a polinomial defined by (3). Suppose that $\alpha_2 \neq 1, |\alpha| = |\alpha_1| > |\alpha_2| > |\alpha_i|$ for $3 \leq i \leq s$, $G_n \neq a\alpha^n$ for $n > c_1$ and $p \leq qc_2$. Then all integer solution $n, |x| > 1, q \geq 2$ of the equation (4) satisfy $q < c_3$, where c_1, c_2 and c_3 are effectively computable positive constants depending on the parameters of the sequence G and the polynomial $F(x)$.

P. Kiss [2] showed that some conditions of the above result can be left out.

We prove a theorem which investigates a similar property of the product of the terms of two different linear recurrences. In the theorem and its proof $c_4, c_5, \dots$ will denote effectively computable positive constants which depend on the sequences, the polynomial $F(x)$ and the constants in the following theorem.

Theorem. *Let G and H be linear recursive sequences satisfying the above conditions. Let $K > 1$ and δ $(0 < \delta < 1)$ be real numbers. Furthermore let $F(x)$ be a polynomial defined in (3) with the condition $p < \delta q$. Assume that $G_i \neq a\alpha^i$, $H_j \neq b\beta^j$ if $i, j > n_0$ and $\alpha \notin \mathbf{Z}$ or $\beta \notin \mathbf{Z}$. Then the equation*

$$G_n H_m = F(x) \tag{5}$$

in positive integers n, m, x for which $m \leq n < Km$, implies that $q < q_0$ $(n_0, G, H, K, F, \delta)$, where q_0 is an effectively computable number (which depends on only n_0, G, H K, F and δ).

In the proof of the Theorem we shall use the following result due to A. Baker (see Theorem 1. in [1] with $\delta = \frac{1}{\delta}$). In this lemma the height of an algebraic number means the height of the minimal defining polynomial of the algebraic number.

Lemma. *Let $\pi_1, \pi_2, \ldots, \pi_r$ be non-zero algebraic numbers of heights not exceeding $M_1, M_2, \ldots, M_r$ respectively ($M_r \geq 4$). Further let $b_1, \ldots, b_{r-1}$ be rational integers with absolute values at most B and let b_r be a non-zero rational integer with absolute value at most B' ($B' \geq 3$). Then there exists a computable constant $C = C(r, M_1, \ldots, M_{r-1}, \pi_1, \ldots, \pi_r)$ such that the inequalities*

$$0 \neq \left|\sum_{i=1}^{r} b_i \log \pi_i\right| > e^{-C(\log M_r \log B' + \frac{B}{B'})}$$

are satisfied. (It is assumed that the logarithms have their principal values.)

Proof. Suppose that (5) holds with the conditions given in the Theorem. We may assume without loss of generality that $|\alpha| \geq |\beta|$ and that the terms of the sequences G, H are positive and $x > 1$. We may assume that $n > n_0$ and $m > n_0$. By (1), (2) and (5) we have

$$F(x) = a\alpha^n \left(1 + \frac{r_2(n)}{a}\left(\frac{\alpha_2}{\alpha}\right)^n + \cdots\right) b\beta^m \left(1 + \frac{q_2(m)}{b}\left(\frac{\beta_2}{\beta}\right)^m + \cdots\right).$$

By the assumption $|\alpha| > |\alpha_i|$ and $|\beta| > |\beta_i|$ we obtain that

$$\left(1 + \frac{r_2(n)}{a}\left(\frac{\alpha_2}{\alpha}\right)^n + \cdots\right) \to 1 \quad \text{as} \quad n \to \infty, \tag{6}$$

and

$$\left(1 + \frac{q_2(m)}{b}\left(\frac{\beta_2}{\beta}\right)^m + \cdots\right) \to 1 \quad \text{as} \quad m \to \infty. \tag{7}$$

Then (5) can be written in the form

$$\begin{aligned} \frac{ab\alpha^n\beta^m}{dx^q} &= (1+\varepsilon_1)((1+\varepsilon_2)(1+\varepsilon_3))^{-1} = \left(1 + \sum_{i=0}^{p} \frac{d_i}{d} x^{i-q}\right) \\ &\left(\left(1+\frac{1}{a}\sum_{i=2}^{s} r_i(n)\left(\frac{\alpha_i}{\alpha}\right)^n\right)\left(1 + \frac{1}{b}\sum_{i=2}^{t} q_i(m)\left(\frac{\beta_i}{\beta}\right)^m\right)\right)^{-1} \end{aligned} \tag{8}$$

where

$$|\varepsilon_1| = \left|\frac{d_p}{d}\left(\frac{1}{x}\right)^{q-p}\right| \left|1 + \frac{d_{p-1}}{d_p}\left(\frac{1}{x}\right) + \cdots\right| \tag{9}$$

$$|\varepsilon_2| = \left|\frac{r_2(n)}{a}\left(\frac{\alpha_2}{\alpha}\right)^n\right| \left|1 + \frac{r_3(n)}{r_2(n)}\left(\frac{\alpha_3}{\alpha_2}\right)^n + \cdots\right| \tag{10}$$

and

$$(11)\qquad |\varepsilon_3| = \left|\frac{q_2(m)}{b}\left(\frac{\beta_2}{\beta}\right)^m\right|\left|1+\frac{q_3(m)}{q_2(m)}\left(\frac{\beta_3}{\beta_2}\right)^m+\cdots\right|$$

Using (8), (9), (10), (11) and $m \le n < Km$ we have

$$(12)\qquad c_4 x^{\frac{1}{2}} < |\alpha|^{\frac{n}{q}} < x^{c_5}.$$

Therefore by (9), (10), (11) and (12) we have the following inequalities

$$(13)\qquad |\varepsilon_1| < \left|\frac{1}{\alpha}\right|^{c_6\frac{q-p}{q}n}, \quad |\varepsilon_2| < \left|\frac{\alpha_2}{\alpha}\right|^{c_7 n}, \quad |\varepsilon_3| < \left|\frac{\beta_2}{\beta}\right|^{c_8 n}.$$

We distinguish two cases. First we suppose that

$$x^q = \frac{ab}{d}\alpha^n\beta^m,$$

moreover, without loss of generality we may assume that $\alpha \notin \mathbf{Z}$. Let $\alpha' \neq \alpha$ be any conjugate of α and let φ be an automorphism of $\overline{\mathbf{Q}}$ with $\varphi(\alpha) = \alpha'$. Then $\varphi(\beta) = \beta'$ is a conjugate of β and $|\beta'| \le |\beta|$, $|\alpha'| < |\alpha|$. Moreover,

$$\frac{ab}{c}\alpha^n\beta^m = \varphi\left(\frac{ab}{c}\right)(\varphi(\alpha))^n(\varphi(\beta))^m.$$

Thus

$$\left|\left(\frac{\alpha}{\varphi(\alpha)}\right)^n\right| = \left|\frac{c\varphi(ab)}{ab\varphi(c)}\left(\frac{\varphi(\beta)}{\beta}\right)^m\right| \le \left|\frac{c\varphi(ab)}{ab\varphi(c)}\right|,$$

whence n is bounded, which implies that q is bounded.

Now we can suppose that $dx^q \neq ab\alpha^n\beta^m$. Applying the Lemma with $M_6 = x$, $B' = q$ and $B = n$, it follows that

$$\begin{aligned} L := \left|\log\frac{dx^q}{a\alpha^n b\beta^m}\right| &= |q\log x - \log a - n\log\alpha - \log b - m\log\beta| \\ &> e^{-C(\log x\log q + \frac{n}{q})}. \end{aligned}$$

On the other hand, using that $\frac{q-p}{q} > 1-\delta$ and (13) we can derive an upper bound for L

$$L < 2|\varepsilon_1| + 2|\varepsilon_2| + 2|\varepsilon_3| < e^{-c_9 n}$$

and it follows that

$$(14)\qquad c_{10}(\log x\log q + \frac{n}{q}) > c_9 n.$$

By (12) we have

$$c_{11} \log x < \frac{n}{q} < c_{12} \log x, \tag{15}$$

so by (14) and (15)

$$\log q \log x > c_{12} n > c_{13} q \log x$$

and

$$\frac{\log q}{q} > c_{13}.$$

This can be satisfied only by finitely many positive integer q so our theorem is proved.

References

[1] A. Baker, A sharpening of the bounds for linear forms in logarithms II, *Acta Arithm.*, **24** (1973), 33–36.

[2] P. Kiss, Note on a result of I. Nemes and A. Pethő concerning polynomial values in linear recurrences, (to appear).

[3] I. Nemes and A. Pethő, Polynomial values in linear recurrences, *Publ. Math. Debrecen,* **31** (1984), 229–233.

[4] T. N. Shorey and C. L. Stewart, On the Diophantine equation $ax^{2t} + bx^t y + cy^2 = d$ and pure powers in recurrence sequences, *Math. Scand.,* **52** (1983), 24–36.

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