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# AN ASSOCIATIVE ALGORITHM

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**Abstract:** In this note we introduce the concept of the associative algorithm with respect to an interval filling sequence, we characterize it and we show that the regular algorithm is associative with respect to any interval filling sequence.

## 1. Introduction

Let $\Lambda$ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $L(\lambda) := \sum_{n=1}^{\infty} \lambda_n < +\infty$. A sequence $(\lambda_n) \in \Lambda$ is called *interval filling* if, for any $x \in [0, L(\lambda)]$, there exists a sequence $(\delta_n)$ such that $\delta_n \in \{0,1\}$ for all $n \in I\!N$ (the set of all positive integers) and $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$. This concept has been introduced and discussed in Daróczy–Járai–Kátai [3]. It is known also from [3] that $\lambda = (\lambda_n) \in \Lambda$ is interval filling if and only if $\lambda_n \le L_{n+1}(\lambda)$ for all $n \in I\!N$ where $L_m(\lambda) = \sum_{i=m}^{\infty} \lambda_i$, $(m \in I\!N)$. The set of the interval filling sequences will be denoted by $IF$.

An *algorithm* (with respect to $\lambda = (\lambda_n) \in IF$) is defined as a sequence of functions $\alpha_n \colon [0, L(\lambda)] \to \{0,1\}$ $(n \in I\!N)$ for which

$$x = \sum_{n=1}^{\infty} \alpha_n(x) \lambda_n \qquad (x \in [0, L(\lambda)]).$$

We denote the set of algorithms (with respect to $\lambda = (\lambda_n) \in IF$) by $\mathcal{A}(\lambda)$. Obviously, $\mathcal{A}(\lambda) \neq \emptyset$ for all $\lambda \in IF$. Namely, it was proved in [3] that, if $\lambda = (\lambda_n) \in IF$ and

$$\mathcal{E}_n(x) = \begin{cases} 0, & \text{if } x < \sum_{i=1}^{n-1} \mathcal{E}_i(x)\lambda_i + \lambda_n, \\ 1, & \text{if } x \ge \sum_{i=1}^{n-1} \mathcal{E}_i(x)\lambda_i + \lambda_n, \qquad (n \in I\!N,\ x \in [0, L(\lambda)]) \end{cases}$$

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or

$$\mathcal{E}'_n(x) = \begin{cases} 0, & \text{if } x \le \sum_{i=1}^{n-1} \mathcal{E}'_i(x)\lambda_i + L_{n+1}(\lambda), \\ 1, & \text{if } x > \sum_{i=1}^{n-1} \mathcal{E}'_i(x)\lambda_i + L_{n+1}(\lambda), \end{cases} \qquad (n \in I\!N,\ x \in [0, L(\lambda)])$$

then $\mathcal{E} = (\mathcal{E}_n) \in \mathcal{A}(\lambda)$ and $\mathcal{E}' = (\mathcal{E}'_n) \in \mathcal{A}(\lambda)$. The algorithms $\mathcal{E}$ and $\mathcal{E}'$ are called *regular* and *anti-regular* algorithms, respectively. In general, there are much more algorithms with respect to an interval filling sequence. They are described and characterized in Daróczy–Maksa–Szabó [4]. The purpose of this paper is to introduce the concept of associative algorithm with respect to an interval filling sequence, to characterize it, and to show that the regular algorithm is associative with respect to any interval filling sequence.

## 2. The regular algorithm is associative

**Definition.** Let $\lambda = (\lambda_n) \in IF$ and $(\alpha_n) \in \mathcal{A}(\lambda)$. Then the algorithm $(\alpha_n)$ is *associative* if the binary operation $\circ : [0, L(\lambda)] \times [0, L(\lambda)] \to [0, L(\lambda)]$ defined by

$$(1) \qquad x \circ y = \sum_{n=1}^{\infty} \alpha_n(x)\alpha_n(y)\lambda_n \qquad (x, y \in [0, L(\lambda)])$$

is associative, that is,

$$(x \circ y) \circ z = x \circ (y \circ z) \qquad (x, y, z \in [0, L(\lambda)]).$$

Obviously, the operation $\circ$ is well-defined by (1) and it is commutative, i.e., $x \circ y = y \circ x$ for all $x, y \in [0, L(\lambda)]$, and idempotent, i.e., $x \circ x = \sum_{n=1}^{\infty} \alpha_n(x)^2\lambda_n = \sum_{n=1}^{\infty} \alpha_n(x)\lambda_n = x$ for all $x \in [0, L(\lambda)]$. First we prove the following

**Theorem 1.** *Let $\lambda = (\lambda_n) \in IF$, $\alpha = (\alpha_n) \in \mathcal{A}(\lambda)$. Then $\alpha$ is associative, if and only if, $\alpha(x \circ y) = \alpha(x)\alpha(y)$, that is,*

$$(2) \qquad \alpha_n(x \circ y) = \alpha_n(x)\alpha_n(y) \qquad (n \in I\!N;\ x, y \in [0, L(\lambda)]).$$

**Proof.** Suppose that (2) holds. Then, for all $x, y, z \in [0, L(\lambda)]$, we have

$$(x \circ y) \circ z = \sum_{n=1}^{\infty} \alpha_n(x \circ y)\alpha_n(z)\lambda_n = \sum_{n=1}^{\infty} \alpha_n(x)\alpha_n(y)\alpha_n(z)\lambda_n = \\ = \sum_{n=1}^{\infty} \alpha_n(x)\alpha_n(y \circ z)\lambda_n = x \circ (y \circ z).$$

On the other hand, suppose that $\alpha$ is associative. Then, by idempotency, $x \circ y = (x \circ x) \circ y = x \circ (x \circ y)$, that is,

$$\sum_{n=1}^{\infty} \alpha_n(x \circ y)\lambda_n = \sum_{n=1}^{\infty} \alpha_n(x)\alpha_n(x \circ y)\lambda_n \qquad (x, y \in [0, L(\lambda)])$$

whence

$$0 = \sum_{n=1}^{\infty} (1 - \alpha_n(x))\alpha_n(x \circ y)\lambda_n \qquad (x, y \in [0, L(\lambda)]).$$

This implies that $(1 - \alpha_n(x))\alpha_n(x \circ y) = 0$, that is,

$$\alpha_n(x \circ y) = \alpha_n(x)\alpha_n(x \circ y) \qquad (n \in I\!N;\ x, y \in [0, L(\lambda)]) \tag{3}$$

and, by interchanging $x$ and $y$, we obtain

$$\alpha_n(x \circ y) = \alpha_n(y)\alpha_n(x \circ y) \qquad (n \in I\!N;\ x, y \in [0, L(\lambda)]). \tag{4}$$

Since $\alpha_n^2(t) = \alpha_n(t) \in \{0, 1\}$ for all $t \in [0, L(\lambda)]$ and for all $n \in I\!N$, (3) and (4) yield

$$\alpha_n(x \circ y) = \alpha_n^2(x \circ y) = \alpha_n(x)\alpha_n(y)\alpha_n^2(x \circ y) \leq \alpha_n(x)\alpha_n(y) \tag{5}$$

for all $x, y \in [0, L(\lambda)]$ and $n \in I\!N$. Therefore, by (1),

$$0 = x \circ y - (x \circ y) = \sum_{n=1}^{\infty} \alpha_n(x)\alpha_n(y)\lambda_n - \sum_{n=1}^{\infty} \alpha_n(x \circ y)\lambda_n = \\ = \sum_{n=1}^{\infty} (\alpha_n(x)\alpha_n(y) - \alpha_n(x \circ y))\lambda_n$$

whence, by (5), (2) follows. Thus the proof is complete.

The following characterization of the regular algorithm, which is due to Daróczy, Járai, Kátai and Szabó (personal communication), is the other tool for proving the associativity of the regular algorithm.

**Theorem 2.** *Let $\lambda = (\lambda_n) \in IF$ and $x = \sum_{n=1}^{\infty} t_n\lambda_n$ with some $(t_n)\colon I\!N \to \{0,1\}$. Then $t_n = \mathcal{E}_n(x)$ for all $n \in I\!N$, if and only if,*

$$k \in I\!N \text{ and } t_k = 0 \text{ imply that } \lambda_k > \sum_{i=k+1}^{\infty} t_i\lambda_i. \tag{6}$$

**Proof.** First suppose that $t_n = \mathcal{E}_n(x)$ for all $n \in I\!N$ and let $k \in I\!N$ be fixed. Then, by definition, $\mathcal{E}_k(x) = 0$ implies that

$$x = \sum_{i=1}^{\infty} \mathcal{E}_i(x)\lambda_n < \sum_{i=1}^{k-1} \mathcal{E}_i(x)\lambda_i + \lambda_k,$$

whence

$$\lambda_k > \sum_{i=k}^{\infty} \mathcal{E}_i(x)\lambda_i = \sum_{i=k+1}^{\infty} \mathcal{E}_i(x)\lambda_i = \sum_{i=k+1}^{\infty} t_i\lambda_i,$$

that is, (6) holds.

Next, suppose (6) to be hold. Futrhermore suppose, in the contrary, that $t_{n_0} \neq \mathcal{E}_{n_0}(x)$ for some $n_0 \in I\!N$ while $t_i = \mathcal{E}_i(x)$, $i \in \{1, \ldots, n_0 - 1\}$ ($\{1, \ldots, n_0 - 1\} = \emptyset$ if $n_0 = 1$). Then

$$\begin{aligned} x &= \sum_{i=1}^{n_0-1} t_i\lambda_i + t_{n_0}\lambda_{n_0} + \sum_{i=n_0+1}^{\infty} t_i\lambda_i = \\ &= \sum_{i=1}^{n_0-1} t_i\lambda_i + \mathcal{E}_{n_0}(x)\lambda_{n_0} + \sum_{i=n_0+1}^{\infty} \mathcal{E}_i(x)\lambda_i. \end{aligned} \tag{7}$$

If $\mathcal{E}_{n_0}(x) = 1$ then $t_{n_0} = 0$, so (7) gives the contradiction to (6):

$$\sum_{i=n_0+1}^{\infty} t_i\lambda_i - \lambda_{n_0} = \sum_{n_0+1}^{\infty} \mathcal{E}_i(x)\lambda_i \geq 0.$$

If $\mathcal{E}_{n_0}(x) = 0$ then $t_{n_0} = 1$ so, by the definition of $\mathcal{E}_{n_0}(x)$, we have

$$\sum_{i=1}^{n_0-1} t_i\lambda_i + \lambda_{n_0} + \sum_{i=n_0+1}^{\infty} t_i\lambda_i = x < \sum_{i=1}^{n_0-1} \mathcal{E}_i(x)\lambda_i + \lambda_{n_0} = \sum_{i=1}^{n_0-1} t_i\lambda_i + \lambda_{n_0}$$

which is impossible again. Thus the theorem is proved.

Now we are ready to prove our main result.

**Theorem 3.** *The regular algorithm $\mathcal{E} = (\mathcal{E}_n)$, with respect to any interval filling sequence $\lambda = (\lambda_n)$ is associative.*

**Proof.** We shall prove that

$$\mathcal{E}_n(x \circ y) = \mathcal{E}_n(x)\mathcal{E}_n(y) \qquad (n \in I\!N;\ x, y \in [0, L(\lambda)]).$$

Let $x, y \in [0, L(\lambda)]$, $k \in I\!N$ and $\mathcal{E}_k(x)\mathcal{E}_k(y) = 0$. Then, by Theorem 2,

$$\lambda_k > \sum_{i=k+1}^{\infty} \mathcal{E}_i(x)\lambda_i \geq \sum_{i=k+1}^{\infty} \mathcal{E}_i(x)\mathcal{E}_i(y)\lambda_i \qquad \text{if} \quad \mathcal{E}_k(x) = 0 \quad \text{and}$$

$$\lambda_k > \sum_{i=k+1}^{\infty} \mathcal{E}_i(y)\lambda_i \geq \sum_{i=k+1}^{\infty} \mathcal{E}_i(x)\mathcal{E}_i(y)\lambda_i \qquad \text{if} \quad \mathcal{E}_k(y) = 0.$$

Therefore, in both cases,

$$\lambda_k > \sum_{i=k+1}^{\infty} \mathcal{E}_i(x)\mathcal{E}_i(y)\lambda_i.$$

Applying Theorem 2 again, we have that $\mathcal{E}_n(x \circ y) = \mathcal{E}_n(x)\mathcal{E}_n(y)$ for all $n \in I\!N$. Finally, the associativity of $\mathcal{E}$ follows from Theorem 1.

## 3. Remarks

**1.** If $\lambda = (\lambda_n) \in I\!F$, $\mathcal{E} = (\mathcal{E}_n) \in \mathcal{A}(\lambda)$ is the regular algorithm and

$$(8) \qquad x \circ y = \sum_{n=1}^{\infty} \mathcal{E}_n(x)\mathcal{E}_n(y)\lambda_n \qquad (x, y \in [0, L(\lambda)])$$

then $([0, L(\lambda)], \circ)$ is an *abelian semigroup* with unit element $L(\lambda)$ in which $x \circ x = x$ for all $x \in [0, L(\lambda)]$, that is, each element is *idempotent*. While the semigroup operation $\circ$ is continuous in bot variables at all but at most countably many points and also it is "strictly monotonic" in some sense (see [4]) in both variables, *it cannot be representable in the form*

$$x \circ y = \varphi^{-1}(\varphi(x) + \varphi(y)) \qquad (x, y \in [0, L(\lambda)])$$

with some injective function $\varphi: [0, L(\lambda)] \to I\!R$ (cf. Aczél [1] or Craigen–Páles [2]) because of the idempotency.

**2.** *Let* $\lambda = (\lambda_n) \in I\!F$ *and* $\mathcal{E}' = (\mathcal{E}'_n)$ *be the anti-regular algorithm with respect to it*. We shall show that $\mathcal{E}'$ *is associative, i.e., the operation*

$$x * y = \sum_{n=1}^{\infty} \mathcal{E}'_n(x)\mathcal{E}'_n(y)\lambda_n \qquad (x, y \in [0, L(\lambda)])$$

*is associative, if and only if,*

$$\mathcal{E}_n(x + y - x \circ y) + \mathcal{E}_n(x \circ y) = \mathcal{E}_n(x) + \mathcal{E}_n(y) \tag{9}$$

*holds for all* $n \in I\!N$; $x, y \in [0, L(\lambda)]$ *where* $\circ$ *is the associative operation defined by* (8) *and* $(\mathcal{E}_n)$ *is the regular algorithm.*

Indeed, the connection

$$\mathcal{E}'_n(x) = 1 - \mathcal{E}_n(L(\lambda) - x) \qquad (n \in I\!N;\ x \in [0, L(\lambda)]) \tag{10}$$

between the regular and anti-regular algorithms can easily be seen. On the other hand, by (10) and (8), we have

$$L(\lambda) - [(L(\lambda) - x) * (L(\lambda) - y)] = \sum_{n=1}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \mathcal{E}'_n(L(\lambda) - x)\mathcal{E}'_n(L(\lambda) - y)\lambda_n =$$

$$= \sum_{n=1}^{\infty} \left[1 - (1 - \mathcal{E}_n(x))(1 - \mathcal{E}_n(y))\right] \lambda_n = x + y - x \circ y$$

for all $x, y \in [0, L(\lambda)]$. Thus, again by (10),

$$\mathcal{E}_n(x + y - x \circ y) = 1 - \mathcal{E}'_n((L(\lambda) - x) * (L(\lambda) - y)). \tag{11}$$

Therefore, by Theorems 1 and 3, (11) and (10) imply that $(\mathcal{E}'_n)$ is associative, if and only if,

$$\begin{aligned} \mathcal{E}_n(x + y - x \circ y) &= 1 - \mathcal{E}'_n(L(\lambda) - x)\mathcal{E}'_n(L(\lambda) - y) = \\ &= 1 - (1 - \mathcal{E}_n(x))(1 - \mathcal{E}_n(y)) = \\ &= \mathcal{E}_n(x) + \mathcal{E}_n(y) - \mathcal{E}_n(x)\mathcal{E}_n(y) = \\ &= \mathcal{E}_n(x) + \mathcal{E}_n(y) - \mathcal{E}_n(x \circ y) \end{aligned}$$

for all $x, y \in [0, L(\lambda)]$; $n \in I\!N$ which proves (9).

**3.** The existence of associative algorithms different from the regular one is still unknown. We have only partial results. To present these, we need the following concept (see [4]): If $\lambda = (\lambda_n) \in I\!F$, $a \in [0, L(\lambda)]$ and $\mathcal{E}_n(a) = \alpha_n(a)$ for all $(\alpha_n) \in \mathcal{A}(\lambda)$ and $n \in I\!N$ (where $(\mathcal{E}_n)$ is the regular algorithm), we say that the number $a$ is *uniquely representable*. The set of uniquely representable numbers will be denoted by $U(\lambda)$.

(a) *The only associative algorithm with respect to the interval filling sequence* $\lambda = \left(\frac{1}{2^n}\right)$ *is the regular one.*

Indeed, suppose that $(\alpha_n) \in \mathcal{A}(\lambda)$, $(\alpha_n)$ is associative and $(\alpha_n(x)) \neq (\mathcal{E}_n(x))$ for some $x \in [0,1]$. Then, by the definition of the regular algorithm, there exists $n_0 \in I\!N$ such that $\alpha_i(x) = \mathcal{E}_i(x)$ for $i \in \{1,\ldots,n_0-1\}$ and $\alpha_{n_0}(x) = 0$, $\mathcal{E}_{n_0}(x) = 1$. Therefore

$$x = \sum_{i=1}^{\infty} \mathcal{E}_i(x)\frac{1}{2^i} = \sum_{i=1}^{\infty} \alpha_i(x)\frac{1}{2^i}$$

implies that

$$\frac{1}{2^{n_0}} = \sum_{i=n_0+1}^{\infty} (\alpha_i(x) - \mathcal{E}_i(x))\frac{1}{2^i}.$$

This holds only if $\alpha_i(x) - \mathcal{E}_i(x) = 1$, that is, $\alpha_i(x) = 1$ and $\mathcal{E}_i(x) = 0$ for $i > n_0$, so $x = \sum\limits_{i=1}^{n_0-1} \mathcal{E}_i(x)\frac{1}{2^i} + \frac{1}{2^{n_0}}$. Define the numbers $a = \sum\limits_{i=1}^{n_0-1} \mathcal{E}_i(x)\frac{1}{2^i} + \frac{1}{2^{n_0}} + \sum\limits_{i=n_0+1}^{\infty} \delta_i \frac{1}{2^i}$ and $b = \sum\limits_{i=1}^{n_0} \frac{1}{2^i} + \sum\limits_{i=n_0+1}^{\infty} (1-\delta_i)\frac{1}{2^i}$ where $\delta_{n_0+i} = 0$ if $i$ is odd and $\delta_{n_0+i} = 1$ if $i$ is even positive integer. Obviously $a, b \in U(\lambda)$ and $a \circ b = \sum\limits_{n=1}^{\infty} \alpha_n(a)\alpha_n(b)\frac{1}{2^n} = \sum\limits_{n=1}^{\infty} \mathcal{E}_n(a)\mathcal{E}_n(b)\frac{1}{2^n} = x$. Thus, applying Theorem 1, we get

$$1 = \alpha_{n_0+1}(x) = \alpha_{n_0+1}(a \circ b) = \alpha_{n_0+1}(a)\alpha_{n_0+1}(b) = \delta_{n_0+1}(1-\delta_{n_0+1}) = 0$$

which is a contradiction.

(b) *IF* $\lambda = (\lambda_n) \in IF$ *and* $]0, L(\lambda)[\, \cap U(\lambda) \neq \emptyset$, *that is, there exists uniquely representable number in the interior of* $[0, L(\lambda)]$, *then the anti-regular algorithm* $\mathcal{E}' = (\mathcal{E}'_n)$ *is not associative.*

Indeed, suppose in the contrary that $\mathcal{E}'$ is associative. Then, by our second remark, we have (9). Let furthermore $x \in ]0, L(\lambda)[\, \cap U(\lambda)$ and $N \in I\!N$ such that $\mathcal{E}_N(x) = 0$ and $\mathcal{E}_{N+1}(x) = 1$. The numbers

$$u = \sum_{i=N+1}^{\infty} \mathcal{E}_i(x)\lambda_i \quad \text{and} \quad v = \sum_{i=N+1}^{\infty} (1 - \mathcal{E}_i(x))\lambda_i$$

are uniquely representable again,

$$u \circ v = \sum_{n=1}^{\infty} \mathcal{E}'_n(u)\mathcal{E}'_n(v)\lambda_n = \sum_{n=1}^{\infty} \mathcal{E}_n(u)\mathcal{E}_n(v)\lambda_n =$$
$$= \sum_{n=N+1}^{\infty} \mathcal{E}_n(x)(1-\mathcal{E}_n(x))\lambda_n = 0$$

and $u+v = \sum_{n=N+1}^{\infty} \lambda_n = L_{n+1}(\lambda)$. Thus, by (9),

$$\mathcal{E}_n(L_{N+1}(\lambda)) = \mathcal{E}_n(u+v-u\circ v) + \mathcal{E}_n(u\circ v) = \mathcal{E}_n(u)+\mathcal{E}_n(v) = \begin{cases} 0, & \text{if } n \le N \\ 1, & \text{if } n > N. \end{cases}$$

Taking into consideration the definition of the regular algorithm, this implies that $L_{N+1}(\lambda) < \lambda_N$ which contradicts the interval fillingness of $(\lambda_n)$.

## References

[1] Aczél, J., Lecetures on Functional Equations and Their Applications, *Academic Press, New York and London*, 1966.

[2] Craigen, R. and Páles, Zs., The associativity equation revisited, *Aequationes Math.*, **37** (1989), 306–312.

[3] Daróczy, Z., Járai, A. and Kátai, I., Intervallfüllende Folgen und volladditive Funktionen, *Acta Sci. Math.*, **50** (1986), 337–350.

[4] Daróczy, Z., Maksa, Gy. and Szabó, T., Some regularity properties of algorithms and additive functions with respect to them, *Aequationes Math.*, **41** (1991), 111–118.

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