ON THE ASSOCIATIVITY OF ALGORITHMS

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Abstract: In this paper we extend the concept of the associative algorithm to the case of interval filling sequences of order N. We show that the regular algorithm is associative, and, answering a question of Gy. Maksa, we prove that there exist interval filling sequences for which the regular algorithm is not the only associative one.

1. Introduction

Throughout this paper let N be a fixed positive integer and $\mathcal{N} = \{0, 1, \ldots, N\}$. Let Λ be the set of the strictly decreasing sequences $\lambda = (\lambda_n)$ of positive real numbers for which $\sum_{n=1}^{\infty} \lambda_n < +\infty$. Let $L(\lambda) = N \cdot \sum_{n=1}^{\infty} \lambda_n$. A sequence $(\lambda_n) \in \Lambda$ is called an *interval filling sequence of order* N if, for any $x \in [0, L(\lambda)]$, there exists a sequence (δ_n) such that $\delta_n \in \mathcal{N}$ for all $n \in \mathbb{N}$ (the set of all positive integers) and $x = \sum_{n=1}^{\infty} \delta_n \lambda_n$. This concept has been introduced in Daróczy–Járai–Kátai [1] for $N = 1$ and in Kovács–Maksa [2] in the general case. It is known also from [2] that $\lambda = (\lambda_n) \in \Lambda$ is an interval filling sequence of order N if and only if $\lambda_n \le L_{n+1}(\lambda)$ for all $n \in \mathbb{N}$ where $L_m(\lambda) = N \cdot \sum_{i=m}^{\infty} \lambda_i$, $(m \in \mathbb{N})$. The set of the interval filling sequences of order N will be denoted by IF_N.

The notions of algorithms, associative algorithms, the regular, the quasi-regular and the anti-regular algorithm were introduced in [3], [4] and [5] for interval filling sequences of order 1, now we will extend them to the case of arbitrary $N \in \mathbb{N}$.

2. The associativity of the regular algorithm

Definition. An *algorithm* (with respect to $\lambda = (\lambda_n) \in IF_N$) is defined as a sequence of functions $\alpha_n \colon [0, L(\lambda)] \to \mathcal{N}$ $(n \in \mathbb{N})$ for which

$$x = \sum_{n=1}^{\infty} \alpha_n(x) \lambda_n \qquad (x \in [0, L(\lambda)]).$$

We denote the set of algorithms (with respect to $\lambda = (\lambda_n) \in IF_N$) by $\mathcal{A}_N(\lambda)$.

It is easy to prove that $\mathcal{A}_N(\lambda) \neq \emptyset$ for all $\lambda \in I\!F_N$, namely if $\lambda = (\lambda_n) \in I\!F_N$, $x \in [0, L(\lambda)]$, $n \in I\!N$ and

$$\mathcal{E}_n(x) = \max\Big\{ j \in \mathcal{N} \;\Big|\; \sum_{i=1}^{n-1} \mathcal{E}_i(x)\lambda_i + j \cdot \lambda_n \leq x \Big\},$$

or

$$\mathcal{E}^\star_n(x) = \max\Big\{ j \in \mathcal{N} \;\Big|\; \sum_{i=1}^{n-1} \mathcal{E}^\star_i(x)\lambda_i + j \cdot \lambda_n < x \Big\},$$

or

$$\mathcal{E}'_n(x) = \min\Big\{ j \in \mathcal{N} \;\Big|\; \sum_{i=1}^{n-1} \mathcal{E}'_i(x)\lambda_i + j \cdot \lambda_n + \sum_{i=n+1}^{\infty} N\lambda_i \geq x \Big\},$$

then $\mathcal{E} = (\mathcal{E}_n) \in \mathcal{A}_N(\lambda)$, $\mathcal{E}^\star = (\mathcal{E}^\star_n) \in \mathcal{A}_N(\lambda)$ and $\mathcal{E}' = (\mathcal{E}'_n) \in \mathcal{A}_N(\lambda)$. The algorithms $\mathcal{E}$, $\mathcal{E}^\star$ and $\mathcal{E}'$ are called *regular (or greedy)*, *quasi-regular* and *anti-regular (or lazy)* algorithms, respectively.

Definition. Let $\lambda = (\lambda_n) \in I\!F_N$ and $(\alpha_n) \in \mathcal{A}_N(\lambda)$. Then the algorithm (α_n) is *associative* if the binary operation $\circ\colon [0, L(\lambda)] \times [0, L(\lambda)] \to [0, L(\lambda)]$ defined by

$$x \circ y = \sum_{n=1}^{\infty} \min\{\alpha_n(x), \alpha_n(y)\}\lambda_n \qquad (x, y \in [0, L(\lambda)]) \tag{1}$$

is associative, that is,

$$(x \circ y) \circ z = x \circ (y \circ z) \qquad (x, y, z \in [0, L(\lambda)]).$$

Remark. This is a generalization of the notion defined by Gy. Maksa [5], because in the set $\{0, 1\}$ the minimum of any two elements is equal to the product of them.

Obviously, the operation $\circ$ is commutative, i.e., $x \circ y = y \circ x$ for all $x, y \in [0, L(\lambda)]$, idempotent, i.e., $x \circ x = \sum_{n=1}^{\infty} \alpha_n(x)^2\lambda_n = \sum_{n=1}^{\infty} \alpha_n(x)\lambda_n = x$ for all $x \in [0, L(\lambda)]$ and $x \circ y \leq \min\{x, y\}$ for all $x, y \in [0, L(\lambda)]$. Now we will characterize the associative algorithms.

Theorem 1. *Let $\lambda = (\lambda_n) \in I\!F_N$, $\alpha = (\alpha_n) \in \mathcal{A}_N(\lambda)$. Then α is associative if and only if*

$$\alpha_n(x \circ y) = \min\{\alpha_n(x), \alpha_n(y)\} \qquad (n \in I\!N;\; x, y \in [0, L(\lambda)]). \tag{2}$$

Proof. Suppose that (2) holds. Then, for all $x, y, z \in [0, L(\lambda)]$, we have

$$\begin{aligned}(x \circ y) \circ z &= \sum_{n=1}^{\infty} \min\{\alpha_n(x \circ y), \alpha_n(z)\}\lambda_n \\ &= \sum_{n=1}^{\infty} \min\bigl\{\min\{\alpha_n(x), \alpha_n(y)\}, \alpha_n(z)\bigr\}\lambda_n = \\ &= \sum_{n=1}^{\infty} \min\{\alpha_n(x), \alpha_n(y), \alpha_n(z)\}\lambda_n = \cdots = x \circ (y \circ z) = \\ &= \sum_{n=1}^{\infty} \min\bigl\{\alpha_n(x), \min\{\alpha_n(y), \alpha_n(z)\}\bigr\}\lambda_n = \\ &= \sum_{n=1}^{\infty} \min\{\alpha_n(x), \alpha_n(y \circ z)\}\lambda_n = x \circ (y \circ z).\end{aligned}$$

On the other hand, suppose that α is associative. Then, by idempotency, $x \circ y = (x \circ x) \circ y = x \circ (x \circ y)$, that is,

$$\sum_{n=1}^{\infty} \alpha_n(x \circ y)\lambda_n = \sum_{n=1}^{\infty} \min\{\alpha_n(x), \alpha_n(x \circ y)\}\lambda_n \qquad (x, y \in [0, L(\lambda)])$$

whence

$$0 = \sum_{n=1}^{\infty} \bigl(\alpha_n(x \circ y) - \min\{\alpha_n(x), \alpha_n(x \circ y)\}\bigr)\lambda_n \qquad (x, y \in [0, L(\lambda)]),$$

and since the coefficient of λ_n is non-negative for all $n \in I\!N$, we obtain that $\alpha_n(x \circ y) - \min\{\alpha_n(x), \alpha_n(x \circ y)\} = 0$, that is,

$$\alpha_n(x \circ y) = \min\{\alpha_n(x), \alpha_n(x \circ y)\} \qquad (n \in I\!N;\ x, y \in [0, L(\lambda)]) \tag{3}$$

and, by interchanging x and y, we get

$$\alpha_n(x \circ y) = \min\{\alpha_n(y), \alpha_n(x \circ y)\} \qquad (n \in I\!N;\ x, y \in [0, L(\lambda)]). \tag{4}$$

Now (3) and (4) yield

$$\alpha_n(x \circ y) \le \min\{\alpha_n(x), \alpha_n(y)\} \tag{5}$$

for all $x, y \in [0, L(\lambda)]$ and $n \in I\!N$. Therefore, by (1),

$$\begin{aligned}0 = x \circ y - (x \circ y) &= \sum_{n=1}^{\infty} \min\{\alpha_n(x), \alpha_n(y)\}\lambda_n - \sum_{n=1}^{\infty} \alpha_n(x \circ y)\lambda_n = \\ &= \sum_{n=1}^{\infty} \bigl(\min\{\alpha_n(x), \alpha_n(y)\} - \alpha_n(x \circ y)\bigr)\lambda_n,\end{aligned}$$

and, because of (5), we have the non-negativity of the coefficients, so (2) holds. Thus the proof is complete.

The following characterization of the regular algorithm is the other tool for proving the associativity of the regular algorithm.

Theorem 2. *Let $\lambda = (\lambda_n) \in IF_N$ and $x = \sum_{n=1}^{\infty} t_n\lambda_n$ with some $(t_n)\colon I\!N \to \mathcal{N}$. Then $t_n = \mathcal{E}_n(x)$ for all $n \in I\!N$, if and only if,*

$$(6) \qquad k \in I\!N \text{ and } t_k < N \text{ imply that } \lambda_k > \sum_{i=k+1}^{\infty} t_i\lambda_i.$$

Proof. The "only if" part of the assertion is trivial.

For the "if" part, suppose (6) to be hold. Furthermore suppose, in the contrary, that $t_{n_0} \neq \mathcal{E}_{n_0}(x)$ for some $n_0 \in I\!N$ while $t_i = \mathcal{E}_i(x)$, $i \in \{1, \ldots, n_0-1\}$ ($\{1, \ldots, n_0-1\} = \emptyset$ if $n_0 = 1$). Because of the definition of the regular algorithm (the greedy property) we have $t_{n_0} < \mathcal{E}_{n_0}(x)$, so $t_{n_0} < N$, and by (6), $\lambda_{n_0} > \sum_{i=n_0+1}^{\infty} t_i\lambda_i$. Thus

$$x = \sum_{i=1}^{\infty} t_i\lambda_i < \sum_{i=1}^{n_0} t_i\lambda_i + \lambda_{n_0} = \sum_{i=1}^{n_0-1} t_i\lambda_i + (t_{n_0}+1)\lambda_{n_0} \leq$$
$$\leq \sum_{i=1}^{n_0-1} \mathcal{E}_i(x)\lambda_i + \mathcal{E}_{n_0}(x)\lambda_{n_0} \leq \sum_{i=1}^{\infty} \mathcal{E}_i(x)\lambda_i = x,$$

which is a contradiction. Thus the theorem is proved.

Now we are ready to prove the following

Theorem 3. *The regular algorithm $\mathcal{E} = (\mathcal{E}_n)$, with respect to any interval filling sequence $\lambda = (\lambda_n)$, is associative.*

Proof. We shall prove that

$$\min\{\mathcal{E}_n(x), \mathcal{E}_n(y)\} = \mathcal{E}_n(x \circ y) \qquad (n \in I\!N;\; x, y \in [0, L(\lambda)]),$$

that is, for $t_n = \min\{\mathcal{E}_n(x), \mathcal{E}_n(y)\}$ (6) holds. Let $x, y \in [0, L(\lambda)]$, $k \in I\!N$ and $\min\{\mathcal{E}_k(x), \mathcal{E}_k(y)\} < N$. Then, without loss of generality we can assume that $\mathcal{E}_k(x) < N$, from which

$$\lambda_k > \sum_{i=k+1}^{\infty} \mathcal{E}_i(x)\lambda_i \geq \sum_{i=k+1}^{\infty} \min\{\mathcal{E}_i(x), \mathcal{E}_i(y)\}\lambda_i.$$

3. Miscellaneous theorems

Theorem 4. *Let $\lambda = (\lambda_n) \in IF_N$. The quasi-regular algorithm $\mathcal{E}^\star$ with respect to λ is not associative.*

Proof. We will define two sequences $(\alpha_n), (\beta_n) \in \mathcal{N}^{I\!N}$ which are quasi-regular, but $(\min\{\alpha_n, \beta_n\})$ is not quasi-regular. It is clear that there exists a subsequence (l_n) of the increasing sequence of natural numbers for which the following three conditions hold:

$$(a) \qquad l_1 = 2,$$

$$(b) \qquad \lambda_{l_n} + \sum_{i=l_{n+1}}^{\infty} \lambda_i < \lambda_{l_n - 1} \qquad (n \in I\!N),$$

and,

$$(c) \qquad l_{n+1} \geq l_n + 2 \qquad (n \in I\!N).$$

And for such a sequence (l_n) there exists another subsequence (m_n) of the increasing sequence of natural numbers for which the following three conditions hold:

$$(d) \qquad m_1 = 2,$$

$$(e) \qquad \lambda_{m_n} + \sum_{i=m_{n+1}}^{\infty} \lambda_i < \lambda_{m_n - 1} \qquad (n \in I\!N),$$

and,

$$(f) \qquad l_i \neq m_j \qquad \text{if } i, j > 1.$$

Now let

$$\alpha_n = \begin{cases} 1, & \text{if there exists } i \in I\!N \text{ for which } n = l_i \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_n = \begin{cases} 1, & \text{if there exists } i \in I\!N \text{ for which } n = m_i \\ 0, & \text{otherwise.} \end{cases}$$

Condition (b) implies the regularity of (α_n), since if $k \in I\!N$ and n is the minimal index for which $k < l_n$ then

$$\lambda_k \geq \lambda_{l_n - 1} > \lambda_{l_n} + \sum_{i=l_{n+1}}^{\infty} \lambda_i \geq \lambda_{l_n} + \lambda_{l_{n+1}} + \lambda_{l_{n+2}} + \cdots = \sum_{i=k+1}^{\infty} \alpha_i \lambda_i,$$

so (6) holds for $(t_n) = (\alpha_n)$. Since $\alpha_n \neq 0$ for infinitely many indices n, we obtain that (α_n) is quasi-regular. The quasi-regularity of (β_n) can be shown in the same way. But $(\min\{\alpha_n, \beta_n\})$ is not a quasi-regular sequence, since it is equal to $(0, 1, 0, 0, 0, \ldots)$.

Theorem 5. *Let $\lambda = (\lambda_n) \in IF_N$. The anti-regular algorithm $\mathcal{E}'$ with respect to λ is not associative.*

Proof. Our purpose is to define two sequences $(\alpha_n), (\beta_n) \in \mathcal{N}^{I\!N}$ which are anti-regular, but $(\min\{\alpha_n, \beta_n\})$ is not anti-regular. Instead of this, it is obviously enough to define two sequences $(\alpha_n), (\beta_n) \in \mathcal{N}^{I\!N}$ which are regular, but $(\max\{\alpha_n, \beta_n\})$ is not regular. We will use this method.

In the proof we distinguish two cases:

Case 1. *There exists $m \in I\!N$ such that $\mathcal{E}_k^\star(\lambda_m) < N$ for infinitely many values of index k.*

Then let $H = \{k \in I\!N \mid k > m \text{ and } \mathcal{E}_k^\star(\lambda_m) < N\}$. Let A and B be subsets of $I\!N$ for which

$$A \cup B = I\!N, \qquad A \cap B = \emptyset,$$

$$A \cap \{k \in I\!N \mid \mathcal{E}_k^\star(\lambda_m) \neq 0\} \text{ and } B \cap \{k \in I\!N \mid \mathcal{E}_k^\star(\lambda_m) \neq 0\} \text{ are infinite sets,}$$

$$(i \in A \text{ and } i+1 \in B) \text{ or } (i \in B \text{ and } i+1 \in A) \Longrightarrow i \in H.$$

The existence of such sets A and B is clear. Now let

$$\alpha_k = \begin{cases} \mathcal{E}_k^\star(\lambda_m), & \text{if } k \in A \\ 0, & \text{if } k \in B \end{cases}$$

$$\beta_k = \begin{cases} \mathcal{E}_k^\star(\lambda_m), & \text{if } k \in B \\ 0, & \text{if } k \in A \end{cases}$$

for all $k \in I\!N$. The regularity of (α_n) and (β_n) follows from the definition of $(\mathcal{E}_n^\star(\lambda_m))$ and the infinite cardinality of $A \cap \{k \in I\!N \mid \mathcal{E}_k^\star(\lambda_m) \neq 0\}$ and $B \cap \{k \in I\!N \mid \mathcal{E}_k^\star(\lambda_m) \neq 0\}$. The "pointwise" maximum of (α_n) and (β_n) is not regular since it is equal to $(\mathcal{E}_n^\star(\lambda_m))$.

Case 2. *For every $n \in I\!N$ $\mathcal{E}_k^\star(\lambda_n) = N$ for all but finitely many values of index k.*

Then, for an arbitrary positive integer K, if m denotes the maximal index for which $\mathcal{E}_m^\star(\lambda_K) < N$ then $\lambda_m = L_{m+1}$ follows from the the quasi-regularity of $\mathcal{E}^\star$. Thus we obtain that $H = \{n \in I\!N \mid \lambda_n = L_{n+1}\}$ is an infinite set. Let A and B be subsets of $I\!N$ for which

$$A \cup B = \{n \in I\!N \mid n > \min H\}, \qquad A \cap B = \emptyset,$$

$$(i \in A \text{ and } i+1 \in B) \text{ or } (i \in B \text{ and } i+1 \in A) \Longleftrightarrow i \in H \setminus \{\min H\}.$$

The existence of such sets A and B is clear. Now let

$$\alpha_k = \begin{cases} N, & \text{if } k \in A \\ 0, & \text{otherwise} \end{cases}$$

$$\beta_k = \begin{cases} N, & \text{if } k \in B \\ 0, & \text{otherwise} \end{cases}$$

for all $k \in I\!N$. Then $(\max\{\alpha_n, \beta_n\}) = (\mathcal{E}_n^\star(\lambda_{\min H}))$, which is not a regular sequence, but the regularity of (α_n) and (β_n) follows from the quasi-regularity of $(\mathcal{E}_n^\star(\lambda_{\min H}))$.

Theorem 6. *In the case of* $(\lambda_n) = \left(\frac{1}{(N+1)^n}\right) \in I\!F_N$ *the only associative algorithm is the regular one.*

Proof. Let $x \in [0, L(\lambda)]$. If $\mathcal{E}_n(x) = 0$ except of a finite set of indices n then x will be called a *finite* number. In the case of $(\lambda_n) = \left(\frac{1}{(N+1)^n}\right)$ if an $x \in [0, L(\lambda)]$ has more than one representations of the form

$$x = \sum_{n=1}^{\infty} \delta_n \lambda_n \qquad (\delta_n \in \mathcal{N} \text{ for all } n \in I\!N)$$

then x is a finite number,

$$x = \sum_{n=1}^{m} \mathcal{E}_n(x) \lambda_n \qquad \text{where } \mathcal{E}_m(x) \neq 0,$$

and x has exactly one representation different from the regular one:

$$x = \sum_{n=1}^{m-1} \mathcal{E}_n(x)\lambda_n + (\mathcal{E}_m(x) - 1)\lambda_m + \sum_{n=m+1}^{\infty} N \cdot \lambda_n.$$

We will show that if $\alpha = (\alpha_n)$ is an associative algorithm and x is a finite number then $\alpha_n(x) = \mathcal{E}_n(x)$ for all $n \in I\!N$. Let $x = \sum\limits_{n=1}^{m} \mathcal{E}_n(x)\lambda_n$ where $\mathcal{E}_m(x) \neq 0$. Then

$$x_1 = x + \sum_{n=1}^{\infty} \lambda_{m+2n-1} \quad \text{and} \quad x_2 = x + \sum_{n=1}^{\infty} \lambda_{m+2n}$$

are uniquely representable numbers, so if $\in I\!N$ and $j \in \{1, 2\}$ then

$$\alpha_n(x_j) = \begin{cases} \mathcal{E}_n(x), & \text{if } n \leq m \\ 1, & \text{if } n > m \ \text{ and } \ n - m - j \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $x = x_1 \circ x_2$, thus

$$\alpha_n(x) = \alpha_n(x_1 \circ x_2) = \min\{\alpha_n(x_1), \alpha_n(x_2)\} = \mathcal{E}_n(x) \qquad (n \in I\!N).$$

Theorem 7. *Let $\lambda = (\lambda_n) \in IF_N$ for which there exists $M \in I\!N, M > 1$ such that $\lambda \setminus \{\lambda_M\}$ is still an interval filling sequence of order N. Then the regular algorithm with respect to λ is not the only associative one.*

Proof. Let $\mathcal{E}$ denote the regular algorithm with respect to $\lambda \setminus \{\lambda_M\}$, and let $\diamond$ be the operation defined by $\mathcal{E}$. We define an associative algorithm α with respect to λ which is different from the regular one. If $x \in [0, L(\lambda)]$ then let

$$\alpha_n(x) = \begin{cases} \max\{k \in \mathcal{N} \mid \quad k\lambda_M \le x\}, & \text{if } n = M \\ \mathcal{E}_{\varphi(n)}\big(x - \alpha_M(x)\lambda_M\big), & \text{if } n \ne M, \end{cases}$$

where φ: $I\!N \setminus \{M\} \to I\!N$,

$$\varphi(n) = \begin{cases} n, & \text{if } n < M \\ n - 1, & \text{if } n > M. \end{cases}$$

The condition that $\lambda \setminus \{\lambda_M\}$ is an interval filling sequence implies that α is an algorithm. This algorithm is obviously different from the regular one since it "begins with index M". If $\circ$ denotes the operation defined by α then we have

$$\text{(7)} \qquad x \circ y = \min\{\alpha_M(x), \alpha_M(y)\} \cdot \lambda_M + \big(x - \alpha_M(x)\lambda_M\big) \diamond \big(y - \alpha_M(y)\lambda_M\big).$$

Furthermore, we know that

$$\text{(8)} \qquad \big(x - \alpha_M(x)\lambda_M\big) \diamond \big(y - \alpha_M(y)\lambda_M\big) \le \min\{x - \alpha_M(x)\lambda_M, y - \alpha_M(y)\lambda_M\}.$$

Our purpose is to prove that (2) holds for α. It is obviously true for $n = M$, and if $n \ne M$ then with the help of (7) and (8) we obtain

$$\begin{gathered} \alpha_n(x \circ y) = \\ = \alpha_n\big(\min\{\alpha_M(x), \alpha_M(y)\} \cdot \lambda_M + (x - \alpha_M(x)\lambda_M) \diamond (y - \alpha_M(y)\lambda_M\big) = \\ = \mathcal{E}_{\varphi(n)}\big((x - \alpha_M(x)\lambda_M) \diamond (y - \alpha_M(y)\lambda_M)\big) = \\ = \min\big\{\mathcal{E}_{\varphi(n)}\big(x - \alpha_M(x)\lambda_M\big), \mathcal{E}_{\varphi(n)}\big(y - \alpha_M(y)\lambda_M\big)\big\} = \\ = \min\{\alpha_n(x), \alpha_n(y)\}. \end{gathered}$$

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