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A MOMENT INEQUALITY FOR THE MAXIMUM PARTIAL SUMS WITH A GENERALIZED SUPERADDITIVE STRUCTURE

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Abstract: F. A. Móricz, R. J. Serfling and W. F. Stout (1982) proved a moment inequality with superadditive function. The theorem of this paper extends this result to multidimensional sequence.

1. Notations

In the following $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and d denotes the set of integers, positive integers, real numbers and a fixed positive integer. We define $\underline{1} = (1, 1, \ldots, 1) \in \mathbb{N}^d$ and if $\underline{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$, $\underline{l} = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d$, $k_i \leq l_i$ for each $1 \leq i \leq d$ then $\underline{k} \leq \underline{l}$. The $\underline{k} < \underline{l}$ relation is defined similarly. If there exists an i index such that $k_i \geq l_i$ then we write $\underline{k} \not< \underline{l}$. Denote $|\underline{k}| = \prod_{i=1}^d k_i$ and let $\{X_{\underline{k}} \colon \underline{k} \in \mathbb{N}^d\}$ be a d-multiple sequence of random variables. $S_{\underline{n}}$ will denote the sum $\sum_{\underline{k} \leq \underline{n}} X_{\underline{k}}$ if $\underline{n} \in \mathbb{N}^d$, otherwise $S_{\underline{n}} = 0$. Finally $\mathbb{E}X$ will denote the expectation of the random variable X.

2. Preliminary results

Let $g \colon \mathbb{N}^2 \to \mathbb{R}$ be a nonnegative function. If $g(i,j) + g(j+1,k) \leq g(i,k)$ for all $1 \leq i \leq j < k$ then we say that g is superadditive. F. A. Móricz, R. J. Serfling and W. F. Stout (1982) proved the next theorem: If $\{X_l \colon l \in \mathbb{N}\}$ sequence of random variables, $\alpha > 1$, $r \geq 1$, g is a superadditive function and

$$\mathbb{E}\left|\sum_{l=i}^{j} X_l\right|^r \leq g^\alpha(i,j)$$

for all $1 \leq i \leq j$ integers then there exists a constant $A_{\alpha,r}$ (what depends on α and r) such that for each $n \in \mathbb{N}$

$$\mathbb{E}\left(\max_{k \leq n}\left|\sum_{l=1}^{k} X_l\right|\right)^r \leq A_{\alpha,r} g^\alpha(1,n).$$

F. A. Móricz (1983) generalized the definition of superadditive function for d-dimension as follows. Let $g\colon \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{R}$ be a nonnegative function. If

$$g(\underline{i}, \hat{\underline{j}}) + g(\hat{\underline{i}}, \underline{j}) \le g(\underline{i}, \underline{j}), \tag{2.1}$$

where $\underline{i}, \underline{j} \in \mathbb{N}^d$, $\underline{i} \le \underline{j}$, $1 \le l \le d$, $i_l \le k_l \le j_l$ and

$$\hat{\underline{i}} = (i_1, \dots, i_{l-1}, k_l + 1, i_{l+1}, \dots, i_d),$$
$$\hat{\underline{j}} = (j_1, \dots, j_{l-1}, k_l, j_{l+1}, \dots, j_d)$$

then we say that the g is superadditive. F. A. Móricz (1983) proved the next theorem what is generalization of the previous theorem. If g is a superadditive function, $\alpha > 1$, $r \ge 1$ and

$$\mathbb{E}\left|\sum_{\underline{i} \le \underline{l} \le \underline{j}} X_{\underline{l}}\right|^r \le g^\alpha(\underline{i}, \underline{j}) \tag{2.2}$$

for all $\underline{i}, \underline{j} \in \mathbb{N}^d$, $\underline{i} \le \underline{j}$ then there exists a constant $A_{\alpha,r,d}$ (what depends on α, r and d) such that

$$\mathbb{E}\left(\max_{\underline{k} \le \underline{n}} |S_{\underline{k}}|\right)^r \le A_{\alpha,r,d}\, g^\alpha(\underline{1}, \underline{n})$$

for all $\underline{n} \in \mathbb{N}^d$. This paper discuss another generalization.

2. Main result

Theorem. *Let $g\colon \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{R}$ be a nonnegative function, $\alpha > 1$ and $r \ge 1$. Assume that for each $\underline{1} \le \underline{i} \le \underline{j} < \underline{k}$*

$$g(\underline{i}, \underline{j}) + g(\underline{j} + \underline{1}, \underline{k}) \le g(\underline{i}, \underline{k}), \tag{3.1}$$

and

$$\mathbb{E}|S_{\underline{j}} - S_{\underline{i}-\underline{1}}|^r \le g^\alpha(\underline{i}, \underline{j}). \tag{3.2}$$

Then

$$\mathbb{E}\left(\max_{\underline{k} \le \underline{n}} |S_{\underline{k}}|\right)^r \le A_{\alpha,r}\, g^\alpha(\underline{1}, \underline{n}) \tag{3.3}$$

for all $\underline{n} \in \mathbb{N}^d$ where $A_{\alpha,r} = \left(1 - \frac{1}{2^{(\alpha-1)/r}}\right)^{-r}$.

Remark. This theorem is generalization of result of F. A. Móricz, R. J. Serfling and W. F. Stout (1982). We remark that condition (2.1) implies (3.1) on the other hand if $X_{\underline{k}}$ ($\underline{k} \in \mathbb{N}^d$) nonnegative random variables then (3.2) implies (2.2) moreover the constant is not depending on d.

Proof of Theorem. Assume that $\underline{1} < \underline{N} = (N, N, \ldots, N) \in \mathbb{N}^d$ and $\underline{n} \in \mathbb{N}^d$ where $\underline{n} \le \underline{N}$ and $\underline{n} \not< \underline{N}$. If $|\underline{j}| = 0$ then let $g(\underline{1}, \underline{j}) = 0$. With these notations, since

$$g(\underline{i}, \underline{j}) \le g(\underline{i}, \underline{j}) + g(\underline{j} + \underline{1}, \underline{k}) \le g(\underline{i}, \underline{k}) \quad \forall\ \underline{1} \le \underline{i} \le \underline{j} < \underline{k}, \tag{3.4}$$

there exists $m \ge 0$ integer having the property that

$$g(\underline{1}, \underline{m} - \underline{1}) \le \frac{1}{2} g(\underline{1}, \underline{n}) \le g(\underline{1}, \underline{m}), \tag{3.5}$$

where $\underline{m} = \underline{n} - m \cdot \underline{1}$. So if $\underline{m} < \underline{n}$ then

$$\frac{1}{2} g(\underline{1}, \underline{n}) + g(\underline{m} + \underline{1}, \underline{n}) \le g(\underline{1}, \underline{m}) + g(\underline{m} + \underline{1}, \underline{n}) \le g(\underline{1}, \underline{n}).$$

Consequently we have

$$g(\underline{m} + \underline{1}, \underline{n}) \le \frac{1}{2} g(\underline{1}, \underline{n}), \quad \text{if } \underline{m} < \underline{n}. \tag{3.6}$$

Let us define sets

$$B = \{\underline{k} \in \mathbb{N}^d : \ \underline{k} < \underline{m}\}$$
$$C = \{\underline{k} \in \mathbb{N}^d : \ \underline{k} \le \underline{n}, \underline{k} \not< \underline{m}, \underline{m} \not< \underline{k}\}$$
$$D = \{\underline{k} \in \mathbb{N}^d : \ \underline{m} < \underline{k} \le \underline{n}\}$$

Let $\underline{k}_1 \in D$ such that $|S_{\underline{k}_1}| = \max_{\underline{k} \in D} |S_{\underline{k}}|$. If $D = \emptyset$ (other words $\underline{m} = \underline{n}$) then let $\underline{k}_1 = \underline{m}$. Let $\underline{k}_2 \in C$ such that $|S_{\underline{k}_2}| = \max_{\underline{k} \in C} |S_{\underline{k}}|$. With these notations we have

$$\begin{aligned}
\max_{\underline{k} \le \underline{n}} |S_{\underline{k}}| = & \max\{\max_{\underline{k} < \underline{m}} |S_{\underline{k}}|, |S_{\underline{k}_1}|, |S_{\underline{k}_2}|\} \le \\
& \max\{\max_{\underline{k} < \underline{m}} |S_{\underline{k}}|, |S_{\underline{m}}| + |S_{\underline{k}_1} - S_{\underline{m}}|, |S_{\underline{k}_2}|\} \le \\
& |S_{\underline{k}_2}| + \max\{\max_{\underline{k} < \underline{m}} |S_{\underline{k}}|, |S_{\underline{k}_1} - S_{\underline{m}}|\} \le \\
& |S_{\underline{k}_2}| + \left(\max_{\underline{k} < \underline{m}} |S_{\underline{k}}|^r + |S_{\underline{k}_1} - S_{\underline{m}}|^r \right)^{1/r}.
\end{aligned} \tag{3.7}$$

The Minkowski's inequality states that

$$(\mathbb{E}|X + Y|^r)^{1/r} \le (\mathbb{E}|X|^r)^{1/r} + (\mathbb{E}|Y|^r)^{1/r}$$

where X, Y random variables and $r \geq 1$. Therefore with $X = |S_{\underline{k}_2}|$ and $Y = \max_{\underline{k}\leq\underline{n}} |S_{\underline{k}}| - |S_{\underline{k}_2}|$ substitutions the Minkowski's inequality and (3.7) imply

$$\mathbb{E}\left(\max_{\underline{k}\leq\underline{n}} |S_{\underline{k}}|^r\right)^{1/r} \leq \left(\mathbb{E}|S_{\underline{k}_2}|^r\right)^{1/r} + \left(\mathbb{E}\left|\max_{\underline{k}<\underline{n}} |S_{\underline{k}}| - |S_{\underline{k}_2}|\right|^r\right)^{1/r} \leq$$

$$\left(\mathbb{E}|S_{\underline{k}_2}|^r\right)^{1/r} + \left(\mathbb{E}(\max_{\underline{k}<\underline{m}} |S_{\underline{k}}|^r) + \mathbb{E}|S_{\underline{k}_1} - S_{\underline{m}}|^r\right)^{1/r}. \tag{3.8}$$

By condition (3.2) and (3.4) we get

$$\left(\mathbb{E}|S_{\underline{k}_2}|^r\right)^{1/r} \leq g^{\alpha/r}(\underline{1}, \underline{k}_2) \leq g^{\alpha/r}(\underline{1}, \underline{n}). \tag{3.9}$$

An elementary computation shows that $A_{\alpha,r} \geq 1$ so (3.2), (3.4) and (3.6) imply

$$\begin{aligned}\mathbb{E}|S_{\underline{k}_1} - S_{\underline{m}}|^r \leq & g^{\alpha}(\underline{m}+\underline{1}, \underline{k}_1) \leq g^{\alpha}(\underline{m}+\underline{1}, \underline{n}) \leq \\ & \frac{1}{2^{\alpha}} g^{\alpha}(\underline{1}, \underline{n}) \leq A_{\alpha,r}\frac{1}{2^{\alpha}} g^{\alpha}(\underline{1}, \underline{n}),\end{aligned} \tag{3.10}$$

if $D \neq \emptyset$ (what means $\underline{m} < \underline{k}_1$).

After these we prove the theorem by d-dimensional induction.

$$\mathbb{E}(\max_{\underline{k}\leq\underline{1}} |S_{\underline{k}}|^r) = \mathbb{E}|S_{\underline{1}}|^r \leq g^{\alpha}(\underline{1}, \underline{1}) \leq A_{\alpha,r} g^{\alpha}(\underline{1}, \underline{1})$$

therefore $\underline{n} = \underline{1}$ satisfies (3.3). Now, assume that (3.3) is true if $\underline{n} < \underline{N}$. Thus (3.5) implies

$$\mathbb{E}(\max_{\underline{k}<\underline{m}} |S_{\underline{k}}|^r) \leq A_{\alpha,r} g^{\alpha}(\underline{1}, \underline{m}-\underline{1}) \leq A_{\alpha,r}\frac{1}{2^{\alpha}} g^{\alpha}(\underline{1}, \underline{n}). \tag{3.11}$$

Finally by (3.8), (3.9), (3.10) and (3.11) we obtain

$$\mathbb{E}(\max_{\underline{k}\leq\underline{n}} |S_{\underline{k}}|^r)^{1/r} \leq g^{\alpha/r}(\underline{1}, \underline{n}) + \left(A_{\alpha,r}\frac{1}{2^{\alpha-1}} g^{\alpha}(\underline{1}, \underline{n})\right)^{1/r} = A_{\alpha,r}^{1/r} g^{\alpha/r}(\underline{1}, \underline{n})$$

therefore (3.3) is true for each $\underline{n}$ with $\underline{n} \leq \underline{N}$ and $\underline{n} \not< \underline{N}$. This completes the proof of the theorem.

References

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[2] F. A. Móricz, A general moment inequality for the maximum of the rectangular partial sums of multiple series, *Acta Math. Hung.*, **41 (3–4)** (1983), 337–346.

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