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# Functions having quadratic differences in a given class

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**Abstract.** Starting from a problem of Z. Daróczy we define the quadratic difference property and show that the class of all real-valued continuous functions on $\mathbf{R}$ and some of its subclasses have this property while the class of all bounded functions does not have.

## 1. Introduction

For a function $f\colon \mathbf{R} \to \mathbf{R}$ (the reals) and for a fixed $y \in \mathbf{R}$ define the function $\Delta_y f$ on $\mathbf{R}$ by $\Delta_y f(x) = f(x+y) - f(x)$, $x \in \mathbf{R}$. The functions $A, N\colon \mathbf{R} \to \mathbf{R}$ are said to be additive and quadratic if

$$A(x+y) = A(x) + A(y) \qquad x, y \in \mathbf{R}$$

and

$$N(x+y) + N(x-y) = 2N(x) + 2N(y) \qquad x, y \in \mathbf{R},$$

respectively. It is well-known (see [1], [5], [2]) that, if an additive function is bounded from one side on an interval of positive length then $A(x) = cx$, $x \in \mathbf{R}$ for some $c \in \mathbf{R}$ and there are discontinuous additive functions. Similarly, if a quadratic function is bounded on an interval of positive length then $N(x) = dx^2$, $x \in \mathbf{R}$ for some $d \in \mathbf{R}$ and there are discontinuous quadratic functions.

In [4] Z. Daróczy asked that for which properties $T$ the following statement is true:

$(*)$ Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function such that for all fixed $y \in \mathbf{R}$ the function $\Delta_y \Delta_{-y} f$ has the property $T$. Then

$$f = f^\star + N + A \qquad \text{on} \quad \mathbf{R} \tag{1}$$

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where $f^\star$ has the property $T$, $N$ is a quadratic function and $A$ is an additive function.

In this note we prove that, if $T$ is the $k$-times continuously differentiability ($k \geq 0$ integer) or $k$-times differentiability ($k > 0$ integer or $k = +\infty$) or being polynomial then the statement $(*)$ is true while if $T$ is the boundedness then $(*)$ is not true.

## 2. Preliminary results

The following lemma will play an important role in our investigations.

**Lemma 1.** *For all functions $f\colon \mathbf{R} \to \mathbf{R}$ and for all $u, v, x \in \mathbf{R}$ we have*

$$\begin{aligned} \Delta_u \Delta_v f(x) = {} & \Delta_{\frac{u-v}{2}} \Delta_{-\frac{u-v}{2}} f\left(x + \frac{u+v}{2}\right) \\ & - \Delta_{\frac{u+v}{2}} \Delta_{-\frac{u+v}{2}} f\left(x + \frac{u+v}{2}\right). \end{aligned} \tag{2}$$

The proof is a simple computation therefore it is omitted.

An other basic tool we will use is the following result of DE BRUIJN ([3] Theorem 1.1.)

**Theorem 1.** *Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is a function such that the function $\Delta_y f$ is continuous for all fixed $y \in \mathbf{R}$. Then $f = f^\star + A$ on $\mathbf{R}$ with some continuous $f^\star\colon \mathbf{R} \to \mathbf{R}$ and additive $A\colon \mathbf{R} \to \mathbf{R}$.*

Finally we will need the following two lemmata.

**Lemma 2.** *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function such that $\Delta_u \Delta_v f$ is continuous for all fixed $u, v \in \mathbf{R}$. Define*

$$H(x,u,v) = \Delta_u \Delta_v f(x) - f(u+v) + f(u) + f(v) \qquad x, u, v \in \mathbf{R}. \tag{3}$$

*Then the function $(x,u) \to H(x,u,v)$, $(x,u) \in \mathbf{R}^2$ is continuous for all fixed $v \in \mathbf{R}$.*

**Proof.** Let $v \in \mathbf{R}$ be fixed. Since $\Delta_u(\Delta_v f)$ is continuous for all fixed $u \in \mathbf{R}$, Theorem 1 implies that $\Delta_v f = f_v^\star + A_v$ on $\mathbf{R}$ where $f_v^\star\colon \mathbf{R} \to \mathbf{R}$ is continuous and $A_v$ is additive. Thus, by (3),

$$\begin{aligned} H(x,u,v) &= \Delta_v f(x+u) - \Delta_v f(x) - \Delta_v f(u) + f(v) \\ &= f_v^\star(x+u) - f_v^\star(x) - f_v^\star(u) + f(v) \end{aligned}$$

whence the continuity of $(x,u) \to H(x,u,v)$, $(x,u) \in \mathbf{R}^2$ follows.

**Lemma 3.** *Suppuse that $L$ is one of the classes of the real-valued functions defined on $\mathbf{R}$ which are $k$-times continuously differentiable for some $k \geq 0$ integer or $k$-times differentiable for some $1 \leq k \leq +\infty$ or polynomials. If $f: \mathbf{R} \to \mathbf{R}$ is continuous and $\Delta_y f \in L$ for all fixed $y \in \mathbf{R}$ then $f \in L$.*

**Proof.** If $L$ is the class of the continuous functions ($k = 0$) or of the polynomials, furthermore $f$ is continuous and $\Delta_y f \in L$ for all fixed $y \in \mathbf{R}$ then, by Theorem 1 and by [3] page 203, respectively, $f = f^\star + A$ for some $f^\star \in L$ and additive function $A$. Therefore, by continuity of $f$, $A(x) = cx$, $x \in \mathbf{R}$ with some $c \in \mathbf{R}$ whence $f \in L$ follows.

The remaining statement of Lemma 3 is just Lemma 3.1. in [3].

## 3. The main results

For the formulation of our main results let us begin with the following

**Definition.** A subset $E$ of the set of all functions $f: \mathbf{R} \to \mathbf{R}$ is said to have the quadratic difference property if for all $f: \mathbf{R} \to \mathbf{R}$, with $\Delta_y \Delta_{-y} f \in E$ for all $y \in \mathbf{R}$, the decomposition (1) holds true on $\mathbf{R}$ where $f^\star \in E$, $N$ is a quadratic function and $A$ is an additive function.

First we prove the following

**Theorem 2.** *The class of all continuous functions $f: \mathbf{R} \to \mathbf{R}$ has the quadratic difference property.*

**Proof.** By (2) in Lemma 1 we have that $\Delta_u \Delta_v f$ is continuous for all fixed $u, v \in \mathbf{R}$. In particular, $\Delta_u(\Delta_1 f)$ is continuous for all fixed $u \in \mathbf{R}$. Applying Theorem 1 to $\Delta_1 f$ we have

$$\Delta_1 f = f_0 + a \qquad \text{on } \mathbf{R} \tag{4}$$

with some continuous $f_0: \mathbf{R} \to \mathbf{R}$ and additive $a: \mathbf{R} \to \mathbf{R}$. Define the function $B$ on $\mathbf{R}^2$ by

$$B(u,v) = \int_0^1 \Delta_u \Delta_v f - \int_0^{u+v} f_0 + \int_0^u f_0 + \int_0^v f_0, \qquad (u,v) \in \mathbf{R}^2. \tag{5}$$

Obviously, $B$ is symmetric. Now we show that $B$ is additive in its first variable. For all $u$, $t$ and $v$, we have

$$B(u+t,v) - B(u,v) = \int_0^1 \Delta_{u+t} \Delta_v f - \int_0^{u+t+v} f_0 + \int_0^{u+t} f_0 + \int_0^v f_0$$

$$- \int_0^1 \Delta_u \Delta_v f + \int_0^{u+v} f_0 - \int_0^u f_0 - \int_0^v f_0$$
$$= \int_u^{u+1} \Delta_t \Delta_v f - \int_0^{u+t+v} f_0 + \int_0^{u+t} f_0 + \int_0^{u+v} f_0 - \int_0^u f_0.$$

Since $\Delta_t \Delta_v f$ and $f_0$ are continuous functions, the right hand side is continuously differentiable with respect to $u$ then so is the left hand side. Differentiating both sides with respect to $u$ and taking into consideration (4) we obtain that

$$\begin{aligned} \frac{\partial}{\partial u}[B(u+t,v) - B(u,v)] &= \Delta_t \Delta_v \Delta_1 f(u) - f_0(u+t+v) + f_0(u+t) \\ &\quad + f_0(u+v) - f_0(u) \\ &= \Delta_t \Delta_v (f_0 + a)(u) - \Delta_t \Delta_v f_0(u) \\ &= \Delta_t \Delta_v a(u) = 0 \qquad (a \text{ being additive}). \end{aligned}$$

Therefore

$$B(u+t,v) - B(u,v) = B(t,v) - B(0,v) = B(t,v),$$

that is, $B$ is additive in its first (and by the symmetry also in its second) variable. Thus, it is well-known (see [2]) and easy to see that, the function $N\colon \mathbf{R} \to \mathbf{R}$ defined by $N(u) = \frac{1}{2}B(u,u)$, $u \in \mathbf{R}$ is quadratic and

$$B(u,v) = N(u+v) - N(u) - N(v) \qquad u,v \in \mathbf{R}. \tag{6}$$

Define the function $H\colon \mathbf{R}^3 \to \mathbf{R}$ by (3) and apply Lemma 2 to get the continuity of the function $(x,u) \to H(x,u,v)$, $(x,u) \in \mathbf{R}^2$ for all fixed $v \in \mathbf{R}$. This implies that the function $s\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$s(u,v) = \int_0^1 H(x,u,v)dx \qquad (u,v) \in \mathbf{R}^2$$

is continuous in its first variable (for all fixed $v \in \mathbf{R}$). Therefore, by (3), (5) and (6) we have

$$s(u,v) = \int_0^1 \Delta_u \Delta_v f - f(u+v) + f(u) + f(v)$$

$$
\begin{aligned}
&= B(u,v) + \int_0^{u+v} f_0 - \int_0^{u} f_0 - \int_0^{v} f_0 - f(u+v) + f(u) + f(v) \\
&= N(u+v) - f(u+v) - (N(u) - f(u)) - (N(v) - f(v)) \\
&\quad + \int_0^{u+v} f_0 - \int_0^{u} f_0 - \int_0^{v} f_0 \\
&= -\Delta_v(f-N)(u) - (N(v) - f(v)) + \int_0^{u+v} f_0 - \int_0^{u} f_0 - \int_0^{v} f_0.
\end{aligned}
$$

This implies that $\Delta_v(f-N)$ is continuous for all fixed $v \in \mathbf{R}$ and Theorem 1 can be applied again to get the decomposition $f - N = f^\star + A$ on $\mathbf{R}$ with some continuous $f^\star\colon \mathbf{R} \to \mathbf{R}$ and additive function $A$, that is, (1) holds and the proof is complete.

**Theorem 3.** *Let $L$ be as in Lemma 3. Then $L$ has the quadratic difference property.*

**Proof.** If $L$ is the class of all continuous functions then the statement is proved by Theorem 2. In the remaining cases, since all functions in $L$ are continuous, Theorem 2 implies the decomposition (1) with continuous $f^\star$, quadratic $N$ and additive $A$. We now prove that $f^\star \in L$. For all $y \in \mathbf{R}$ we get from (1) that

$$
\Delta_y \Delta_{-y} f = \Delta_y \Delta_{-y} f^\star + 2N(y). \tag{7}
$$

Therefore $\Delta_y \Delta_{-y} f^\star \in L$ for all fixed $y \in \mathbf{R}$. Applying (2) in Lemma 1 we obtain that $\Delta_u(\Delta_v f^\star) \in L$ for all fixed $u, v \in \mathbf{R}$. Obviously $\Delta_v f^\star$ is continuous thus, by Lemma 3, $\Delta_v f^\star \in L$. Since $f^\star$ is continuous, Lemma 3 can be applied again to get $f^\star \in L$.

**Remark.** The set of all bounded functions $f\colon \mathbf{R} \to \mathbf{R}$ does not have the quadratic difference property. Indeed, let

$$
f(x) = x \ln(x^2+1) + 2 \operatorname{arc\,tg} x - 2x, \qquad x \in \mathbf{R}.
$$

Applying the Lagrangian mean value theorem with fixed $u, v, x \in \mathbf{R}$ we have

$$
\Delta_u \Delta_v f(x) = u \Delta_v f'(\xi) = uv f''(\eta) \tag{8}
$$

for some $\xi, \eta \in \mathbf{R}$. Since $|f''(\eta)| = \frac{2|\eta|}{\eta^2+1} \leq 1$, (8) implies that $|\Delta_y \Delta_{-y} f(x)| \leq y^2$ for all $x, y \in \mathbf{R}$, that is, $\Delta_y \Delta_{-y} f$ is bounded for all fixed $y \in \mathbf{R}$. Suppose that $f$ has the decomposition (1) for some bounded $f^\star\colon \mathbf{R} \to \mathbf{R}$, quadratic $N$ and additive $A$. Then $N + A$ must be bounded on any bounded interval. Thus $N(x) + A(x) = \alpha x^2 + \beta x$, $x \in \mathbf{R}$ for some $\alpha, \beta \in \mathbf{R}$. This and (1) imply that

$$f^\star(x) = x \ln(x^2+1) + 2 \operatorname{arc\,tg} x - \alpha x^2 - (2+\beta)x, \qquad x \in \mathbf{R}. \tag{9}$$

Since $f^\star$ is bounded, $0 = \lim\limits_{x \to +\infty} \frac{f^\star(x)}{x^2} = -\alpha$ and thus

$$0 = \lim_{x \to +\infty} \frac{f^\star(x) - 2 \operatorname{arc\,tg} x}{x} = \lim_{x \to +\infty} \left( \ln(x^2+1) - (2+\beta) \right),$$

which is a contradiction. This shows that the set of all bounded functions does not have the quadratic difference property.

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