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The generalization of Pascal's triangle from algebraic point of view

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Abstract. In this paper we generalize Pascal's Triangle and examine the connections between the generalized triangles and powering integers and polynomials respectively.

The interesting and really romantic Pascal's Triangle is a favourite research field of mathematicians for a very long time. The table of binomial coefficients has been named after Blaise Pascal, a French scientist, but was known already by the ancient Chinese and others before Pascal (Edwards [1]).

Among the elements of the triangle a lot of interesting connections exist. One of them is that from the n-th row of the triangle with positional addition we get the n-th power of 11 (Figure 1.), where n is a non-negativ integer, and the indices in the rows and columns run from 0.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \dots & & & & & \end{array}$$

$$1 = 11^0,\ 11 = 11^1,\ 121 = 11^2,\ 1331 = 11^3,\ 14641 = 11^4,\ 161051 = 11^5, \dots$$

Figure 1: The powers of 11 in Pascal's triangle

This comes immediately from the binomial equality

$$\binom{n}{0}10^n + \binom{n}{1}10^{n-1} + \binom{n}{2}10^{n-2} + \dots + \binom{n}{n-1}10^1 + \binom{n}{n}10^0 = 11^n$$

An interesting way of generalizing is if we construct triangles in which the powers of other numbers appear. To achieve this, let us consider Pascal's Triangle as the 11-based triangle, and take the following.

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Definition. Let a and b integers, with $0 \leq a, b \leq 9$. Then we can get the k-th element in the n-th row of the ab-based triangle if we add the $k-1$-th elemetn in the $n-1$-th row b-times to the k-th element in the $n-1$-th row a-times. If $k-1<0$ or $k>n-1$ (id est the element in the $n-1$-th row does not exist according to the traditional implementation) then we consider this element to be 0 (Figure 2.). The indices in the rows and columns of the triangle run from 0.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 4 & & 7 & & \\ & 16 & & 56 & & 49 & \\ 64 & & 336 & & 588 & & 343 \\ & & & \dots & & & \end{array}$$

Figure: 2: The 47-based triangle

Example. In the third row of the 47-based triangle $64 = 7 \cdot 0 + 4 \cdot 16$ and $336 = 7 \cdot 16 + 4 \cdot 56$.

Proposition 1. *By positional addition from the n-th row of the ab-based triangle we get the n-th power of $ab(10a+b)$.*

Proof. From the expansion of $(10a+b)^n$ we get

$$(10a+b)^n = \binom{n}{0}a^n 10^n + \binom{n}{1}a^{n-1}b10^{n-1} + \cdots + \binom{n}{n-1}ab^{n-1}10 + \binom{n}{n}b^n.$$

This is exactly the number we get after positional addition from the n-th row.

The structure of the ab-based triangle is relatively simple. We have the following.

Proposition 2. *The k-th element in the n-th row of the ab-based triangle is $a^{n-k}b^k C_n^k$, where C_n^k (the number of combinations of n things taken k at a times) is the k the element in the n-th row of Pascal's Triangle.*

Proof. We prove by induction. In the first row we have $a = a^1 \cdot 1$ and $b = b^1 \cdot 1$. Let us now assume, that the $k-1$-th element in the $n-1$-th row is $a^{n-k}b^{k-1}C_{n-1}^{k-1}$ and the k-th element in the $n-1$-th row is $a^{n-k-1}b^k C_{n-1}^k$. Then the k-th element in the n-th row by definition is

$$\begin{aligned} ba^{n-k}b^{k-1}C_{n-1}^{k-1} + aa^{n-k-1}b^k C_{n-1}^k &= a^{n-k}b^k C_{n-1}^{k-1} + a^{n-k}b^k C_{n-1}^k \\ &= a^{n-k}b^k \left(C_{n-1}^{k-1} + C_{n-1}^k\right) = a^{n-k}b^k C_n^k. \end{aligned}$$

Proposition 3. Connection with the binomial theorem.
The elements in the n-th row of the ab-based triangle are the coefficients of the polynomials $(ax + by)^n$.

Proof. If we substitute ax with $10a$ and by with b in the Proof of Proposition 1, and use that the k-th element in the n-th row of the ab-based triangle is $a^{n-k}b^kC_n^k$ we get the statement.

Example. From the 47-based triangle $(4x + 7y)^3 = 64x^3 + 336x^2y + 588xy^2 + 343y^3$.

The base-number of the triangle can consist of not only 2, but arbitrarily many digits.

Definition. Let $0 \le a_0, a_1, a_2, \ldots, a_{m-2}, a_{m-1} \le 9$ be integers. Then we can get the k-th element in the n-th row of the $a_0a_1a_2 \ldots a_{m-2}a_{m-1}$-based triangle if we multiply the $k - m$-th element in the $n - 1$-th row by a_{m-1}, the $k - m + 1$-th element in the $n - 1$-th row by a_0, and add the products. If for some i we have $k - m + i < 0$ or $k - m + i > n - 1$ (id est some element in the $n - 1$-th row does not exists according to the traditional implementation) then we consider this element to be 0. The indices in the rows and columns of the triangle run from 0 (Figure 3.).

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 4 & 3 & 5 & & \\
 & 16 & 24 & 49 & 30 & 25 & \\
64 & 144 & 348 & 387 & 435 & 225 & 125 \\
 & & & \ldots & & &
\end{array}$$

Figure 3: The 435-based triangle

Remarks. In the above definition we can allow for the base-number not only $0 \le a_0, a_1, a_2, \ldots, a_{m-2}, a_{m-1} \le 9$ digits, but arbitrary integers, rational and irrational numbers. Thus for example we can build triangles with base of root expressions (Figure 4.).

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \sqrt{2} & \sqrt{3} & \sqrt{5} & & \\
 & 2 & 2\sqrt{6} & 2\sqrt{10}+3 & 2\sqrt{15} & 5 & \\
2\sqrt{2} & 6\sqrt{3} & 6\sqrt{5}+9\sqrt{30} & 6\sqrt{30}+3\sqrt{3} & 9\sqrt{5}+15\sqrt{2} & 15\sqrt{3} & 5\sqrt{5} \\
 & & & \ldots & & &
\end{array}$$

Figure 4: The $\sqrt{2}\ \sqrt{3}\ \sqrt{5}$-based triangle

In the combinatorical literature the $11\dots1$-based triangles (k pieces of 1 digits) with name order k (Vilenkin [2]) or k-th Pascal's triangle (Gerőcs [3]) can be found. The authors gave this different definition because of different approach.

Theorem 1. *From the n-th row of the $a_0a_1a_2\dots a_{m-2}a_{m-1}$-based triangle after positional addition we get the n-th power of the base-number $a_0a_1a_2\dots a_{m-2}a_{m-1}$ is obviously in the first row of the triangle.*

Let us now assume, that in the $n-1$-th row ($n>1$) we have the elements $b_0, b_1, b_2, \dots, b_{p-1}, b_p$ (where p equals to $m+(m-1)(n-2)-1 = mn-m-n+1$, because in the first row there are m pieces of elements and in every new row there are $m-1$ pieces more), and from these elements with positional addition we get the $n-1$-th power of the number $a_0a_1a_2\dots a_{m-2}a_{m-1}$. Then we can write out $(a_0 10^{m-1} + a_1 10^{m-2} + \dots + a_{m-2}10 + a_{m-1})^n$ as

$$\begin{aligned}(b_0 10^p + b_1 10^p + b_1 10^{p-1} + \dots + b_{p-1}10 + b_p)&a_0 10^{m-1}\\ &+ (b_0 10^p + b_1 10^{p-1} + \dots + b_{p-1}10 + b_p)a_1 10^{m-2}\\ &\dots\\ &+ (b_0 10^p + b_1 10^{p-1} + \dots + b_{p-1}10 + b_p)a_{m-2}10\\ &+ (b_0 10^p + b_1 10^{p-1} + \dots + b_{p-1}10 + b_p)a_{m-1}.\end{aligned}$$

By adding these expressions (using that $p = mn-m-n+1$) we get

$$\begin{aligned}&a_0b_0 10^{mn-n} + (a_0b_1 + a_1b_0)10^{mn-n-1} + (a_0b_2 + a_1b_1 + a_2b_0)10^{mn-n-2}\\ &+ \dots + (a_{m-1}b_0 + a_{m-2}b_1 + a_{m-3}b_2 \dots + a_1b_{m-2} + a_0b_{m-1})10^{mn-m-n-1}\\ &+ \dots + (a_{m-2}b_p + a_{m-1}b_{p-1})10 + a_{m-1}b_p.\end{aligned}$$

And this is exactly the number we get after positional addition from the n-th row of the triangle.

Consideration of effectivity. This method is easy to algorithmize, it is enough to store the proceeding row and the base-number to determine one row of the triangle. In the row we work with relatively small numbers (compare with the final result), and we have to multiply only with digits. However, to reach the n-th row we need to determine $(2+(n-1)(m-1))\frac{n}{2} = O(n^2m)$ elements. So obviously, if we need only the $n$the power of the base-number some other methods are more effective (Knuth [4]). However, if we need all the (non-negative integer) powers up to n of the base-number this method is competitive. It is especially interesting that with this method the first some powers of a base number of a few digits can even be determined by heart. It is similar to some methods of by heart calculate artists (Surányi [5]).

In Proposition 3 we have seen a connection of the ab-based triangle with the binomial theorem. Thus, we expect for the $a_0a_1a_2\dots a_{m-2}a_{m-1}$-based triangle a relation with the polynomial theorem. However, the structure of the latter triangle is much more complicated. See for example the triangle with abc base-number (Figure 5.) The elements in the n-th row are some sums of the coefficients of the polynomials $(ax+by+cz)^n$.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & a & b & c & & \\
 & a^2 & 2ab & 2ac+b^2 & 2bc & c^2 & \\
a^3 & 2a^2b & 3a^2c+3ab^2 & 6abc+b^3 & 3ac^2+3b^2c & 3bc^2 & c^3 \\
 & & & \dots & & &
\end{array}$$

Figure 5: The abc-based triangle

To discover the connection of the general triangle with the polynomial theorem we need the following.

Definition. For the digits of the base-number let he weight of a digit be its distance from the centerline. So $w(a_0) = -w(a_{m-1}), w(a_1) = -w(a_{m-2})$, etc. If the base number is odd, then $w(a_{(m-1)/2}) = 0$. Let the unit of the weights be the distance of two neighbouring elements in the triangle, id est $w(a_i) = w(a_{i+1}) = 1$.

Example. In the abc-based triangle $w(a) = -1$, $w(b) = 0$ and $w(c) = 1$, in the $abcd$-based triangle $w(a) = -1.5$, $w(b) = -0.5$, $w(c) = 0.5$ and $w(d) = 1.5$.

We would like to extend this idea to the elements of the other rows. Because the elements of the triangle are sums, consider first the parts of them. For such an expression let the weight of the part be the sum of the weights of its digits. If a digit is on the i-th power then we count its weight i-times.

Example. One part of the third element in the third row of the abc-based triangle is $3a^2c$ (Figure 5.). For this expression we have $q(3a^2c) = 2w(a) + w(c) = -1$.

Lemma 1. *In an element of the general triangle the weights of the parts are identical, and this weight is the distance of the element from the centerline.*

Proof. We get this result by induction immediately from the construction of the triangle.

Lemma 2. *Let us consider an expression* $a_0^{i_0}a_1^{i_1}\cdots a_{m-2}^{i_{m-2}}a_{m-1}^{i_{m-1}}$, *for which* $i_0+i_1+\cdots+i_{m-2}+i_{m-1}=n$. *Then we can find this expression with some coefficient as a part of the element with the same weight in the n-th row of the general triangle.*

Proof. Let us assume indirectly that this expression does not exist in the n-th row of the general triangle as a part of the element with corresponding weight. We should get this expression from parts of elements of the previous row

$$\Big(a_0^{i_0-1}a_1^{i_1}\cdots a_{m-2}^{i_{m-2}}a_{m-1}^{i_{m-1}}\,,a_0^{i_0}a_1^{i_1-1}\cdots a_{m-2}^{i_{m-2}}a_{m-1}^{i_{m-1}},$$

$$a_0^{i_0}a_1^{i_1}\cdots a_{m-2}^{i_{m-2}-1}a_{m-1}^{i_{m-1}},\,a_0^{i_0}a_1^{i_1}\cdots a_{m-2}^{i_{m-2}}a_{m-1}^{i_{m-1}-1}\Big)$$

with multiplication (by a_0, $a_1,\dots,a_{m-2},a_{m-1}$). Thus, these parts of the elements can't exist in the previous row. Proceeding backwards with this method we conclude that in the first line some digits of the base-number do not exist, and this is a contradiction.

Lemma 3. *For the coefficient* e *of the expression* $ea_0^{i_0}a_1^{i_1}\cdots a_{m-1}^{i_{m-1}}$ *with* $i_0+i_1+\cdots+i_{m-1}=n$ *in the n-th row of the general triangle we have* $e=\frac{n!}{i_0!i_1!\cdots i_{m-1}!}$.

Proof. We prove with induction. In the first row the statement is true. Let us now assume that in the $n-1$-th row there are the following expressions as parts of the elements:

$$e_0a_0^{i_0-1}a_1^{i_1}\cdots a_{m-1}^{i_{m-1}},\ e_1a_0^{i_0}a_1^{i_1-1}\cdots a_{m-1}^{i_{m-1}},\dots,\ e_{m-1}a_0^{i_0}a_1^{i_1}\cdots a_{m-1}^{i_{m-1}-1},$$

with coefficients

$$e_0=\frac{(n-1)!}{(i_0-1)!i_1!\cdots i_{m-1}!}\ ,\quad e_1=\frac{(n-1)!}{i_0!(i_1-1)!\cdots i_{m-1}!},\dots,$$

$$e_{m-1}=\frac{(n-1)!}{i_0!i_1!\cdots(i_{m-1}-1)!}$$

(by the induction assumption). Thus, the coefficient e of the expression $ea_0^{i_0}a_1^{i_1}\cdots a_{m-1}^{i_{m-1}}$ is the sum of the coefficients e_i $(0\le i\le m-1)$

$$e=e_0+e_1+\cdots+e_{m-1}=\frac{(n-1)!}{(i_0-1)!(i_1-1)!\cdots(i_{m-1}-1)!}$$

$$\left(\frac{1}{i_1 i_2 \cdots i_{m-1}} + \frac{1}{i_0 i_2 \cdots i_{m-3} i_{m-1}} + \cdots + \frac{1}{i_0 \cdots i_{m-3} i_{m-2}}\right) =$$

$$\frac{(n-1)!}{(i_0-1)!(i_1-1)!\cdots(i_{m-1}-1)!}\left(\frac{i_0+i_1+\cdots+i_{m-1}}{i_0 i_1 \cdots i_{m-1}}\right) = \frac{n!}{i_0! i_1! \cdots i_{m-1}!}.$$

By these three Lemmas we have proved the following.

Theorem 2. *The elements in the n-th row of the $a_0a_1a_2\cdots a_{m-2}a_{m-1}$-based triangle are exactly the sums of the coefficients of the polynomial $(a_0x_0 + a_1x_1 + a_2x_2 + \cdots + a_{m-2}x_{m-2} + a_{m-1}x_{m-1})^n$, in which the weights of the parts are identical.*

Like among the binomial coefficients in Pascal's triangle (for example Edwards [1] and Vilenkin [2]), in the general triangle there are also interesting connecitons among the elements. One of them comes immediately from the second Theorem.

Corollary. *In the n-th row of the $a_0a_1a_2\ldots a_{m-2}a_{m-1}$-based triangle the sum of the elements (with normal addition) is $(a_0 + a_1 + a_2 + \cdots + a_{m-2} + a_{m-1})^n$.*

Proof. If we set in the polynomial

$$(a_0x_0 + a_1x_1 + a_2x_2 + \cdots + a_{m-2}x_{m-2} + a_{m-1}x_{m-1})^n,$$

$1 = x_0 = x_1 = x_2 = \cdots = x_{m-2} = x_{m-1}$, then from Theorem 2 in the n-th row of the triangle there are the coefficients of the "polynomial" $(a_0 + a_1 + a_2 + \cdots + a_{m-2} + a_{m-1})^2$.

Remark. In Pascal's triangle from this Corollary we get the well known combinatorical equality

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Another possibility to power polynomials is that we extend the property for the general triangle, that the elements in the n-th row of Pascal's Triangle are the coefficients of the binomial $1 + x$.

Proposition 4. *The elements in the n-th row of the general triangle are exactly the coefficients of the polynomials $(a_0 + a_1x + a_2x^2 + \cdots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1})^n$, the k-th element is the coefficient of x^k.*

Proof. We prove by induction. In the first row the statement is true. Let us now assume, that in the $n-1$-th row there are the coefficients of the polynomial $(a_0 + a_1x + a_2x^2 + \cdots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1})^{n-1}$, the k-th element is the coefficient of x^k. If we multiply this polynomial by $a_0 + a_1x + a_2x^2 + \cdots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1}$, and add up the results (similarly as in the proof of Theorem 1), we get the n-th power of the basic polynomial. But according to the forming rules of the triangle, the coefficients of this polynomial are exactly the elements of the n-th row.

Example. From the third row of the 435-based triangle (Figure 3.)

$$(4 + 3x + 5x^2)^3 = 64 + 144x + 348x^2 + 387x^3 + 435x^4 + 225x^5 + 125x^6.$$

Consideration of effectivity. The powering of polynomials is considerably more complex operation as powering of (integer) numbers. However, the consideration above applies here, too. So if we need only the n-th power of the base-polynomial some other methods are more effective (Knuth [4], Geddes [6].) However, if we need all the (non-negativ integer) powers up to n then this method is competitive.

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