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Residual Lie nilpotence of the augmentation ideal

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Abstract. In this paper we give necessary and sufficient conditions for the residual Lie nilpotence of the augmentation ideal for an arbitrary group ring RG except for the case when the derived group of G is with no generalized torsion elements with respect to the lower central series of G and the torsion subgroup of the additive group of R contains a non-trivial element of infinite height. From this results we get the residual Lie nilpotence of the augmentation ideal of the p-adic integer group rings.

1. Introduction

Let R be a commutative ring with identity, G a group and RG its group ring. The group ring RG may be considered as a Lie algebra, with the usual bracket operation. The study of this Lie algebra was initiated by I. B. S. Passi, D. S. Passman and S. K. Sehgal [5]. Additional results on the Lie structure of RG may be found in [4] and [6].

Let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the homomorphism RG onto R which sends each group element to 1. It is easy to see that as R-module $A(RG)$ is a free module with elements $g-1\,(g \in G)$ as a basis.

There are many problems and results relating to $A(RG)$ ([4], [6]). In particular, it is an interesting problem to characterize the group rings whose augmentation ideal satisfy some conditions. In this paper, we treat the Lie property.

The Lie powers $A^{[\lambda]}(RG)$ of $A(RG)$ are defined inductively: $A^{[1]}(RG) = A(RG)$, $A^{[\lambda+1]}(RG) = [A^{[\lambda]}(RG), A(RG)]{\cdot}RG$, if λ is not a limit ordinal, and for the limit ordinal λ, $A^{[\lambda]}(RG) = \cap_{\nu<\lambda} A^{[\nu]}(RG)$, where $[K, M]$ denotes the R-submodule of RG generated by $[k, m] = km - mk$ ($k \in K \subseteq RG$, $m \in M \subseteq RG$), and for $K \cdot RG$ denotes the right ideal generated by K in RG.

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For the first limit ordinal ω we adopt the notation:

$$A^{[\omega]}(RG) = \bigcap_{i=1}^{\infty} A^{[i]}(RG).$$

The ideal $A(RG)$ of the group ring RG is said to be *residually Lie nilpotent* if $A^{[\omega]}(RG) = 0$.

In this paper we give necessary and sufficient conditions for the residual Lie nilpotence of the augmentation ideal for an arbitrary group ring RG except for the case when the derived group of G is with no generalized torsion elements with respect to the lower central series of G and the torsion subgroup of the additive group of R contains a non-trivial element of infinite height.

Our main results are given in section 3. These results (Theorem A, B and C) are rather technical so they are not stated in the introduction.

2. Notations and some known facts

If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short) denotes the ideal of RG generated by elements of the form $h-1, (h \in H)$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi}: RG \to RG/H$ induced by the group homomorphism ϕ of G onto G/H. It is clear that $I(RG) = A(RG)$.

Let F be a free group on the free generators x_i $(i \in I)$ and ZF be its integral group ring (Z denotes the ring of rational integers). Then every homomorphism $\phi: F \to G$ induces a ring homomorphism $\overline{\phi}: ZF \to RG$ by letting $\overline{\phi}(\sum n_y y) = \sum n_y \phi(y)$. If $f \in ZF$, we denote by $A_f(RG)$ the two-sided ideal of RG generated by the elements $\overline{\phi}(f)$, $\phi \in \mathrm{Hom}(F,G)$, the set of homomorphism from F to G. In other words $A_f(RG)$ is the ideal generated by the values of f in RG as the elements of G are substituted for the free generators x_i-s.

An ideal J of RG is called a *polynomial ideal* if $J = A_f(RG)$ for some $f \in ZF$. It is easy to see that the augmentation ideal $A(RG)$ is a polynomial ideal. Really, $A(RG)$ is generated as an R-module by elements $g-1$ $(g \in G)$, i.e. by the values of the polynomial $x-1$.

We also use the following

Lemma 2.1. ([4], Proposition 1.4., page 2.) *Let $f \in ZF$. Then f defines a polynomial ideal $A_f(RG)$ in every group ring RG. Further, if $\theta: RG \to KH$*

is a ring homomorphism induced by a group homomorphism $\phi: G \to H$ and a ring homomorphism $\psi: R \to K$, then

$$\theta(A_f(RG)) \subseteq A_f(KH).$$

(It is assumed here that $\psi(1_R) = 1_K$, where 1_R and 1_K are identities of rings R and K respectively.)

For every natural number n $A^{[n]}(RG)$ is a polynomial ideal (see in particular [4], Corollary 1.9., page 6.) and by Lemma 2.1.

$$\overline{\phi}(A^{[n]}(RG)) \subseteq A^{[n]}(RG/L)$$

for every n. From this inclusion it can be obtained easily that

$$\overline{\phi}(A^{[\omega]}(RG)) \subseteq A^{[\omega]}(RG/L). \tag{1}$$

If $\mathcal{K}$ denotes a class of groups we define the class $\mathbf{R}\mathcal{K}$ of residually-$\mathcal{K}$ groups by letting $G \in \mathbf{R}\mathcal{K}$ if and only if: whenever $1 \neq g \in G$, there exists a normal subgroup H_g of the group G such that $G/H_g \in \mathcal{K}$ and $g \notin H_g$. It is easy to see that $G \in \mathbf{R}\mathcal{K}$ if and only if there exists a family $\{H_i\}_{i\in I}$ of normal subgroups G such that $G/H_i \in \mathcal{K}$ for every $i \in I$ and $\cap_{i\in I} H_i = \langle 1\rangle$.

A group G is said to be *discriminated* by $\mathcal{K}$ if for every finite set $g_1, g_2, \ldots, g_n$ of distinct elements of G, there exists a group $H \in \mathcal{K}$ and a homomorphism $\phi: G \to H$ such that $\phi(g_i) \neq \phi(g_j)$ if $i \neq j, (1 \leq i, j \leq n)$.

Lemma 2.2. *Let a class of groups $\mathcal{K}$ be closed with respect to forming subgroups and finite direct products and let G be a residually-$\mathcal{K}$ group. Then G is discriminated by $\mathcal{K}$.*

The proof can be obtained easily.

It is easy to show that if G is discriminated by a class of groups $\mathcal{K}$ and if x is a non-zero element of RG, then there exists a group $H \in \mathcal{K}$ and a homomorphism ϕ of RG to RH such that $\phi(x) \neq 0$.

From this fact and from inclusion (1) we have

Lemma 2.3. *If G is discriminated by a class of groups $\mathcal{K}$ and for each $H \in \mathcal{K}$ the equation $A^{[\omega]}(RH) = 0$ holds, then $A^{[\omega]}(RG) = 0$.*

We use the following notations for standard group classes:
$\mathcal{D}_0$ — the class of those nilpotent groups whose derived groups are torsion-free.
$\mathcal{D}_p$ — the class of nilpotent groups whose derived groups are p-groups of bounded exponent.

$\mathcal{N}_0$ — the class of torsion-free nilpotent groups.
$\mathcal{N}_p$ — the class of nilpotent p-groups of bounded exponent.
$\mathcal{N}_\Omega = \cup_{p\in\Omega}\mathcal{N}_p$ and
$\mathcal{D}_\Omega = \cup_{p\in\Omega}\mathcal{D}_p$, where Ω is a subset of the set of primes.

The ideal $J_p(R)$ of a ring R is defined by $J_p(R) = \cap_{n=1}^{\infty} p^n R$.

Theorem 2.4. ([4], Theorem 2.13., page 85.) *Let G be a residually $\mathcal{D}_p$-group and $J_p(R) = 0$. Then $A^{[\omega]}(RG) = 0$.*

We shall use the following lemma, which gives some elementary properties of the Lie powers of $A(RG)$.

Lemma 2.5. ([4], Proposition 1.7., page 4.) *For arbitrary natural numbers n and m are true:*
(1) $I(\gamma_n(G)) \subseteq A^{[n]}(RG)$,
(2) $[A^{[n]}(RG), A^{[m]}(RG)] \subseteq A^{[n+m]}(RG)$,
(3) $A^{[n]}(RG)\cdot A^{[m]}(RG) \subseteq A^{[n+m-1]}(RG)$,
where $\gamma_n(G)$ is the nth term of the lower central series of G.

We write $D_{[n]}(RG)$ for the nth *Lie dimension subgroup* $D_{[n]}(RG)$ of G over R. That is

$$D_{[n]}(RG) = \{g \in G | g - 1 \in A^{[n]}(RG)\}.$$

By Lemma 2.5. it follows that for every natural number n the inclusion

$$\gamma_n(G) \subseteq D_{[n]}(RG)$$

holds.

We also use the following theorems

Theorem 2.6. ([1], Theorem 3.2.) *Let a group G contain a non-trivial generalized torsion element. Then $A(RG)$ is residually nilpotent if and only if there exists a non-empty subset Ω of the set of primes such that $\cap_{p\in\Omega} J_p(R) = 0$, G is discriminated by the class $\mathcal{N}_\Omega$ and for every proper subset Λ of the set Ω at least one of the conditions*
(1) $\cap_{p\in\Lambda} J_p(R) = 0$
(2) *G is discriminated by the class of groups $\mathcal{N}_{\Omega\setminus\Lambda}$*
holds.

Let $T(R^+)$ denote the torsion subgroup of the additive group R^+ of a ring R and let $A^\omega(RG) = \cap_{i=1}^{\infty} A^n(RG)$, where $A^n(RG)$ is the nth associative power of $A(RG)$.

Theorem 2.7. ([4], Theorem 2.7., page 87.) *If $G \in \mathbf{R}\mathcal{N}_0$ and R is a ring with identity such that its additive group R^+ is torsion-free, then $A^\omega(RG) = 0$.*

3. Residual Lie nilpotence

It is clear, that $A^{[2]}(RG) = 0$ if and only if G is an Abelian group. Therefore we may assume that the derived group $G' = \gamma_2(G)$ of G is non-trivial.

For a nilpotent group G the following inclusion is true

$$A^{[\omega]}(RG) \subseteq A^\omega(RG')RG \tag{2}$$

(see in particular [4]). For every natural number $i > 1$ we define the normal subgroup

$$L_i = \{g \in G' | g^k \in \gamma_i(G) \text{ for a suitable } k \geq 1\}$$

of G. It is easy to see that $\gamma_i(G) \subseteq L_i$ and also that $G/L_i \in \mathcal{D}_0$ for every $i > 1$.

An element g of a group G is called a *generalized torsion element* with respect to the lower central series of G if for every n the order of the elements $g\gamma_n(G)$ of the factor group $G/\gamma_n(G)$ is finite.

We recall that if the derived group G' of G contains no generalized torsion elements with respect to the lower central series of G, then G' has no generalized torsion elements with respect to the lower central series of G'.

Theorem A. *Let R be a commutative ring with identity, $T(R^+) = 0$ and let G' be with no generalized torsion elements with respect to the lower central series of G. Then $A^{[\omega]}(RG) = 0$ if and only if G is a residually-$\mathcal{D}_0$ group.*

Proof. Since G' is with no generalized torsion elements with respect to the lower central series of G, then $\cap_{i=2}^\infty L_i = \langle 1 \rangle$ and so, $G \in \mathbf{R}\mathcal{D}_0$.

Conversely. Let $G \in \mathbf{R}\mathcal{D}_0$ and $T(R^+) = 0$. Since class $\mathcal{D}_0$ is closed with respect to forming subgroups and finite direct products, by Lemmas 2.2. and 2.3. it is enough to show that $A^{[\omega]}(RG) = 0$ for all $G \in \mathcal{D}_0$. So let $G \in \mathcal{D}_0$. Then by (2)

$$A^{[\omega]}(RG) \subseteq A^\omega(RG')RG.$$

Because G' is a torsion-free nilpotent group, by Theorem 2.7. $A^\omega(RG') = 0$, and so, $A^{[\omega]}(RG) = 0$. The proof is completed.

Let p be a prime and n a natural number. Then G^{p^n} is the subgroup of G generated by all elements of the form g^{p^n}, $g \in G$.

For a prime p and a natural number k the normal subgroup $G_{[p,k]}$ of G is defined by

$$G_{[p,k]} = \bigcap_{n=1}^{\infty} (G')^{p^n} \gamma_k(G).$$

We have the following sequence

$$G = G_{[p,1]} \supseteq G_{[p,2]} \supseteq \ldots \supseteq G_{[p]}$$

of normal subgroups $G_{[p,k]}$ of G, where

$$G_{[p]} = \bigcap_{k=1}^{\infty} G_{[p,k]}.$$

It is clear, that $G/(G')^{p^n}\gamma_k(G)$ are in $\mathcal{D}_p$, and $G/G_{[p,k]}$ and $G/G_{[p]}$ are residually-$\mathcal{D}_p$ groups for every k and n.

Lemma 3.1. *If $n \geq ks$ and $h \in (G')^{p^n}\gamma_k(G)$, then*

$$h - 1 \equiv p^s X(k,h) \pmod{A^{[k]}(RG)}$$

for a suitable $X(k,h) \in A^{[2]}(RG)$.

Proof. Let $h \in (G')^{p^n}\gamma_k(G)$. We can write element h as

$$h = h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k$$

where $h_i \in G', y_k \in \gamma_k(G)$. Using the identity

$$ab - 1 = (a-1)(b-1) + (a-1) + (b-1) \tag{3}$$

to $h-1$ we have that

$$h - 1 = (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)(y_k - 1) + (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) + (y_k - 1).$$

By Lemma 2.5. $I(\gamma_k(G)) \subseteq A^{[k]}(RG)$ and hence $y_k - 1 \in A^{[k]}(RG)$. Therefore

$$h - 1 \equiv (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) \pmod{A^{[k]}(RG)}.$$

Applying identity (3) repeatedly to $(h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)$ from the previous congruence it follows that

$$h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv \sum_{i=1}^{m} \sum_{j=1}^{p^n} \binom{p^n}{j} (h_i - 1)^j b_i \pmod{A^{[k]}(RG)},$$

where $b_i \in RG$. Because $h_i \in G' = \gamma_2(G)$, from Lemma 2.5. (cases 1 and 3) we obtain that $(h_i - 1)^j \in A^{[j+1]}(RG)$ for every i and j. If $n \geq sk$, then p^s divides $\binom{p^n}{j}$ for every $j = 1, 2, \ldots, k-1$. Therefore

$$\begin{aligned} h - 1 &\equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv p^s \sum_{i=1}^{m} \sum_{j=1}^{k-1} d_j (h_i - 1)^j b_i \\ &\equiv p^s X(k, h) \pmod{A^{[k]}(RG)}, \end{aligned}$$

where $X(k,h) = \sum_{i=1}^{m} \sum_{j=k}^{p^n} d_j (h_i - 1)^j b_i$, $b_i \in RG$, $p^s d_j = \binom{p^n}{j}$. The Lemma is proved.

It is easy to show that if $g \in G'$ and $g^{p^n} \in D_{[k]}(RG)$ then

$$p^m (g - 1) \in A^{[k]}(RG) \tag{4}$$

for a large enough m.

Lemma 3.2. ([1], Lemma 3.6.) *Let $\mathcal{K}$ be a class of groups and $\{G_\alpha\}_{\alpha \in I}$ a family of normal subgroups of G such that for all α ($\alpha \in I$) the conditions*

(1) $G/G_\alpha \in \mathcal{K}$

(2) G_α is torsion-free

hold. If G is not discriminated by $\mathcal{K}$ then there exists a finite set of distinct elements $g_1, g_2, \ldots, g_s$ from G such that the non-zero element $y = (g_1 - 1)(g_2 - 1) \cdots (g_s - 1)$ lies in the ideal $\cap_{\alpha \in I} I(G_\alpha)$.

The torsion subgroup $T(R^+)$ of the additive group R^+ of a ring R is the direct sum of its p-primary components $S_p(R^+)$. Let Π be the set of those primes for which the p-primary components $S_p(R^+)$ of $T(R^+)$ are non-zero.

An element a of an additive Abelian group A is called an *element of infinite p-height* for a prime p, if the equation $p^n x = a$ has a solution in A for every natural number n.

Proposition 3.3. ([1], Theorem 3.3.) *Let $T(R^+) \neq 0$, and suppose that for some $p \in \Pi$ group $T(R^+)$ has no element of infinite p-height.*

Further let G be a group with no generalized torsion elements. Then $A^{\omega}(RG) = 0$ if and only if G is a residually-$\mathcal{N}_p$ group for all $p \in \Pi$.

Theorem B. *Let $T(R^+) \neq 0$. If G' is with no generalized torsion elements with respect to the lower central series of G and $T(R^+)$ is with no non-trivial elements of infinite p-height then $A^{[\omega]}(RG) = 0$ if and only if G is a residually-$\mathcal{D}_p$ group for all $p \in \Pi$.*

Proof. Let p an arbitrary prime of Π, $A^{[\omega]}(RG) = 0$, and let p^s $(s \geq 1)$ be the order of element $a \in T(R^+)$. Since the equation

$$G_{[p]} = \bigcap_{k=1}^{\infty} G[p,k] = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} ((G')^{p^n} \gamma_k(G)) = \langle 1 \rangle$$

implies that $G \in \mathbf{R}\mathcal{D}_p$, it is enough to show, that $G_{[p]} = \langle 1 \rangle$.

Suppose that $g \in G_{[p]}$. Then $g \in (G')^{p^n} \gamma_k(G)$ for every n and k and by Lemma 3.1. we have that

$$g - 1 \equiv p^s X(k,g) \pmod{A^{[k]}(RG)}$$

for every k. From $p^s a = 0$ it follows that $a(g-1) \in A^{[k]}(RG)$ for every k. Hence $a(g-1) \in A^{[\omega]}(RG)$ and $a(g-1) = 0$. This implies that $g = 1$. Consequently $G_{[p]} = \langle 1 \rangle$. This means that G is a residually-$\mathcal{D}_p$ group for all $p \in \Pi$.

Conversely. Let $G \in \mathbf{R}\mathcal{D}_p$ for $p \in \Pi$ and let $1 \neq g$ be an arbitrary element of G'. Then there exists a normal subgroup H of G such that $G/H \in \mathcal{D}_p$ and $g \notin H$. Since $G/H \in \mathcal{D}_p$ then $(G/H)' \in \mathcal{N}_p$. By the isomorphism $G'H/H \cong G'/H \cap G'$ we have that $\overline{g} = g(H \cap G') \neq \overline{1}$. This means that if $G \in \mathbf{R}\mathcal{D}_p$ then $G' \in \mathbf{R}\mathcal{N}_p$. Using Proposition 3.3. we have that $A^{\omega}(RG') = 0$ and from (2) it follows that $A^{[\omega]}(RG) = 0$.

Lemma 3.4. *Let*

$$y \in \bigcap_{p \in \Gamma} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} I((G')^{p^n} \gamma_j(G)).$$

Then for a prime $p \in \Gamma$ and arbitrary natural numbers k and s

$$y \equiv p^s Y(p,k,s,y) \pmod{A^{[k]}(RG)},$$

where $Y(p,k,s,y) \in RG$ and Γ is a subset of the set of prime numbers.

Proof. Let $p \in \Gamma$. For every natural n we can express y as

$$y = \sum_{i=1}^{l} \alpha_i z_i (h_i - 1),$$

where $h_i \in (G')^{p^n}\gamma_k(G)$, $\alpha_i \in R$ and every z_i is from a set of coset representatives of $(G')^{p^n}\gamma_k(G)$ in G. For a large enough n by Lemma 3.1.

$$h_i - 1 \equiv p^s X(k, h_i) \pmod{A^{[k]}(RG)}$$

for every i $(i = 1, 2, \dots, l)$ and the proof follow.

If $g \in G'$ is a generalized torsion element of a group G then Ω_g denotes the set of the prime divisors of the order of the elements $g\gamma_k(G) \in G/\gamma_k(G)$ for every $k = 2, 3, \dots$.

Lemma 3.5. *Let $g \in G'$ be a generalized torsion element of a group G, Λ an arbitrary subset of Ω_g, $a \in \cap_{p\in\Lambda} J_p(R)$ and let*

$$x \in \bigcap_{p\in\Omega_g\setminus\Lambda} \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} I((G')^{p^i}\gamma_k(G)).$$

Then one of the following statements
(1) *if Λ is a proper subset of Ω_g, then $a(g-1)x \in A^{[\omega]}(RG)$*
(2) *if $\Lambda = \Omega_g$, then $a(g-1) \in A^{[\omega]}(RG)$*
(3) *if $\Lambda = \emptyset$, then $(g-1)x \in A^{[\omega]}(RG)$*
holds.

Proof. It is enough to show that for an arbitrary natural number k the elements $a(g-1), (g-1)x, a(g-1)x$ are in the ideal $A^{[k]}(RG)$.

If $g \in \gamma_k(G)$ then by Lemma 2.5. $(g-1) \in A^{[k]}(RG)$, and the statements follow. Now let $g \notin \gamma_k(G)$ and let

$$n_k = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$$

be the prime factorization of the order of the elements $g\gamma_k(G)$ of the nilpotent group $G/\gamma_k(G)$. It is clear that $p_i \in \Omega_g$ for every $i = 1, 2, \dots, s$. Let Λ a subset of Ω_g. With loss of generality we may assume that $p_1, p_2, \dots, p_l \in \Lambda$ and $p_i \notin \Lambda$ for $i > l$.

Let $g = g_1 g_2 \cdots g_s \gamma_k(G)$ be the decomposition of the element $g\gamma_k(G)$ of the nilpotent group $G/\gamma_k(G)$ in the product of p_i-elements $g_i\gamma_k(G)$ $(i = 1, 2, \dots, s)$. Then

$$g = g_1 g_2 \cdots g_s y_k, \quad g_i \in G', i = 1, 2, \dots, s$$

for a suitable $y_k \in \gamma_k(G)$. Then there exists m_i $(i = 1, 2, \dots, s)$ such that

$$g_i^{{p_i}^{m_i}} \in \gamma_k(G).$$

Using identity (3) repeatedly to $(g-1)$ we conclude that

$$g-1 \equiv v+w+(y_k-1) \equiv v+w \pmod{A^{[k]}(RG)},$$

where $v=\sum\limits_{i=1}^{l}(g_i-1)x_i$, $w=\sum\limits_{i=l+1}^{s}(g_i-1)x_i$ and $x_i \in RG$. In the case when $\Lambda \cap \{p_1,p_2,\dots,p_s\}=\emptyset$ we assume that $v=0$, and if $\Lambda \cap \{p_1,p_2,\dots,p_s\}=\{p_1,p_2,\dots,p_s\}$ we put $w=0$. Because

$$g_i^{p_i{}^{m_i}} \in \gamma_k(G) \subseteq D_{[k]}(G)$$

and $g_i \in G'$ for every $i=1,2,\dots,s$, we conclude from (4) that there exists a natural number r_i $(i=1,2,\dots,s)$ such that

$$p_i{}^{r_i}(g_i-1) \in A^{[k]}(RG). \tag{5}$$

Also, since

$$a \in \bigcap_{p\in\Lambda} J_p(R) \subseteq \bigcap_{i=1}^{l} J_p(R)$$

we can express a as $a=p_i^{r_i}a_i$ $(a_i \in R)$ for each $i \le l$. Then by (5)

$$av \equiv \sum_{i=1}^{l} a_i p_i^{r_i}(g_i-1)x_i \equiv 0 \pmod{A^{[k]}(RG)}.$$

Therefore

$$a(g-1) \equiv av+aw \equiv aw \pmod{A^{[k]}(RG)}. \tag{6}$$

If $\Lambda=\Omega_g$ then $w=0$ and case 2) is proved.

By Lemma 3.4.

$$x \equiv p_i^{r_i}Y(p_i,k,r_i,x) \pmod{A^{[k]}(RG)},$$

and so,

$$wx \equiv \sum_{i=l+1}^{s} p_i^{r_i}(g_i-1)x_iY(p_i,k,r_i,x) \pmod{A^{[k]}(RG)}.$$

Hence by (5)

$$wx \equiv 0 \pmod{A^{[k]}(RG)}. \tag{7}$$

If $\Lambda = \emptyset$, then $v = 0$, and so,

$$(g-1)x \equiv vx + wx \equiv wx \equiv 0 \pmod{A^{[k]}(RG)}$$

and case 3) is proved.

Also, since

$$a(g-1)x \equiv avx + awx \pmod{A^{[k]}(RG)}$$

from congruences (6) and (7) the proof (of case 1)) follows.

We recall that for a prime p $\mathcal{N}_p$ denotes the class of nilpotent groups whose derived groups are p-groups of bounded exponent, and if Ω a subset of the set of primes, then $\mathcal{N}_\Omega = \cup_{p\in\Omega}\mathcal{N}_p$ and $\mathcal{D}_\Omega = \cup_{p\in\Omega}\mathcal{D}_p$.

Let a group G be discriminated by the class of groups $\mathcal{D}_\Gamma$ $(\Gamma \neq \emptyset)$ and let $g_1, g_2, \dots, g_n$ be a finite set of distinct elements of G'. Then there exists a normal subgroup H of G such that $g_iH \neq g_jH$ if $i \neq j$ and $G/H \in \mathcal{D}_\Gamma$. Therefore $(G/H)' \in \mathcal{N}_p$ for any prime $p \in \Gamma$. By the isomorphism $G'H/H \cong G'/H \cap G'$ we have $g_iH(\cap G') \neq g_j(H \cap G')$ if $i \neq j$ $(i, j = 1, 2, \dots, n)$. This means, that if G is discriminated by the class $\mathcal{D}_\Gamma$, then G' is discriminated by the class of groups $\mathcal{N}_\Gamma$.

Lemma 3.6. *Let Ω be a non-empty subset of the set of primes such that*
$\cap_{p\in\Omega} J_p(R) = 0$ and a group G is discriminated by the class of groups $\mathcal{D}_\Omega$. If for every proper subset Λ of the set Ω at least one of the conditions

(1) $\cap_{p\in\Lambda} J_p(R) = 0$

(2) *G is discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$*

holds, then $A^{[\omega]}(RG) = 0$.

Proof. Let

$$x = \sum_{i=1}^{n} \alpha_i g_i \in A^{[\omega]}(RG).$$

By Lemma 2.3. it is enough to show that $A^{[\omega]}(RG) = 0$ for all groups $G \in \mathcal{D}_\Omega$. So let $G \in \mathcal{D}_\Omega$. Then G is a nilpotent group and by (2)

$$A^{[\omega]}(RG) \subseteq A^\omega(RG')RG.$$

Clearly, $G' \in \mathcal{N}_\Omega$. If G is discriminated by the class of groups $\mathcal{D}_\Gamma$, where Γ is an arbitrary non-empty subset of Ω, then G' is discriminated by the clas $\mathcal{N}_\Gamma$, which was showed above. Then G' satisfies Theorem 2.6. and so, $A^\omega(RG') = 0$. Consequently $A^{[\omega]}(RG) = 0$.

Theorem C. *Let the derived group G' contain a generalized torsion element of G with respect to the lower central series of G. Then $A(RG)$ is residually Lie nilpotent if and only if there exists a non-empty subset Ω of the set of primes such that $\cap_{p\in\Omega} J_p(R)=0$, G is discriminated by the class of groups $\mathcal{D}_\Omega$ and every proper subset Λ of the set Ω at least one of the conditions*

(1) $\cap_{p\in\Lambda} J_p(R)=0$

(2) *G is discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$*

holds.

Proof. Let $A^{[\omega]}(RG)=0$. Let us first consider the case when G' contains a non-trival torsion element. Then there exists a p-element g in G' with $p\in\Omega$. Then by (4) for every k there exists a natural number m such that

$$p^m(g-1)\in A^{[k]}(RG). \tag{8}$$

If $a\in J_p(R)$, then for each m we can write element a as $a=p^m a_m$ $(a_m\in R)$. Therefore $a(g-1)\in A^{[k]}(RG)$ for every k, that is $a(g-1)\in A^{[\omega]}(RG)$. Hence $a(g-1)=0$ and so, $a=0$. Consequently $J_p(R)=0$.

Now we show, that G is discriminated by $\mathcal{D}_{\{p\}}$. Let

$$h\in\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty}(G')^{p^i}\gamma_k(G).$$

Then

$$h-1\in\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty}I((G')^{p^i}\gamma_k(G))$$

and by Lemma 3.4. for every k and m

$$h-1\equiv p^m Y(p,k,m,h-1)\pmod{A^{[k]}(RG)}. \tag{9}$$

By (8) and (9) we have that

$$(g-1)(h-1)\equiv p^m(g-1)(h-1)Y(p,m,k,h-1)\pmod{A^{[k]}(RG)}$$

for every k. This implies that

$$(g-1)(h-1)\in A^{[\omega]}(RG)\text{ and so, }(g-1)(h-1)=0.$$

From this equation we have that the characteristic of R is p $(=2)$ and from (9) it follows that $h-1\in A^{[\omega]}(RG)$. Therefore $h=1$ and so

$$\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty}(G')^{p^i}\gamma_k(G)=\langle 1\rangle.$$

For every k and i $G/(G')^{p^i}\gamma_k(G) \in \mathcal{D}_{\{p\}}$. The class $\mathcal{D}_{\{p\}}$ is closed with respect to forming subgroups and finite direct products, and by Lemma 2.2. G is discriminated by $\mathcal{D}_{\{p\}}$. Consequently we can choose the set $\Omega = \{p\}$.

Let us consider the case when G' is a torsion-free group and $1 \neq g \in G'$ is a generalized torsion element of G. We put $\Omega = \Omega_g$. From Lemma 3.5. (case 2) it follows that

$$\bigcap_{p\in\Omega} J_p(R) = 0.$$

From Lemma 3.2. (here we put $\{G_\alpha\}_{\alpha\in I} = \{(G')^{p^n}\gamma_k(G), k, n=1,2,\dots\}_{p\in\Omega}$) and Lemma 3.5. (case 3) we have that G is discriminated by the class $\mathcal{D}_\Omega$.

Let Λ be an arbitrary subset of Ω and let $\cap_{p\in\Lambda} J_p(R) \neq 0$. If G is not discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$, then by Lemma 3.2. there exists a set of elements $g_1, g_2, \dots, g_n$ $(g_i \in G)$ of infinite orders such that

$$0 \neq (g_1-1)(g_2-1)\cdots(g_n-1) \in \bigcap_{p\in\Omega\setminus\Lambda}\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty} I((G')^{p^i}\gamma_k(G)).$$

By Lemma 3.5. (case 1) for every element $a \in \cap_{p\in\Lambda} J_p(R)$

$$a(g-1)(g_1-1)(g_2-1)\cdots(g_n-1) \in A^{[\omega]}(RG).$$

Because $A^{[\omega]}(RG) = 0$ we have that

$$a(g-1)(g_1-1)(g_2-1)\cdots(g_n-1) = 0.$$

Since element g_i $(i = 1, 2, \dots, n)$ has infinite order and so has zero left (and right) annihilator in RG, then for g_n we have

$$a(g-1)(g_1-1)(g_2-1)\cdots(g_{n-1}-1) = 0.$$

Continuing this procedure for $i = n-1, n-2, \dots, 1$ on the last step we get that

$$a(g-1) = 0.$$

Since the element g has infinite order, its left annihilator is zero in RG, which implies $a = 0$. Consequently, if G is not discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$, then $\cap_{p\in\Lambda} J_p(R) = 0$.

The sufficiency part is proved in Lemma 3.6.

Corollary. *Let $R = \widehat{Z}_p$, the ring of p-adic integers. Then $A^{[\omega]}(\widehat{Z}_pG) = 0$ if and only if either*

(1) *G is discriminated by the class $\mathcal{D}_0$ or*
(2) *G is discriminated by the class $\mathcal{D}_p$.*

Proof. If G' is with no generalized torsion elements (with respect to the lower central series of G), then by Theorem A $A^{[\omega]}(\widehat{Z}_pG) = 0$ if and only if G is discriminated by the class $\mathcal{D}_0$.

Let us consider the case when G' contains a generalized torsion element.

Let $A^{[\omega]}(\widehat{Z}_pG) = 0$. By Theorem C there exists a non-empty subset Ω of the set of primes, such that $\cap_{q\in\Omega} J_q(\widehat{Z}_p) = 0$. It is known that $J_p(\widehat{Z}_p) = 0$ and for a prime $q \neq p$, $J_q(\widehat{Z}_p) = \widehat{Z}_p$. Therefore $p \in \Omega$. If $\Omega = \{p\}$, then by the last theorem G is discriminated by $\mathcal{D}_p$. If Ω contains a prime $q \neq p$, then we choose $\Lambda \subseteq \Omega$ such that $\Omega \setminus \Lambda = \{p\}$. Then $\cap_{q\in\Lambda} J_q(\widehat{Z}_p) \neq 0$ and by Theorem C G is discriminated by the class $\mathcal{D}_p$.

Conversely. If G is discriminated by the class $\mathcal{D}_p$, we put $\Omega = \{p\}$, and the proof follows from Theorem C.

From Theorem A and C we also get the results of I. Musson and A. Weiss ([2], Theorem A).

References

[1] Király B., The residual nilpotency of the augmentation ideal, *Publ. Math. Debrecen.*, **45** (1994), 133–144.

[2] Musson I., Weiss A., Integral group rings with residually nilpotent unit groups, *Arch. Math.*, **38** (1982), 514–530.

[3] Parmenter, M. M., Passi, I. B. S. and Sehgal, S. K., Polynomial ideals in group rings, *Canad. J. Math.*, **25** (1973), 1174–1182.

[4] Passi, I. B. S., Group ring and their augmentation ideals, *Lecture notes in Math.*, 715, Springer-Verlag, Berlin–Heidelberg–New York, 1979.

[5] Passi, I. B. S., Passman D. S. and *Sehgal S. K.*, Lie solvable group rings, *Canad. J. Math.* **25** (1973), 748–757.

[6] Sehgal S. K., Topics in group rings, Marcel–Dekker Inc., New York–Basel, 1978.

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