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Unitary subgroup of the Sylow 2-subgroup of the group of normalized units in an infinite commutatuve group ring

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Abstract. Let G be an abelian group, K a commutative ring with unity of prime characteristic p and let $V(KG)$ denote the group of normalized units of the group ring KG. An element $u{=}\sum_{g\in G}\alpha_g g{\in}V(KG)$ is called unitary if u^{-1} coincides with the element $u^{\star}{=}\sum_{g\in G}\alpha_g g^{-1}$. The set of all unitary elements of the group $V(KG)$ forms a subgroup $V_{\star}(KG)$.

S. P. Novikov had raised the problem of determining the invariants of the group $V_{\star}(KG)$ when G has a p-power order and K is a finite field of characteristic p. This problem was solved by A. Bovdi and the author. We gave the Ulm–Kaplansky invariants of the unitary subgroup of the Sylow p-subgroup of $V(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring with unity of odd prime characteristic p without nilpotent elements. Here we continue this works describing the unitary subgroup of the Sylow 2-subgroup of the group $V(KG)$ in case when G is an arbitrary abelian group and K is a commutative ring with unity of characteristic 2 without zero divisors.

Let G be an abelian group and K a commutative ring with unity of prime characteristic p. Let, further on, $V(KG)$ denote the group of normalized units (i.e. of augmentation 1) of the group ring KG and $V_p(KG)$ the Sylow p-subgroup of the group $V(KG)$. We say that for $x=\sum_{g\in G}\alpha_g g\in KG$ the element $x^{\star}=\sum_{g\in G}\alpha_g g^{-1}$ is conjugate to x. Clearly, the map $x\to x^{\star}$ is an anti-isomorphism (involution) of the ring KG. An element $u\in V(KG)$ is called unitary if $u^{-1}=u^{\star}$. The set of all unitary elements of the group $V(KG)$ obviously forms a subgroup, which we therefore call the unitary subgroup of $V(KG)$, and we denote it by $V_{\star}(KG)$.

Let G^p denote the subgroup $\{g^p : g\in G\}$ and λ an arbitrary ordinal. The subgroup $G^{p^{\lambda}}$ of the group G is defined by transfinite induction in following way: $G^{p^0}=G$, for a non-limited ordinals $G^{p^{\lambda+1}}=\left(G^{p^{\lambda}}\right)^p$, and if λ is a limited ordinal, then $G^{p^{\lambda}}=\cap_{\nu<\lambda}G^{p^{\nu}}$.

The subring $K^{p^{\lambda}}$ of the ring K is defined similarly. The ring K is called

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p-divisible if $K^p = K$.

Let $G[p]$ denote the subgroup $\{g \in G : g^p = 1\}$ of G. Then the factor-group $G^\lambda[p]/G^{\lambda+1}[p]$ can be considered as a vector space over $GF(p)$ the field of p elements and the cardinality of a basis of this vector space is called the λ-th Ulm–Kaplansky invariant $f_\lambda(G)$ of the group G concerning to p.

S. P. Novikov had raised the problem of determining the invariants of the group $V_\star(KG)$ when G has a p-power order and K is a finite field of characteristic p. This was solved by A. Bovdi and the author in [1]. In [2] we gave the Ulm–Kaplansky invariants of the unitary subgroup $W_p(KG)$ of the group $V_p(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring of odd prime characteristic p without nilpotent elements. Here we continue this works describing the unitary subgroup $W_2(KG)$ of the Sylow 2-subgroup $V_2(KG)$ of the group $V(KG)$ in case when G is an arbitrary abelian group and K is a commutative ring with unity of characteristic 2 without zero divisors.

Note that for the odd primes p the problem of determining the Ulm–Kaplansky invariants of the group $W_p(KG)$ is based, in fact, in the following statement

$$W_p(KG) = \left\{x^{-1}x^\star : x \in V_p(KG)\right\}$$

(see [2]). But in case $p = 2$ this statement is not true and in the characterization of the group $W_2(KG)$ we must keep in mind the following lemma.

Lemma 1. *Let G be an abelian group of exponent 2^{n+1} $(n > 0)$ and K a commutative ring with unity of characteristic 2 without zero divisors. Then $(V_\star(KG))^{2^n} = G^{2^n}$.*

Proof. At first we shall prove the lemma for a finite group G. We shall use induction on the exponent of G.

Let $n = 1$, i.e. G is a group of exponent 4. We shall prove by induction on the order of G that $(V_\star(KG))^2 = G^2$.

Let $G = \langle a : a^4 = 1\rangle$. Then the element

$$x = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 \in V(KG)$$

is unitary if and only if

$$xx^\star = 1 + (\alpha_0 + \alpha_2)(\alpha_1 + \alpha_3)(a + a^3) = 1.$$

Hence $\alpha_0 = \alpha_2$ or $\alpha_1 = \alpha_3$. If $\alpha_1 = \alpha_3$ then, according to the condition $\alpha_0+\alpha_1+\alpha_2+\alpha_3 = 1$, the unitary element x has the form $x = 1+\alpha_2(1+a^2)+\alpha_1(a + a^3)$ and $x^2 = 1$. If $\alpha_0 = \alpha_2$ then $x = \alpha_0(1 + a^2) + \alpha_1 a + (1 + \alpha_1)a^3$.

Therefore $x^2 = a^2$ and the statement is proved for the cyclic group G of order 4.

Let G be a non-cyclic group of exponent 4 and order greater than 4. Then G can be presented as a direct product of a suitable group H and the cyclic group $\langle b\rangle$ which order divides 4.

Suppose that b is an element of second order. Then every $x \in V(KG)$ can be written in the form $x = x_0 + x_1 b$, where $x_0, x_1 \in KH$. If x is a unitary element then

$$xx^\star = x_0{x_0}^\star + x_1{x_1}^\star + ({x_0}^\star x_1 + x_0{x_1}^\star)b = 1$$

and the equations $x_0{x_0}^\star + x_1{x_1}^\star = 1$, ${x_0}^\star x_1 + x_0{x_1}^\star = 0$ hold. Hence $(x_0 + x_1)({x_0}^\star + {x_1}^\star) = 1$ and $y = x_0 + x_1 \in V_\star(KH)$. By the induction hypothesis, $y^2 = h^2$ for some $h \in H$. Obviously $x^2 = h^2$.

Let b be an element of order 4. The element

$$x = x_0 + x_2b^2 + (x_1 + x_3b^2)b \quad (x_i \in KH, i = 0, 1, 2, 3)$$

of the group $V(KG)$ is unitary if and only if

$$(1) \qquad \begin{cases} (x_0 + x_2b^2)({x_0}^\star + {x_2}^\star b^2) + (x_1 + x_3b^2)({x_1}^\star + {x_3}^\star b^2) = 1, \\ (x_0 + x_2b^2)({x_1}^\star + {x_3}^\star b^2) = 0. \end{cases}$$

Let $\chi(x_0 + x_2b^2) = \gamma$ denote the sum of coefficients of the element $x_0 + x_2b^2$. Then $\chi(x_1 + x_3b^2) = 1 + \gamma$ and from the second equation of (1) we have that $\gamma(1 + \gamma) = 0$. Since K without zero divisors, it follows that $\gamma = 0$ or $\gamma = 1$ i.e. one of the elements $x_0 + x_2b^2$ or $x_1 + x_3b^2$ is invertible. Hence for the unitary element x either $x_0 = x_2b^2$ or $x_1 = x_3b^2$. If $x_0 = x_2b^2$ then, by (1), the element $y = x_1 + x_3b^2$ is unitary in the group ring of the group $\widetilde{H} = H \times \langle b^2\rangle$. Then, by the induction hypothesis, $y^2 = h^2$ for some $h \in H$ and obviously $x^2 = y^2b^2 = h^2b^2 \in G^2$. If $x_1 = x_3b^2$ then $y = x_0 + x_2b^2 \in V_\star(K\widetilde{H})$ and $x^2 = y^2 \in G^2$. So $(V_\star(KG))^2 = G^2$ for a finite group G of exponent 4.

Suppose that G is a group of exponent 2^{n+1} $(n > 1)$ and the statement is proved for the groups of exponent less than 2^{n+1}. It is easy to see that $(V_\star(KG))^2 \subseteq V_\star(KG^2)$. From this, useing the induction hypothesis $(V_\star(KG^2))^{2^{n-1}} = (G^2)^{2^{n-1}}$, we have that $V_\star(KG)^{2^n} \subseteq G^{2^n}$. The reverse inclusion is obvious and the lemma is proved for a finite group G.

Let G be an infinite abelian group of exponent 2^{n+1} $(n > 0)$ and $x \in V_\star(KG)$. Then the subgroup $H = \langle \mathrm{supp}\ x\rangle$ of the support of x is finite and, by the statement proved in above, $x^{2^n} \in H^{2^n}$. This completes the proof of the lemma.

Theorem. *Let λ be an arbitrary ordinal, K a commutative ring with unity of characteristic 2 without zero divisors, P the maximal divisible subgroup of the Sylow 2-subgroup S of an abelian group G, $G_\lambda = G^{2^\lambda}$, $S_\lambda = S^{2^\lambda}$, $K_\lambda = K^{2^\lambda}$. Let, further on, $V_2 = V_2(KG)$ denote the Sylow 2-subgroup of the group $V = V(KG)$ of normalized units in the group ring KG and $W = W(KG)$ the unitary subgroup of $V_2(KG)$. In case $P \neq 1$ we assume that the ring K is 2-divisible.*

If $G_\lambda \neq G_{\lambda+1}$, $S_\lambda \neq 1$ and at least one of the ordinals $|K_\lambda|$ and $|G_\lambda|$ is infinite, then the λ-th Ulm–Kaplansky invariant $f_\lambda(W)$ of the group W concerning to 2 is characterized in the following way:

$$f_\lambda(W) = \begin{cases} \max\{|G|,\, |K|\}, & \text{if } \lambda = 0, \\ f_\lambda(V_2) = \max\{|G_\lambda|,\, |K_\lambda|\}, & \text{if } \lambda > 0 \text{ and } G_{\lambda+1} \neq 1, \\ f_\lambda(G), & \text{if } \lambda > 0 \text{ and } G_{\lambda+1} = 1. \end{cases}$$

Proof. It is easy to prove the following statements (see [3]):

1) $|K^2| = |K|$;

2) if n a nonnegative integer and $J(G^{p^n}[p])$ the ideal of the ring $(KG)^{2^n}$ generated by the elements of the form $g-1$ $\left(g \in G^{2^n}[2]\right)$, then $V^{2^n}(KG)[2] = V(K_nG_n)[2] = 1 + J(G^{2^n}[2])$.

Note if $G_\lambda = G_{\lambda+1}$ or $S_\lambda = 1$ then, according to [3], $f_\lambda(V_2) = 0$ and hence $f_\lambda(W) = 0$.

At first we shall prove the theorem for a finite ordinal $\lambda = n$. Suppose that n is a nonnegative integer, the Sylow 2-subgroup S_n of the group G_n is not singular, $G_n \neq G_{n+1}$ and at least one of the ordinals $|K_n|$ and $|G_n|$ is infinite. Since

$$W^{2^n}[2] \subseteq V^{2^n} = V(K_nG_n),$$

it follows that

$$f_n(W) \leq |V^{2^n}| \leq \max\{|K_n|, |G_n|\} = \beta.$$

In the proof of the equation $f_n(W) = \beta$ we shall consider the following cases:

A) $|K_n| \geq |G_n|$,

B) $|G_n| > |K_n|$ and $S_n \neq S_{n+1}$,

C) $|G_n| > |K_n|$ and $S_n = S_{n+1}$,

and in each of this cases we shall construct a set $M \subseteq W^{2^n}(KG)[2]$ of cardinality $\beta = \max\{|K_n|, |G_n|\}$ (if, keeping in mind Lemma 1, it is possible) which elements belong to the different cosets of the group $V^{2^n}(KG)[2]$ by the subgroup $V^{2^{n+1}}(KG)[2]$. This will be sufficient for the proof of the lemma, because the elements of the such constructed set M can be considered as the representatives of the cosets of the group $W^{2^n}(KG)[2]$ by the subgroup $W^{2^{n+1}}(KG)[2]$. Note that the elements of the set M we shall choose in the form $yy^\star$ $\big(y \in V^{2^n}(KG)\big)$.

Let A) holds, i.e. $|K_n| \geq |G_n|$.

It is easy to prove that in this case the Sylow 2-subgroup S_n of the group G_n has such element g of order 2 and there exists an $a \in G_n$ that one of the following conditions holds:

A_1) $G_n \neq \langle g\rangle, a \notin \langle g\rangle$ and $a^2 \notin \langle g\rangle$,

A_2) $G_n \neq \langle g\rangle, a \notin \langle g\rangle$ and $a^2 \in \langle g\rangle$,

A_3) $G_n = \langle g\rangle$

and in cases A_1) and A_2) at least one of the elements a or g do not belong to the subgroup G_{n+1}. Indeed, if $g \in G_{n+1}$ then, by condition $G_n \neq G_{n+1}$, the set $G_n \setminus G_{n+1}$ has a proper element a.

Let A_1) holds. Let α be a nonzero element of the ring K_n and $y_\alpha = 1 + \alpha a(1 + g)$. We shall prove that the set

$$M = \big\{x_\alpha = y_\alpha {y_\alpha}^\star = 1 + \alpha(a + a^{-1})(1 + g) \colon 0 \neq \alpha \in K_n\big\}$$

has the above declared property. Really, since $a^2 \notin \langle g\rangle$, it follows that the elements a and a^{-1} belong to the different cosets of the group G_n by the subgroup $\langle g\rangle$. Hence $x_\alpha \neq 1$. It is easy to see that ${x_\alpha}^\star = x_\alpha = {x_\alpha}^{-1}$. Therefore x_α is a unitary element of second order of the group $V(K_nG_n)$. If $x_\alpha \in V^{2^{n+1}}$ then, from the condition $a^2 \notin \langle g\rangle$, it follows that the elements a and ag belong to the group G_{n+1}, but this contradicts to the choice of elements a and g. Therefore $x_\alpha \in W^{2^n}[2] \setminus W^{2^{n+1}}[2]$.

Suppose that the coset $x_\alpha V^{2^{n+1}}[2]$ coincides with $x_\nu V^{2^{n+1}}[2]$ for a different α and ν from K_n. Then $x_\alpha = x_\nu z$ for a suitable $z \in V^{2^{n+1}}$. Since ${x_\nu}^\star = {x_\nu}^{-1}$, it follows that

$$z = x_\alpha {x_\nu}^\star = 1 + (\alpha + \nu)(a + a^{-1})(1 + g) = x_{\alpha+\nu}$$

and $x_{\alpha+\nu}$ belongs to the subgroup $V^{2^{n+1}}$ what contradicts it which was proved in above. Obviously $|M| = |K_n|$. Therefore the constructed set M has the above declared property.

Let A_2) holds.

It is easy to see that the elements of the set

$$M = \{x_\alpha = 1 + \alpha a(1 + g) \colon 0 \neq \alpha \in K\}$$

belong to the different cosets of the group $V(KG)[2]$ by the subgroup $V^2(KG)[2]$. Indeed, if $x_\alpha \in V^2$ then $a \in G_1$ and $ag \in G_1$. But this contradicts to the choice of the elements a and g and hence $x_\alpha \in W[2] \setminus W^2[2]$.

The equation $x_\alpha = x_\nu z$ $\big(z \in V^2, \alpha \neq \nu\big)$ is impossible since from it follows that $z = x_\alpha x_\nu = 1 + (\alpha + \nu)a(1 + g) = x_{\alpha+\nu}$, and, by proved in above, $x_{\alpha+\nu} \notin V^2$. Obviously $|M| = |K|$ and therefore $f_0(W) = |K|$.

Let us contruct the set M in case $n > 0$.

Since, by Lemma 1, $f_n(W_2) = f_n(G)$ when $G_{n+1} = 1$, it follows that we can assume that $G_{n+1} \neq 1$. Let $|G_n| \neq 4$. Then the set $G_n \setminus G_{n+1}$ has neither element a, which order is not divisible by 2, or element b of order $2^r > 4$, or has a subgroup $\langle c \colon c^4 = 1\rangle \times \langle d \colon d^2 = 1\rangle$. Obviously in the first case $a^2 \notin \langle g\rangle$. If in the other cases we put $a = b, g = b^{2^{r-1}}$ or $a = c, g = d$ respectively then the condition $a^2 \notin \langle g\rangle$ holds and we have the above considered case A$_1$).

Let $G_n = \langle a \colon a^4 = 1\rangle$ and $y_\alpha = 1 + \alpha(a + 1)$. Obviously the element

$$x_\alpha = y_\alpha {y_\alpha}^\star = 1 + (\alpha + \alpha^2)(a + a^3)$$

is unitary. Let L denote a subset of K_n that has a unique representative in every subset of the form $\{\alpha, 1 + \alpha\} \subseteq K_n$. Then the elements of the set

$$M = \big\{x_\alpha = y_\alpha {y_\alpha}^\star = 1 + (\alpha + \alpha^2)(a + a^3) \colon 0 \neq \alpha \in L\big\}$$

belong to the different cosets of the group $W^{2^n}(KG)[2]$ by the subgroup $W^{2^{n+1}}(KG)[2]$. Really, if x_α coincides with x_ν $(\alpha, \nu \in L)$, then $\alpha + \alpha^2 = \nu + \nu^2$. Hence the equation $(\alpha + \nu)(1 + \alpha + \nu) = 0$ holds, but in the ring without zero divisors this is possible for the different α and ν only in the case $\nu = 1 + \alpha$, what contradicts to the choice of the elements of the set L. Obviously $|M| = |L| = |K_n|$. By Lemma 1, $W^{2^{n+1}} = \langle a^2\rangle$. If $x_\alpha W^{2^{n+1}} = x_\nu W^{2^{n+1}}$ $(x_\alpha \neq x_\nu)$ we get the contradictinally equation

$$1 + (\alpha + \alpha^2)(a + a^3) = a^2 + (\nu + \nu^2)(a + a^3).$$

Therefore $x_\alpha W^{2^{n+1}} \neq x_\nu W^{2^{n+1}}$ for $x_\alpha \neq x_\nu$ the case A$_2$) is considered.

Let A$_3$) holds, i.e. $G_n = \langle g\rangle$. Then $G_{n+1} = 1$. If $n = 0$ then $W(KG) = V_2(KG)$ and $f_0(W) = f_0(V_2) = |K|$. If $n > 0$ then, according to Lemma 1, $f_n(W) = f_n(G)$.

Therefore the case A) is fully considered.

Suppose now that B) holds, i.e. $|G_n| > |K_n|$ and the Sylow 2-subgroup S_n of the group G_n does not coincide with the Sylow 2-subgroup S_{n+1} of the group G_{n+1}. Then the set $S_n \setminus S_{n+1}$ has an element g of order $q = 2^r$. Let, further on, $\Pi = \Pi(G_n/\langle g\rangle)$ denote the full set of representatives of the cosets of the group G_n by the subgroup $\langle g\rangle$. Let us consider two disjunct subsets

$$\Pi_1 = \big\{a \in \Pi \colon a^2 \notin \langle g\rangle\big\} \quad \textit{and} \quad \Pi_2 = \big\{a \in \Pi \colon a^2 \in \langle g\rangle\big\}$$

of the set Π. Since G_n is infinite, it is easy to see that $|G_n| = |\Pi| = \max\{|\Pi_1|, |\Pi_2|\}$.

Let us suppose at first that $|G_n| = |\Pi_1|$. Without loss of generality we can assume that the representative of the coset $a^{-1}\langle g\rangle$ is the element a^{-1}. Let E denote the set which has a unique representative in every subset of the form $\{a, a^{-1}\} \subseteq \Pi_1$ and $y_a = 1 + a(1 + g + \cdots + g^{q-1})$. Then $|G_n| = |E|$ and the elements of the set

$$M = \{x_a = y_a y_a{}^\star = 1 + (a + a^{-1})(1 + g + \cdots + g^{q-1}) \colon a \in E\}$$

belong to the different cosets of the group $V^{2^n}[2]$ by the subgroup $V^{2^{n+1}}[2]$. Indeed, from the supposition $x_a \in V^{2^{n+1}}[2]$ it follows that $ag^i \in G_{n+1}$ for every $i = 0, 1, \dots, q-1$, but this contradicts to the choice of the element $g \in G_n \setminus G_{n+1}$. It is easy to see that x_a is a unitary element and so $x_a \in W^{2^n}[2] \setminus W^{2^{n+1}}[2]$. Suppose that a and c are the distinct elements of the set E. If $x_a = x_c z$ for some $z \in V^{2^{n+1}}$ then

$$z = x_a x_c{}^\star = 1 + (a + a^{-1} + c + c^{-1})(1 + g + \cdots + g^{q-1}).$$

According to the choice of the elements of the set E we have that the elements a, a^{-1}, c, c^{-1} belong to the distinct cosets of the group G_n by the subgroup $\langle g\rangle$. Hence from the condition $z \in V^{2^{n+1}}$ it follows that $a \in G_{n+1}$, $ag \in G_{n+1}$, which contradicts to the choice of the element $g \in S_n \setminus S_{n+1}$.

Let be now $|G_n| = |\Pi_2|$. If $G^2 = 1$ then $W(KG) = V(KG)$ and $f_0(W) = f_0(V_2) = |G|$. If $n > 0$ and $G_{n+1} = 1$ then, by Lemma 1, $f_n(W) = f_n(G)$. Suppose that $G_{n+1} \neq 1$. Then the group G_n has such element v of order not equals to 2 that $\langle g\rangle \cap \langle v\rangle = 1$. If a such representative of the coset $a\langle g\rangle$ that $a^2 \in \langle g\rangle$ and $a^2 \neq 1$, then $a^2 = g^i \in G_{n+1}$ and, according to the choice of the element g, the integer i is divisible by 2. In this case in role of the representative of the coset $a\langle g\rangle$ in the set Π_2 we can choose the element $a_1 = ag^{-\frac{i}{2}}$. Therefore, we can assume that the set Π_2 consists of the elements of second order. Since $\langle g\rangle \cap \langle v\rangle = 1$, it follows that from the Π_2 we can choose a subset $\widetilde{\Pi}_2$ which elements belong to the distinct cosets of the group G_n by the subgroup $\langle g, v\rangle$ and $|G_n| = |\widetilde{\Pi}_2|$. Let $y_a = 1 + av(1 + g + \cdots + g^{q-1})$. Then the set

$$M = \left\{x_a = y_a y_a{}^\star = 1 + a(v + v^{-1})(1 + g + \cdots + g^{q-1}) \colon a \in \widetilde{\Pi}_2\right\}$$

has the need property. Indeed, the cosets $x_a V^{2^{n+1}}$ and $x_c V^{2^{n+1}}$ coincide if and only if

$$x_a x_c = 1 + (a + c)(v + v^{-1})(1 + g + \cdots + g^{q-1}) \in V^{2^{n+1}}.$$

Since the elements a and c belong to the distinct cosets of the group G_n by the subgroup $\langle g, v\rangle$, it follows that $av \in G_{n+1}$ and $avg \in G_{n+1}$, but this contradicts to the choice of the element $g \in G_n \setminus G_{n+1}$. So the case B) is fully considered.

Let C) holds, that is $|G_n| > |K_n|$ and the Sylow 2-subgroup S_n of the group G_n is 2-divisible.

Let us fix an element $g \in S_n[2]$ and choose such $v \in G_n \setminus G_{n+1}$ that 2 does not divide the order of element v. Since $|S_n| = [S_n : \langle g\rangle] \geq |\langle v\rangle|$ and $v \notin S_n$, it follows that the cardinality of the full set of representatives of the cosets $\Pi = \Pi(G_n/\langle g, v\rangle)$ of the group G_n by the subgroup $\langle g, v\rangle$ coincides with $|G_n|$. Obviously the set Π decomposes to the two disjunct subsets $\Pi_1 = \{a \in \Pi \colon a^2 \notin \langle v, g\rangle\}$ and $\Pi_2 = \{a \in \Pi \colon a^2 \in \langle v, g\rangle\}$.

Let $\mid G_n \mid = \mid \Pi_1 \mid$, E be the set which has a unique representative in every subset of the form $\{a, a^{-1}\} \subseteq \Pi_1$ and $y_a = 1 + a(1 + v + v^{-1}(1 + g)$. Then the set M can be choosen in the following way:

$$M = \left\{x = y_a {y_a}^\star = 1 + (a + a^{-1})(1 + v + v^{-1})(1 + g) \colon a \in E\right\}.$$

Indeed, from the equation $x_a = x_c z$ $\left(z \in V^{2^{n+1}},\ a \neq c\right)$ follows that

$$z = 1 + (a + a^{-1} + c + c^{-1})(1 + v + v^{-1})(1 + g) \in V^{2^{n+1}}.$$

Hence, according to the construction of the set E, the elements a and av belong to the subgroup G_{n+1}, but this contradicts to the condition $v \notin G_{n+1}$.

Suppose now that $|G_n| = |\Pi_2|$. Then $v^2 \neq 1$. If $a^2 = v^2$ for some $a \in \Pi_2$, then from the condition $v \notin G_{n+1}$ it follows that i is an even number. Let us choose in the role of the representative of the coset $a\langle g, v\rangle$ the element $a_1 = av^{-\frac{i}{2}}$. Hence we can assume that the set Π_2 of the representatives of the group G_n by the subgroup $\langle g, v\rangle$ consists of the elements of the group $S_n = S_{n+1}$. The set

$$M = \left\{x_a = 1 + a(v + v^{-1})(1 + g) \colon a \in \Pi_2\right\}$$

has the need property. Indeed, if $x_a = x_c z$ for the distinct $a, c \in \Pi_2$ and for some $z \in V^{p^{n+1}}$, then $z = x_a x_c = 1 + (a + c)(v + v^{-1})(1 + g)$ and $av \in G_{n+1}$. Hence $v \in G_{n+1}$ because – by the choice – $\Pi_2 \subseteq S_{n+1}$, and so we get the contradiction.

Therefore the case C) is fully considered and the statement is proved for a finite ordinal $\lambda = n$.

Let us consider the case of infinite ordinal λ.

Let λ be an arbitrary infinite ordinal $R = K_\lambda, H = G_\lambda \neq G_{\lambda+1}$ and the Sylow 2-subgroup S_λ of the group G_λ is not singular. Then

$$W(KG)^{2^\lambda} \subseteq W(RH) \subseteq V_2(RH)$$

and by transfinite induction it is easy to prove the equation

$$V_2(KG)^{2^\lambda} = V_2(RH). \tag{2}$$

As compared to the group $V_2(RH)$ we can construct the set M as in the above shown cases $A), B)$ and $C)$. Since in every of this cases the set M consist of the elements of the form $x = y^{-1}y^\star$ and, by (2), y belongs to the group $V_2(RH) = V_2(KG)^{2^\lambda}$, it follows that the elements x are the representatives of the cosets of group $W^{2^\lambda}(KG)[2]$ by the subgroup $W^{2^{\lambda+1}}(KG)[2]$.

Therefore for an arbitrary infinite ordinal λ the Ulm–Kaplansky invariants of the group $W(KG)$ can be calculated in the above shown way for the case $\lambda = n$.

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