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# On the certain subsets of the space of metrics

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**Abstract.** In this note we look at certain subsets of the metric space of metrics for an arbitrary given set $X$ and show that in terms of cardinility these can be very large while being extremely small in the topological point of view.

## Introduction

Let $X$ be a given non-void set. Denote by $\mathcal{M}$ the set of all metrics on $X$ endowed with the metric:

$$d^\star(d_1, d_2) = \min\Big\{1, \sup_{x,y\in X} \{|d_1(x,y) - d_2(x,y)|\} \text{ for } d_1, d_2 \in \mathcal{M}\Big\}.$$

First of all recall some basic definitions and notations.

Suppose $\alpha > 0$ and put

$$\mathcal{H}_\alpha = \Big\{d \in \mathcal{M} : \underset{\substack{x\neq y\\ x,y\in X}}{\forall} d(x,y) \geq \alpha\Big\} \text{ and } \mathcal{H} = \bigcup_{\alpha>0} \mathcal{H}_\alpha.$$

Results shown in [2] include $\mathcal{M}$ is a non-complete Baire space and $\mathcal{H}$ is an open and dense subset of $\mathcal{M}$, thus $\mathcal{M} \mid \mathcal{H}$ is nowhere dense in $\mathcal{M}$. Other results on the metric space of metrics may be found in [2], [3] and [4].

Let $\mathcal{A}$ and $\mathcal{B}$ denote the set of all metrics on $X$ that are unbounded and bounded, respectively. It is proved in [2] (Theorem 5) that $\mathcal{A}, \mathcal{B}$ are non-empty, open subsets of the Baire space $(\mathcal{M}, d^\star)$ (of [2], Theorem 3) provided $X$ is infinite. Thereby $\mathcal{A}, \mathcal{B}$ are sets of the 2-nd cathegory in $\mathcal{M}$, if is infinite. If $X$ is finite, then $\mathcal{B} = \mathcal{M}$ and $\mathcal{A} = \emptyset$.

Now define the mapping

$$f\colon \mathcal{M} \to (0,1],\ g\colon \mathcal{M} \to [0,\infty) \text{ and } h\colon \mathcal{B} \to (0,+\infty)$$

as follows:

$$\begin{aligned} f(d) &= \sup_{x,y\in X} \frac{d(x,y)}{1+d(x,y)} \quad \text{where} \quad d \in \mathcal{M}, \\ g(d) &= \inf_{\substack{x\neq y\\ x,y\in X}} d(x,y) \quad \text{where} \quad d \in \mathcal{M}, \quad \text{and} \\ h(d) &= \sup_{x,y\in X} d(x,y) \quad \text{where} \quad d \in \mathcal{B}. \end{aligned}$$

Obviously $f^{-1}(\{1\}) = \mathcal{A}$ and $g^{-1}(\{0\}) = \mathcal{M} \setminus \mathcal{H}$.

It is purpose of this paper to establish how large sets $f^{-1}(\{t\})$, $g^{-1}(\{t\})$, are given.

In what follows if $\mathcal{U} \subset \mathcal{M}$, then $\mathcal{U}$ is considered as a metrics space with the metric $d^\star |_{\mathcal{U}\times\mathcal{U}}$ (a metric subspace of $\mathcal{M}$).

## Main results

Let $\varphi(t) = \frac{t}{1+t}$ for $t \in [0, +\infty)$. Then $\varphi$ is increasing and continuous function on $[0, +\infty)$. Therefore $f(d) = \sup\limits_{x,y\in X} \varphi(d(x,y))$ for $d \in \mathcal{M}$. The natural question arises wether $f$ is continuous on $\mathcal{M}$, too. The answer of this question is positive. We have

**Lemma.** *The function $f, g$ are uniformly continuous on $\mathcal{M}$ and the function $h$ is uniformly continuous on $\mathcal{B}$.*

**Proof.** Let $0 < \varepsilon < 1$ and $d_1, d_2 \in \mathcal{M}$ such that $d^\star(d_1, d_2) < \varepsilon$. We show

$$|f(d_1) - f(d_2)| \le d^\star(d_1, d_2), \quad |g(d_1) - g(d_2)| \le d^\star(d_1, d_2).$$

We can simply count

$$\begin{aligned}\varphi(d_1(x,y)) &\le \varphi(d_2(x,y)) + |\varphi(d_1(x,y)) - \varphi(d_2(x,y))| \\ &\le \varphi(d_2(x,y)) + d^\star(d_1, d_2)\end{aligned}$$

because

$$\frac{|d_1(x,y) - d_2(x,y)|}{(1 + d_1(x,y))\,(1 + d_2(x,y))} \le d^\star(d_1, d_2).$$

Taking supremum in the previous inequality we obtain $f(d_1) \le f(d_2) + d^\star(d_1, d_2)$, therefore $f(d_1) - f(d_2) \le d^\star(d_1, d_2)$. From symetrics we have $f(d_2) - f(d_1) \le d^\star(d_1, d_2)$ and $|f(d_1) - f(d_2)| \le d^\star(d_1, d_2)$. From this we see that the function $f$ is uniformly continuous on $\mathcal{M}$. Obviously for $x, y \in X$

$$|d_1(x,y) - d_2(x,y)| \ge d_1(x,y) - d_2(x,y) \ge g(d_1) - d_2(x,y).$$

Then

$$g(d_1) - g(d_2) \le \inf_{\substack{x,y\in X\\ x\ne y}} |d_1(x,y) - d_2(x,y)| \le d^\star(d_1, d_2). \tag{1}$$

According the inequality $|d_1(x,y) - d_2(x,y)| \ge d_2(x,y) - d_1(x,y)$, similarly to the previous we get

$$g(d_2) - g(d_1) \le d^\star(d_1, d_2). \tag{2}$$

Then we required inequality follows from (1) and (2).

Analogously $|h(d_1) - h(d_2)| \le d^\star(d_1, d_2)$. ∎

**Remark 1.** The function $h$ can be continuosly continued on $\mathcal{M}$. Because the set $\mathcal{B}$ is closed in $\mathcal{M}$, the Hausdorff's function (see [1], p. 382) is continuous continuation of the function $h$ on $\mathcal{M}$.

**Remark 2.** Because $\mathcal{A} \cup \mathcal{B} = \mathcal{M}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$ and the set $\mathcal{B}$ is closed in $\mathcal{M}$, according the lemma of Uryshon there exists a function $G\colon \mathcal{M} \to [0,1]$ such that $G$ is continuous on $\mathcal{M}$ and $G(\mathcal{A}) = \{0\}$, $G(\mathcal{B}) = \{1\}$. For this reason $G(\mathcal{M}) = \{0,1\}$.

Space $(\mathcal{M}, d^\star)$ is Bair's space, e.g. every non-empty open subset of the set $\mathcal{M}$ is of the 2-nd cathegory in $\mathcal{M}$. The set $\mathcal{A}$ is non-void and open subset in $\mathcal{M}$, then the set $f^{-1}(\{1\}) = \mathcal{A}$ is of the 2-nd cathegory in $\mathcal{M}$. One may ask: Is there any $t \in (0,1)$ such that the set $f^{-1}(\{t\})$ is of the 2-nd cathegory in $\mathcal{M}$? Similarly for $g^{-1}(\{t\})$ and $h^{-1}(\{t\})$. This question is answered in the next theorem.

**Theorem 1.** *We have*

(i) *For arbitratry $t \in (0,1)$ the set $f^{-1}(\{t\})$ is nowhere dense in $\mathcal{M}$.*

(ii) *For arbitrary $t \in [0,+\infty)$ the set $g^{-1}(\{t\})$ is nowhere dense in $\mathcal{M}$.*

(iii) *For arbitrary $t \in [0,+\infty)$ the set $h^{-1}(\{t\})$ is nowhere dense in $\mathcal{M}$.*

**Proof.** (i) Let $0 < t < 1$. According to lemma the set $f^{-1}(\{t\})$ is closed in $\mathcal{M}$. Therefore it is sufficient to prove that the set $\mathcal{M} \setminus f^{-1}(\{t\})$ is dense in $\mathcal{M}$. We will use inequality

$$\frac{t_2}{1+t_2} \ge \frac{t_1}{1+t_1} + \frac{t_2 - t_1}{(1+t_2)^2} \quad \text{for} \ \ 0 \le t_1 \le t_2 \tag{3}$$

(it is equivalent to $(t_2 - t_1)^2 \ge 0$).

Let $d \in f^{-1}(\{t\})$ and $0 < \varepsilon < 1$. Clearly $d \in \mathcal{B}$ and there exists a $K \in R^+$ such that

$$d(x,y) \le K \ \ \text{for every} \ \ x, y \in X. \tag{4}$$

Choose $d' \in \mathcal{M}$ as follows

$$d'(x,y) = \begin{cases} d(x,y) + \frac{\varepsilon}{2}, & \text{if } x, y \in x, x \ne y \\ 0, & \text{if } x = y. \end{cases}$$

Then $d^\star(d, d') < \varepsilon$. We show that $d' \in \mathcal{M} \setminus f^{-1}(\{t\})$. From (3) and (4) for $x, y \in X (x \ne y)$ and $t_1 = d(x,y)$, $t_2 = d'(x,y)$ we have

$$\varphi\left(d'(x,y)\right) \ge \varphi\left(d(x,y)\right) + \frac{\frac{\varepsilon}{2}}{(1 + d'(x,y))^2} \ge \varphi\left(d(x,y)\right) + \frac{\frac{\varepsilon}{2}}{(1+K)^2}.$$

Then $f(d') > f(d)$, so $d' \notin f^{-1}(\{t\})$.

(ii) According to lemma the set $g^{-1}(\{t\})$ is closed in $\mathcal{B}$. It is enough to show that the set $\mathcal{B}\setminus g^{-1}(\{t\})$ is dense in $\mathcal{B}$. Let $d \in g^{-1}(\{t\})$ and $0 < \varepsilon < 1$. Define $d'$ on $X$ as follows:

$$d'(x,y) = \begin{cases} d(x,y) + \frac{\varepsilon}{2}, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Evidently $g(d') = t + \frac{\varepsilon}{2}$, therefore $g' \in \mathcal{B} \setminus g^{-1}(\{t\})$ and $d^\star(d,d') < \varepsilon$.

(iii) We can prove similarly like (ii). ∎

From the above Theorem 1 we can see that the sets $f^{-1}(\{t\})$, $g^{-1}(\{t\})$, $h^{-1}(\{t\})$ are small from the topological point of view but on the other hand we show, that the cardinality of them is equal to the cardinality of the set $\mathcal{M}$.

In [2] is proved: $\mathrm{card}(\mathcal{M}) = c$ if $X$ is a finite set having at least two elements and card $(\mathcal{M}) = 2^{\mathrm{card}(X)}$ if $X$ is infinite set ($c$ denotes the cardinality of the set of all real numbers).

**Theorem 2.** *Let $X$ be an infinite set. Then we have:*

1. $\mathrm{card}(f^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$ *for* $t \in (0,1]$
2. $\mathrm{card}(g^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$ *for* $t \in [0,+\infty)$
3. $\mathrm{card}(h^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$ *for* $t \in (0,+\infty)$.

**Proof.** 1. Let $0 < t < 1$ and $0 < \varepsilon < \frac{1}{2} \cdot \frac{t}{1-t}$. Let $B \subseteq X$ for which $\mathrm{card}(B) \geq 2$. We define the metric on $X$ as follows:

$$\sigma_B(x,y) = \begin{cases} 0, & \text{if } x = y \\ \frac{t}{1-t}, & \text{if } x,y \in B;\ x \neq y \\ \frac{t}{1-t} - \varepsilon, & \text{if } x \notin B \text{ or } y \notin B,\ x \neq y \end{cases}$$

It is to easy to verify that $\sigma_B$ is a metric and that $\sigma_B \neq \sigma'_B$, if $B \neq B'$. Evidently $f(\sigma_B) = t$. There are $2^{\mathrm{card}(X)}$ many choices for $B$ so we can see

$$2^{\mathrm{card}(X)} \leq \mathrm{card}(f^{-1}(\{t\})) \leq \mathrm{card}(\mathcal{M}) \leq 2^{\mathrm{card}(X)}.$$

We get by the Cantor–Bernstein theorem that $\mathrm{card}(f^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$.

Let now $t = 1$ and $X_0 = \{x_1 < x_2 < \cdots < x_n < \cdots\} \subset X$. Define the function $d_B \colon X \times X \to R$:

$$\begin{aligned} d_B(x_n,x_m) &= |n-m| \quad \text{for} \quad n,m = 1,2,\ldots \\ d_B(x,x_n) &= d_B(x_n,x) = n \quad \text{for} \quad x \notin X_0 \\ d_B(x,y) &= d_B(y,x) = 1 \quad \text{for} \quad x,y \notin X_0,\ x \neq y \\ d_B(x,x) &= 0 \quad \text{for} \quad x \in X. \end{aligned}$$

(The same function was used in [2], Theorem 5.) It can be easily verified that $d_B(x_n, x_1) \to \infty (n \to \infty)$, hence $f(d_B) = 1$. Thereby we have $2^{\mathrm{card}(X)}$ possibilities for choosing of $B$, we get that $\mathrm{card}(f^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$.

2. For $t=0$ it has been proved in [4] (Theorem 1), that $\mathrm{card}(g^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$. Let $t > 0$. Let $B \subset X$ is so, that $\mathrm{card}(B) \geq 2$. Define $\rho_B$ on $X$ as follows:

$$\rho_B(x,y) = \begin{cases} 0, & \text{for } x = y \\ t & \text{for } x, y \in B, x \neq y \\ t+1 & \text{otherwise.} \end{cases}$$

Then $\rho_B \in \mathcal{M}$ and $g(\rho_B) = t$.

3. Let $t > 0$ and $0 < \zeta < \frac{t}{2}$. Then the function $\tau_B$ defined on $X$ by this way

$$\tau_B(x,y) = \begin{cases} 0, & \text{for } x = y \\ t, & \text{for } x, y \in B,\ x \neq y \\ t-\zeta & \text{otherwise,} \end{cases}$$

is a metric on $X$ and $h(\tau_B) = t$. ∎

## References


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