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GENERAL HISTORY-DEPENDENT OPERATORS WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a class of nonlinear operators–the class of general history-dependent operators. These are the operators defined on spaces of functions endowed with a structure of Banach space (the case of bounded interval of time) or Fréchet space (the case of unbounded interval of time). We state and prove various properties of such operators, including fixed point properties. Moreover, we also study several classes of differential equations in Banach spaces, for which we our previous results can be applied.

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1. INTRODUCTION

As we know, the term "history-dependent operator" was firstly introduced by Sofonea–Matei [15] on spaces of functions endowed with a structure of Banach space (the case of bounded interval of time) or Fréchet space (the case of unbounded interval of time), because history-dependent operator is useful to represent the models in Contact Mechanics involving both quasistatic frictional and frictionless contact conditions with elastic or viscoelastic materials, for example, mechanical impact problems, electrical circuits with ideal diodes, the Coulomb friction problems for contacting bodies and so on. After the work [15], more and more scholars are attracted to boost the development of theory and applications for history-dependent operator. For instance, see [2, 9, 13, 14, 16, 18]. Very recently, Sofonea and Migórski [17] considered a generalized "history-dependent operator", which is called "almost history-dependent operator".

Our work is motivated by the development of the mathematical theory in Contact Mechanics, which requires new mathematical tools needed for the study of the differential equations. In particular, the theory requires results for new classes of general history-dependent operators under specific assumptions on functionals and operators. Several differential equations, for instance, in [1, 3–6, 8, 12], lead to general history-dependent operators of the form studied in this paper, in which is either the integer-order differential equations or the fractional-order differential equations. The abstract results presented in the paper can be applied to the study of these problems and, therefore, they can be used to prove the unique solvability of the corresponding differential equations. Any progress in the study of above mentioned problems will open avenues for new advances and applications of the history-dependent operators.

The aim of this manuscript is two folds. The first one is to provide the definition for general history-dependent operators, and then we prove an existence and uniqueness result for a new class of general history-dependent operators on spaces of functions endowed with a structure of Banach space (the case of bounded interval of time) or Fréchet space (the case of unbounded interval of time). The second aim is to illustrate our main results to study several classes of fractional differential equations. This gives rise to a new approach on the analysis of the corresponding mathematics models.

The rest of the manuscript is structured as follows. In Section 2, we present a definition for general history-dependent operator and then prove a general fixed point principle in the study of differential equations which gives rise to generic existence and uniqueness results in Theorems 1 and 2. In Section 3, we particularize Theorems 1 and 2 in the study of a specific class of differential equations for which we obtain new results of unique solvability for fractional-order differential equations in Theorem 3 and 4.

2. General history-dependent operators

In this section we review some prerequisites that are necessary in the next sections. Throughout this paper, the norm of a Banach space X will be denoted by $\|\cdot\|_X$. Let $\mathbb{R}_+ = [0,+\infty)$. Below, I denotes either a bounded interval of the form $[0,T]$ with $T>0$, or the unbounded interval $\mathbb{R}_+$. For, $L^p(I;X)$ we denote the Banach space of all Bochner integrable functions from I into X with the norm $\|f\|_{L^p(I,X)} = \left(\int_I \|f(t)\|_X^p ds\right)^{\frac{1}{p}}$. Next, in the case $I=[0,T]$ the space $C(I;X)$ will be equipped with the norm $\|x\|_{C(I;X)} = \max\limits_{t\in I=[0,T]} \|x(t)\|_X$. It is well known that if X is a Banach space, then $C(I;X)$ is also a Banach space. Assume now that $I=\mathbb{R}_+$. It is well known that, if X is a Banach space, then $C(I;X)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Recall that the convergence of a sequence $\{x_k\}_{k\geq 1}$ to the element x, in the space $C(\mathbb{R}_+;X)$, can be described as follows

$$\begin{cases} x_k \to x \ \text{ in } C(\mathbb{R}_+;X), \ \text{ as } n\to\infty \ \text{ if and only if} \\ \max\limits_{t\in[0,n]} \|x_k(t)-x(t)\|_X \to 0, \ \text{ as } k\to\infty, \qquad \forall n\in\mathbb{N}. \end{cases} \tag{2.1}$$

Moreover, we also note that

$$\begin{cases} \{x_k\} \subset C(\mathbb{R}_+;X) \text{ is a Cauchy sequence if and only if} \\ \forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists N = N(\varepsilon, n) \text{ such that} \\ \|x_{k_1} - x_{k_2}\|_{C([0,n];X)} < \varepsilon \qquad \forall k_1, k_2 > N. \end{cases}$$

As we know, the notion of history-dependent operator was introduced in [15] and the definition of history-dependent operator was considered in [17], which have found numerous applications. Here, we define the class of general history-dependent operator which is valid in the case of both bounded and unbounded intervals I as follows.

Definition 1. An operator $F : C(I;X) \to C(I;X)$ is called a general history-dependent operator if for any compact set $K \subset I$, there exist constants $\ell_K \in [0,1), L_K \geq 0$ and $q > -1$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K \|x(t) - y(t)\|_X + L_K \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \qquad (2.2)$$

for all $x, y \in C(I;X)$, $t \in K$.

Remark 1. We specialize the previous definition of general history-dependent operators in the cases $I = [0,T]$ and $I = \mathbb{R}_+$, respectively.

(i) An operator $F : C([0,T];X) \to C([0,T];X)$ is called a general history-dependent operator if there exist constants $\ell \in [0,1)$, $L \geq 0$ and $q > -1$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell \|x(t) - y(t)\|_X + L \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \qquad (2.3)$$

for all $x, y \in C([0,T];X), t \in [0,T]$.

(ii) An operator $F : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$ is called a general history-dependent operator if for any $n \in \mathbb{N}$, there exist constants $\ell_n \in [0,1)$, $L_n \geq 0$ and $q > -1$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_n \|x(t) - y(t)\|_X + L_n \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \qquad (2.4)$$

for all $x, y \in C(\mathbb{R}_+;X), t \in [0,n]$.

Next, let us now consider three special cases of general history-dependent operators.

(i) If $L_K = 0$, then (2.2) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K \|x(t) - y(t)\|_X,$$

which implies that

$$\|Fx - Fy\|_{C(I;X)} \leq \ell_K \|x - y\|_{C(I;X)},$$

i.e., F is a Lipschitz continuous operator on $C(I;X)$ with constant $\ell_K \in [0,1)$.

(ii) If $\ell_K = 0,\ q = 0$, then (2.2) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \le L_K \int_0^t \|x(s) - y(s)\|_X ds,$$

i.e., F is a history-dependent operator on $C(I;X)$ with constant $L_K \ge 0$ (cf. [15]).

(iii) If $q = 0$, then (2.2) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \le \ell_K \|x(t) - y(t)\|_X + L_K \int_0^t \|x(s) - y(s)\|_X ds,$$

i.e., F is an almost history-dependent operator on $C(I;X)$ with constant $\ell_K \in [0,1)$, $L_K \ge 0$ (cf. [17]).

Similarly, general history-dependent operators also have important fixed point properties which are very useful to prove the solvability of various classes of nonlinear equations and variational inequalities. Our aim in what follows is to present the fixed point properties for general history-dependent operators, in the cases of both bounded and unbounded intervals of time. We shall consider separately the case of the bounded interval $I = [0,T]$ and the case of the unbounded interval $I = \mathbb{R}_+$, since the arguments in proof are different. We start with the case of the bounded interval $I = [0,T]$ as follows.

Theorem 1. *If $F : C([0,T];X) \to C([0,T];X)$ is a general history-dependent operator, then F has a fixed point in $C([0,T];X)$.*

Proof. For $\lambda > 0$, we introduce the Bielecki norm

$$\|x\|_\lambda = \max_{t\in[0,T]} e^{-\lambda t} \|x(t)\|_X \qquad \text{for } x \in C([0,T];X). \tag{2.5}$$

Clearly, $\|\cdot\|_\lambda$ defines a norm on the space $C([0,T];X)$ which is equivalent to the usual norm $\|\cdot\|_{C([0,T];X)}$. As a consequence, it results that $C([0,T];X)$ is a Banach space with the norm $\|\cdot\|_\lambda$, too. Let $t \in [0,T]$. From the definition of the general history-dependent operator, it follows that

$$\|(Fx)(t) - (Fy)(t)\|_X \le \ell \|x(t) - y(t)\|_X + L \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds$$

for all $x, y \in C([0,T];X)$ with $\ell \in [0,1)$, $L \ge 0$ and $q > -1$. Then for $\lambda > 0$,

$$\begin{aligned}
& e^{-\lambda t} \|(Fx)(t) - (Fy)(t)\|_X \\
\le\ & \ell e^{-\lambda t} \|x(t) - y(t)\|_X + L e^{-\lambda t} \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \\
=\ & \ell e^{-\lambda t} \|x(t) - y(t)\|_X + L e^{-\lambda t} \int_0^t (t-s)^q e^{\lambda s} (e^{-\lambda s} \|x(s) - y(s)\|_X) ds \\
\le\ & \ell \|x - y\|_\lambda + L e^{-\lambda t} \|x - y\|_\lambda \int_0^t (t-s)^q e^{\lambda t} ds
\end{aligned}$$

$$
\begin{aligned}
&= \ell\|x-y\|_\lambda + Le^{-\lambda t}\|x-y\|_\lambda \int_0^t r^q e^{\lambda(t-r)}dr \qquad (r=t-s)\\
&= \ell\|x-y\|_\lambda + L\|x-y\|_\lambda \int_0^t r^q e^{-\lambda r}dr\\
&= \ell\|x-y\|_\lambda + L\|x-y\|_\lambda \lambda^{-q-1}\int_0^{\lambda t} \theta^q e^{-\theta}d\theta \qquad (\theta=\lambda r)\\
&\le \ell\|x-y\|_\lambda + L\|x-y\|_\lambda \lambda^{-q-1}\int_0^{\infty} \theta^q e^{-\theta}d\theta\\
&= \ell\|x-y\|_\lambda + L\|x-y\|_\lambda \lambda^{-q-1}\Gamma(q+1),
\end{aligned}
$$

and hence

$$\|Fx-Fy\|_\lambda \le (\ell + L\lambda^{-q-1}\Gamma(q+1))\|x-y\|_\lambda.$$

Next, since $\ell \in [0,1)$, we choose λ such that $\lambda > \sqrt[q+1]{\frac{L\Gamma(q+1)}{1-\ell}}$. Then $\ell + L\lambda^{-q-1}\Gamma(q+1) < 1$, which shows that the operator F is a contraction on the space $C([0,T];X)$ endowed with the norm $\|\cdot\|_\lambda$. By applying the Banach fixed point theorem we obtain that F has a unique fixed point $x^* \in C([0,T];X)$, which concludes the proof. □

We now move to the case of the unbounded interval $I = \mathbb{R}_+$.

Theorem 2. *If $F : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$ is a general history-dependent operator, then F has a fixed point in $C(\mathbb{R}_+;X)$.*

Proof. The proof is based on the following two claims.

Claim (1) For every $n \in \mathbb{N}$ there exist two constants $\ell_n \in [0,1)$, $L_n > 0$, which depend on n, such that

$$\|Fx-Fy\|_{\lambda,n} \le (\ell_n + L_n\lambda^{-q-1}\Gamma(q+1))\|x-y\|_{\lambda,n}, \tag{2.6}$$

where $x,y \in C(\mathbb{R}_+;X)$ and

$$\|x\|_{\lambda,n} = \max_{t\in[0,n]} e^{-\lambda t}\|x(t)\|_X, \qquad \text{for } x \in C([0,n];X).$$

The proof of this claim follows from the proof of Theorem 1.

Claim (2) For every $n \in \mathbb{N}$, there exists a unique function $x_n \in C([0,n];X)$ such that

$$(Fx_n)(t) = x_n(t), \qquad \forall\, t \in [0,n]. \tag{2.7}$$

Moreover, if $m,n \in \mathbb{N}$ are such that $m \ge n$, then

$$x_m(t) = x_n(t) \qquad \forall\, t \in [0,n]. \tag{2.8}$$

The proof of this claim is based on recurrence and is a consequence of **Claim (1)**. First, we choose λ_1 such that $\lambda_1 > \sqrt[q+1]{\frac{L_1\Gamma(q+1)}{1-\ell_1}}$ and denote $\alpha_1 = \ell_1 + L_1\lambda_1^{-q-1}\Gamma(q+1)$. Then it follows that $\alpha_1 \in (0,1)$ and, using (2.6) for $n=1$, we have

$$\|Fx-Fy\|_{\lambda_1,1} \le \alpha_1\|x-y\|_{\lambda_1,1}, \qquad \forall x,y \in C([0,1];X).$$

Next, we choose $\lambda_2 > \max\{\lambda_1, \sqrt[q+1]{\frac{L_2\Gamma(q+1)}{1-\ell_2}}\}$ and define $\alpha_2 = \ell_2 + L_2\lambda_2^{-q-1}\Gamma(q+1)$, and we continue by recurrence. As a result, we obtain a sequence $\{\lambda_n\}$ which satisfies $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ and a sequence $\{\alpha_n\}$ which satisfies $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$. Moreover,

$$\|Fx - Fy\|_{\lambda_n,n} \leq \alpha_n \|x-y\|_{\lambda_n,n}, \qquad \forall x,y \in C([0,n];X), n \in \mathbb{N}. \tag{2.9}$$

The first part of the claim is now a consequence of the Banach fixed point theorem. Assume now that $m,n \in \mathbb{N}$ are such that $m \geq n$. Then, equality (2.8) is a consequence of the uniqueness of the fixed point of the operator F on the space $C([0,n];X)$ for all $n \in \mathbb{N}$.

We now have all the ingredients to proceed with the proof of this Theorem.

Existence. **Claim (2)** allows us to consider the sequence $\{\widetilde{x}_n\} \subset C(\mathbb{R}_+;X)$ given by

$$\widetilde{x}_n(t) = \begin{cases} x_n(t) & \text{if } t \in [0,n], \\ x_n(n) & \text{if } t \geq n. \end{cases} \tag{2.10}$$

Let $n \in \mathbb{N}$. It follows from (2.8) and (2.10) that, if $m \in \mathbb{N}$ satisfies $m \geq n$, then

$$\widetilde{x}_m(t) = x_n(t), \qquad \forall t \in [0,n], \tag{2.11}$$

which implies that if $m_1, m_2 \in \mathbb{N}$ are such that $m_1, m_2 \geq n$, and hence

$$\|\widetilde{x}_{m_1} - \widetilde{x}_{m_2}\|_{\lambda_n,n} = 0, \qquad \forall t \in [0,n]. \tag{2.12}$$

Since n is arbitrary, we deduce from (2.12) that $\{\widetilde{x}_m\}$ is a Cauchy sequence in $C(\mathbb{R}_+;X)$. By the completeness of the space $C(\mathbb{R}_+;X)$, there exists $x \in C(\mathbb{R}_+;X)$ such that $\widetilde{x}_m \to x$ as $m \to \infty$. Then we have

$$\lim_{m\to\infty} \|\widetilde{x}_m - x\|_{\lambda_n,n} = 0, \tag{2.13}$$

i.e., the sequence $\{\widetilde{x}_m\}$ converges uniformly on $[0,n]$, for all $n \in \mathbb{N}$. Since the uniform convergence implies the pointwise convergence, we have

$$\lim_{m\to\infty} \widetilde{x}_m(t) = x(t), \qquad \forall t \in [0,n], n \in \mathbb{N}. \tag{2.14}$$

We now combine (2.11) and (2.14) to obtain that

$$x_n(t) = x(t), \qquad \forall t \in [0,n], n \in \mathbb{N}. \tag{2.15}$$

On the other hand, taking into account (2.9) and (2.13) we deduce that

$$\lim_{m\to\infty} \|F\widetilde{x}_m - Fx\|_{\lambda_n,n} = 0 \qquad n \in \mathbb{N}, \tag{2.16}$$

which shows that

$$\lim_{m\to\infty} F\widetilde{x}_m = Fx \qquad \text{in } C(\mathbb{R}_+;X). \tag{2.17}$$

Let $t \in \mathbb{R}_+$ be fixed. Obviously, there exists $n \geq 1$ such that $t \in [0,n]$ and, again, since the uniform convergence on $C([0,n];X)$ implies the pointwise convergence, by (2.17) we have

$$\lim_{m\to\infty} (F\widetilde{x}_m)(t) = (Fx)(t) \qquad \text{in } X. \tag{2.18}$$

Now, using (2.7), (2.11) and (2.15) for $m \geq n$, we obtain

$$(F\widetilde{x}_m)(t) = (Fx_m)(t) = x_n(t) = x(t). \tag{2.19}$$

We pass to the limit, as $m \to \infty$ in (2.19) and use (2.18) to find that $(Fx)(t) = x(t)$. Since t is an arbitrary real positive number, we conclude that $Fx = x$, i.e., $x \in C(\mathbb{R}_+;X)$ is a fixed point of the operator F.

Uniqueness. Assume that there exist $x, x' \in C(\mathbb{R}_+;X)$ such that $x \neq x'$, and

$$Fx = x \qquad \text{and} \qquad Fx' = x'. \tag{2.20}$$

Then, there exists $t_0 \in \mathbb{R}_+$ such that

$$x(t_0) \neq x'(t_0). \tag{2.21}$$

We choose $n \in \mathbb{N}$ such that $t_0 \in [0,n]$. Equations (2.20) imply that the functions $x, x' : [0,n] \to X$ are two fixed points of the operator F on the space $C([0,n];X)$, and therefore, by the uniqueness of the function $x_n \in C([0,n];X)$ introduced in **Claim 2**, we have $x(t) = x'(t) = x_n(t)$ for all $t \in [0,n]$, which contradicts (2.21). We conclude that the fixed point of the operator F is unique. □

3. FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

In this section, we give several fractional-order differential equations to illustrate the applications of the generalized history-dependent operators. We will apply Theorem 1 and Theorem 2 to show the existence and uniqueness results for the equations.

Let us recall some basic statements of fractional calculus from [7, 11].

Definition 2. The fractional integral of order q with the lower limit zero for a function f is defined as

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \qquad t > 0,\ q > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 3. The Caputo fractional derivative of order q for a function $f : [0,\infty) \to R$ is defined as

$${}^cD_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s)ds,\ t > 0,\ 0 < q < 1.$$

Remark 2. (i) If $f \in AC([0,\infty))$, then

$${}^cD_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s)ds = I_t^{1-q} f'(t),\ t > 0,\ 0 < q < 1.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2 and 3 are taken in Bochner's sense.

Consider the fractional neutral differential equations having the following form:

$$\begin{cases} {}^cD_t^q(x(t)-g(t,x(t))) = f(t,x(t)), & t\in I,\ 0<q<1, \\ x(0)=x_0. \end{cases} \tag{3.1}$$

Definition 4. A function $x\in C(I;X)$ is said to be a solution of problem (3.1) on I if

$$x(t) = x_0 - g(0,x_0) + g(t,x(t)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds, \qquad t\in I.$$

The hypotheses on the functions f and g are the following:

$$\begin{cases} g: I\times X\to X \text{ is such that} \\ \quad (\mathrm{a})\ g(\cdot,x) \text{ is measurable for every } x\in X; \\ \quad (\mathrm{b}) \text{ there exists a constant } L_g\in(0,1) \text{ such that} \\ \qquad\qquad \|g(t,x)-g(t,y)\|_X \le L_g\|x-y\|_X \\ \qquad \text{for all } x,y\in X,\ \text{ a.e. } t\in I. \end{cases} \tag{3.2}$$

$$\begin{cases} f: I\times X\to X \text{ is such that} \\ \quad (\mathrm{a})\ f(\cdot,x) \text{ is measurable for every } x\in X; \\ \quad (\mathrm{b}) \text{ there exists a constant } L_f>0 \text{ such that} \\ \qquad\qquad \|f(t,x)-f(t,y)\|_X \le L_f\|x-y\|_X \\ \qquad \text{for all } x,y\in X,\ \text{ a.e. } t\in I. \end{cases} \tag{3.3}$$

Define an operator $F_1: C(I;X)\to C(I;X)$ as follows:

$$(F_1x)(t) = x_0 - g(0,x_0) + g(t,x(t)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds,\ t\in I. \tag{3.4}$$

Then, we have the following existence and uniqueness result.

Theorem 3. *If (3.2), (3.3) hold and and $x_0\in X$, then problem (3.1) has a unique mild solution on $C(I;X)$.*

Proof. It is clear that the operator F_1 defined by (3.4) is a generalized history-dependent operator. □

In what follows, we will study the fractional evolution equations having the following form:

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + f(t,x(t)), & t\in I,\ 0<q<1, \\ x(0)=x_0. \end{cases} \tag{3.5}$$

Based on [10, 19], we shall define the following concept.

Definition 5. A function $x \in C(I;X)$ is said to be a mild solution of problem (3.5) on I if

$$x(t) = S_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s,x(s))ds, \qquad t \in I,$$

where

$$S_q(t) = \int_0^\infty \xi_q(\theta) T(t^q\theta) d\theta,$$
$$T_q(t) = q\int_0^\infty \theta\xi_q(\theta) T(t^q\theta) d\theta,$$
$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \varpi_q(\theta^{-\frac{1}{q}}),$$
$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

ξ_q is a probability density function defined on $(0,\infty)$ (see [19]), that is

$$\xi_q(\theta) \geq 0, \ \theta \in (0,\infty), \quad \text{and} \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

Due to the papers [10, 19] again, we can obtain the following results.

Lemma 1. *$S_q(t)$, $T_q(t)$ have the following properties.*

(i) *For any fixed $t \geq 0$, $S_q(t)$, $T_q(t)$ are linear and bounded operators, such that, for any $x \in X$,*

$$\|S_q(t)x\| \leq M\|x\|,$$
$$\|T_q(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

(ii) *$S_q(t)$, $T_q(t) (t \geq 0)$ are strongly continuous.*

Define an operator $F_2 : C(I;X) \to C(I;X)$ as follows:

$$(F_2x)(t) = S_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s,x(s))ds, \qquad t \in I. \tag{3.6}$$

Then, we have the following existence and uniqueness result.

Theorem 4. *If (3.3) holds and $x_0 \in X$, then problem (3.5) has a unique mild solution on $C(I;X)$.*

Proof. For all $x, y \in C(I;X)$ and $t \in I$, we have

$$\begin{aligned}\|(F_2x)(t) - (F_4y)(t)\| &= \|\int_0^t (t-s)^{q-1} T_q(t-s)(f(s,x(s)) - f(s,y(s)))ds\| \\ &\leq \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s)) - f(s,y(s)))\|ds\end{aligned}$$

$$\leq \frac{ML_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s)-y(s)\| ds,$$

which shows that the operator F_2 is a general history-dependent operator. □

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