Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 401–415

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3065

FEJÉR AND HERMITE-HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE h-CONVEX AND QUASI CONVEX FUNCTIONS WITH APPLICATIONS

SOFIAN OBEIDAT, M. A. LATIF, AND S. S. DRAGOMIR

Received 08 October, 2019

Abstract. In this paper, we establish new weighted integral inequalities for differentiable h-convex and quasi-convex functions. Applications of our findings to continuous random variables and approximation of integrals are given.

2010 *Mathematics Subject Classification:* 26D15; 26A51; 26A33

Keywords: convex functions, Hölder inequality, quasi-convex function, Hermite-Hadamard inequality

1. INTRODUCTION

The following two types of convexity of functions are well known in the literature.

Definition 1 ([7])**.** A function $\alpha : J \subset \mathbb{R} \to \mathbb{R}$ is said to be h-convex, where J is an interval, if the inequality

$$\alpha(tx + (1-t)y) \leq h(t)\alpha(x) + h(1-t)\alpha(y),$$

holds for all x, $y \in J$ and $t \in [0,1]$.

Definition 2 ([4])**.** A function $\alpha : J \subset \mathbb{R} \to \mathbb{R}$ is said to be quasi convex, where J is an interval, if the inequality

$$\alpha(tx + (1-t)y) \leq \max\{\alpha(x), \alpha(y)\}$$

holds for all x, $y \in J$ and $t \in [0,1]$.

Varšanec, in [9], introduced the concept of h-convex functions, which generalizes the concept of a nonnegative convex function, and other types of convexity of functions such as Godunova–Levin, and s-convex functions. We refer the reader to [7] and [8] for further properties of h-convex functions.

The Hermite-Hadamard inequality is one of the most important mathematical inequalities due to its applications in different contexts. It is stated as follows: For an interval $[c,d]$ with $c<d$,

$$\alpha\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d\alpha(x)\,dx\leq\frac{\alpha(c)+\alpha(d)}{2},$$

where $\alpha:[c,d]\to\mathbb{R}$ is a convex function. For extensions, refinements and applications of this inequality, see [3] and [6].

Recently, several authors established a number of inequalities to estimate the difference between $\alpha\left(\frac{c+d}{2}\right)$ and $\frac{1}{d-c}\int_c^d\alpha(x)\,dx$ for differentiable convex mappings, see for example [1,4] and [5]. In [2], Hwang obtained the following two theorems.

Theorem 1 ([2]). *Let $\alpha:J\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on J°, where J is an interval, and $w:[u,v]\to[0,\infty)$ be a continuous and symmetric mapping with respect to $\frac{u+v}{2}$, where $u,\ v\in J^\circ$ with $u<v$. If $\alpha'\in L_1([u,v])$ and $|\alpha'|$ is convex on $[u,v]$, then*

$$\left|\alpha\left(\frac{u+v}{2}\right)\int_u^v w(x)\,dx-\int_u^v w(x)\,\alpha(x)\,dx\right|\leq\frac{(v-u)}{2}\left[\left|\alpha'(u)\right|+\left|\alpha'(v)\right|\right]\int_0^1 M(w,u,v,t)\,dt,\quad(1.1)$$

where

$$M(w,u,v,t)=\int_u^{l(u,v,t)}w(x)\,dx\ \text{ and }\ l(u,v,t)=\frac{1+t}{2}u+\frac{1-t}{2}v.$$

Theorem 2 ([2]). *Let $\alpha:J\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on J°, where J is an interval, and $w:[u,v]\to[0,\infty)$ be a continuous and symmetric mapping with respect to $\frac{u+v}{2}$, where $u,\ v\in J^\circ$ with $u<v$. If $\alpha'\in L_1([u,v])$ and $|\alpha'|^q$ is convex on $[u,v]$ for $q\geq1$, then*

$$\left|\alpha\left(\frac{u+v}{2}\right)\int_u^v w(x)\,dx-\int_u^v w(x)\,\alpha(x)\,dx\right|\leq(v-u)\left[\frac{|\alpha'(u)|^q+|\alpha'(v)|^q}{2}\right]^{\frac{1}{q}}\int_0^1 M(w,u,v,t)\,dt,\quad(1.2)$$

where

$$M(w,u,v,t)=\int_u^{l(u,v,t)}w(x)\,dx\ \text{ and }\ l(u,v,t)=\frac{1+t}{2}u+\frac{1-t}{2}v.$$

Clearly, inequalities (1.1) and (1.2) give and upper bound for the difference between $\int_u^v w(x)\alpha(x)\,dx$ and $\alpha\left(\frac{u+v}{2}\right)\int_u^v w(x)\,dx$. In this paper, we generalize the inequalities obtained by Hwang in [2] for h-convex and quasi convex functions. Then we introduce an application of these results to continuous random variables whose probability density functions are continuous, and give another application to approximation of integrals.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $J \subset \mathbb{R}$ denotes an interval, and $h : (0,1) \to \mathbb{R}$ denotes a non-negative function and non-zero function. In addition, for any positive integer k, and $u, v \in J^\circ$ with $u < v$, the functions $\theta_{k,u,v}, \tilde{\theta}_{k,u,v} : [0,k] \to \mathbb{R}$ are defined as:

$$\theta_{k,u,v}(t) = \left(\frac{k+t}{2k}\right)u + \frac{k-t}{2k}v, \theta^*_{k,u,v}(t) = u + v - \theta_{k,u,v}(t).$$

The discrete power mean inequality will be used in the next section, and it is stated as follows:

$$x_1^\gamma + x_2^\gamma \le 2^{1-\gamma}(x_1+x_2)^\gamma, \tag{1.3}$$

where $x_1 > 0, x_2 > 0, \gamma < 1$.

2. Main Results

We start this section with the following lemma which will be used repeatedly in the sequel.

Lemma 1. *Let $\alpha : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on J° and u, $v \in J^\circ$ with $u < v$. Suppose that $g : [u,v] \to [0,\infty)$ is a differentiable mapping. If $\alpha' \in L_1([u,v])$, then for any positive integer k,*

$$\begin{aligned}
&g(u)\left(\frac{\alpha(u)+\alpha(v)}{2}\right) - g(v)\alpha\left(\frac{u+v}{2}\right) \\
&\quad + \left(\frac{v-u}{4k}\right)\int_0^k \left[g'(\theta_{k,u,v}(t)) + g'\left(\theta^*_{k,u,v}(t)\right)\right]\left[\alpha(\theta_{k,u,v}(t)) + \alpha\left(\theta^*_{k,u,v}(t)\right)\right]dt \\
&= \left(\frac{v-u}{4k}\right)\int_0^k \left[g(\theta_{k,u,v}(t)) - g\left(\theta^*_{k,u,v}(t)\right) + g(v)\right] \\
&\quad \times \left[\alpha'\left(\theta^*_{k,u,v}(t)\right) - \alpha'(\theta_{k,u,v}(t))\right]dt.
\end{aligned} \tag{2.1}$$

Proof. Let

$$S_1 = -\int_0^k \left[g(\theta_{k,u,v}(t)) - g\left(\theta^*_{k,u,v}(t)\right) + g(v)\right]\alpha'(\theta_{k,u,v}(t))\,dt$$

and

$$S_2 = \int_0^k \left[g(\theta_{k,u,v}(t)) - g\left(\theta^*_{k,u,v}(t)\right) + g(v)\right]\alpha'\left(\theta^*_{k,u,v}(t)\right)dt.$$

Using integration by parts,

$$\begin{aligned}
S_1 &= \frac{-2k}{u-v}\left[g\left(\theta_{k,u,v}(t)\right)-g\left(\theta^*_{k,u,v}(t)\right)+g(v)\right]\alpha\left(\theta_{k,u,v}(t)\right)\Big|_0^k \\
&\quad+\frac{2k}{u-v}\int_0^k\left[g'\left(\theta_{k,u,v}(t)\right)\frac{u-v}{2k}-g'\left(\theta^*_{k,u,v}(t)\right)\frac{v-u}{2k}\right]\alpha\left(\theta_{k,u,v}(t)\right)dt \\
&=\frac{2k}{v-u}\left(g(u)\alpha(u)-g(v)\alpha\left(\frac{u+v}{2}\right)\right) \\
&\quad+\int_0^k\left[g'\left(\theta_{k,u,v}(t)\right)+g'\left(\theta^*_{k,u,v}(t)\right)\right]\alpha\left(\theta_{k,u,v}(t)\right)dt,
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \frac{2k}{v-u}\left(g(u)\alpha(v)-g(v)\alpha\left(\frac{u+v}{2}\right)\right) \\
&\quad+\int_0^k\left[g'\left(\theta_{k,u,v}(t)\right)+g'\left(\theta^*_{k,u,v}(t)\right)\right]\alpha\left(\theta^*_{k,u,v}(t)\right)dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\frac{v-u}{4k}\right)[S_1+S_2]=g(u)\left(\frac{\alpha(u)+\alpha(v)}{2}\right)-g(v)\alpha\left(\frac{u+v}{2}\right) \\
&+\left(\frac{v-u}{4k}\right)\int_0^k\left[g'\left(\theta_{k,u,v}(t)\right)+g'\left(\theta^*_{k,u,v}(t)\right)\right]\left[\alpha\left(\theta_{k,u,v}(t)\right)+\alpha\left(\theta^*_{k,u,v}(t)\right)\right]dt,
\end{aligned}$$

and hence the result follows. □

Theorem 3. *Let $\alpha : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on J° and u, $v \in J^\circ$ with $u < v$. Suppose that $p : [u,v] \to [0,\infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha' \in L_1([u,v])$ and $\left|\alpha'\right|$ is h-convex on $[u,v]$, then for any positive integer k,*

$$\begin{aligned}
&\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right| \\
&\le\left(\frac{v-u}{k}\right)\left(\frac{|\alpha'(u)|+|\alpha'(v)|}{2}\right)\int_0^k W_{k,u,v}(t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt \qquad (2.2)
\end{aligned}$$

where $W_{k,u,v}(t)=\int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Proof. Define $g:[u,v]\to[0,\infty)$ as follows:

$$g(t)=\int_u^t p(x)\,dx, t\in[u,v].$$

Clearly, w is differentiable with $w'=p$. Using Lemma 1, we have

$$\begin{aligned}
&g(u)\left(\frac{\alpha(u)+\alpha(v)}{2}\right)-g(v)\alpha\left(\frac{u+v}{2}\right)\\
&\quad+\left(\frac{v-u}{4k}\right)\int\limits_0^k\left[g'(\theta_{k,u,v}(t))+g'\left(\theta^*_{k,u,v}(t)\right)\right]\left[\alpha(\theta_{k,u,v}(t))+\alpha\left(\theta^*_{k,u,v}(t)\right)\right]dt\\
&=\left(\frac{v-u}{4k}\right)\int_0^k\left[g(\theta_{k,u,v}(t))-g\left(\theta^*_{k,u,v}(t)\right)+g(v)\right]\left[\alpha'\left(\theta^*_{k,u,v}(t)\right)-\alpha'(\theta_{k,u,v}(t))\right]dt.
\end{aligned}$$

But

$$g(u)=0, g(v)=\int_u^v p(x)\,dx,$$

and

$$g'(\theta_{k,u,v}(t))=p(\theta_{k,u,v}(t)), g'\left(\theta^*_{k,u,v}(t)\right)=p\left(\theta^*_{k,u,v}(t)\right),$$

which imply that

$$\begin{aligned}
&\left(\frac{v-u}{4k}\right)\int\limits_0^k\left[p(\theta_{k,u,v}(t))+p\left(\theta^*_{k,u,v}(t)\right)\right]\left[\alpha(\theta_{k,u,v}(t))+\alpha\left(\theta^*_{k,u,v}(t)\right)\right]dt\\
&\quad-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx \qquad (2.3)\\
&=\left(\frac{v-u}{4k}\right)\times\int_0^k\left[g(\theta_{k,u,v}(t))-g\left(\theta^*_{k,u,v}(t)\right)+g(v)\right]\left[\alpha'\left(\theta^*_{k,u,v}(t)\right)-\alpha'(\theta_{k,u,v}(t))\right]dt.
\end{aligned}$$

Note that,

$$\begin{aligned}
g(\theta_{k,u,v}(t))-g\left(\theta^*_{k,u,v}(t)\right)+g(v)&=\int_u^{\theta_{k,u,v}(t)}p(x)\,dx-\int_u^{\theta^*_{k,u,v}(t)}p(x)\,dx+\int_u^v p(x)\,dx\\
&=\int_u^{\theta_{k,c,u}(t)}p(x)\,dx+\int_{\theta^*_{k,u,v}(t)}^v p(x)\,dx. \qquad (2.4)
\end{aligned}$$

Since p is symmetric about $\frac{u+v}{2}$, we have

$$\begin{aligned}
&\left(\frac{v-u}{4k}\right)\int\limits_0^k\left[p(\theta_{k,u,v}(t))+p\left(\theta^*_{k,u,v}(t)\right)\right]\left[\alpha(\theta_{k,u,v}(t))+\alpha\left(\theta^*_{k,u,v}(t)\right)\right]dt\\
&=\left(\frac{v-u}{2k}\right)\int\limits_0^k p(\theta_{k,u,v}(t))\,\alpha(\theta_{k,u,v}(t))\,dt+\left(\frac{v-u}{2k}\right)\int\limits_0^k p\left(\theta^*_{k,u,v}(t)\right)\alpha\left(\theta^*_{k,u,v}(t)\right)dt
\end{aligned}$$

$$= \int_u^c p(x)\alpha(x)\,dx + \int_c^v p(x)\alpha(x)\,dx = \int_u^v p(x)\alpha(x)\,dx, \tag{2.5}$$

and

$$\int_u^{\theta_{k,u,v}(t)} p(x)\,dx = \int_{\theta^*_{k,u,v}(t)}^v p(x)\,dx, \tag{2.6}$$

for each $t \in [0,k]$. This implies that

$$\left(\frac{v-u}{4k}\right)\int_0^k \left[g(\theta_{k,u,v}(t)) - g\left(\theta^*_{k,u,v}(t)\right) + g(v)\right]\left[\alpha'\left(\theta^*_{k,u,v}(t)\right) - \alpha'(\theta_{k,u,v}(t))\right]dt$$
$$= \left(\frac{v-u}{2k}\right)\int_0^k \left[\int_u^{\theta_{k,u,v}(t)} p(x)\,dx\right] \times \left[\alpha'\left(\theta^*_{k,u,v}(t)\right) - \alpha'(\theta_{k,u,v}(t))\right]dt. \tag{2.7}$$

Combining (2.3), (2.4), (2.5), (2.6) and (2.7), we get that

$$\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx$$
$$= \left(\frac{v-u}{2k}\right)\int_0^k W_{k,u,v}(t)\left[\alpha'\left(\theta^*_{k,u,v}(t)\right) - \alpha'(\theta_{k,u,v}(t))\right]dt.$$

Using the triangle inequality and the h-convexity of $\left|\alpha'\right|$, we have

$$\left|\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\le \left(\frac{v-u}{2k}\right)\int_0^k W_{k,u,v}(t)\left(\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right| + \left|\alpha'(\theta_{k,u,v}(t))\right|\right)dt$$
$$\le \left(\frac{v-u}{2k}\right)\left(\left|\alpha'(u)\right| + \left|\alpha'(v)\right|\right)\int_0^k W_{k,u,v}(t)\left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right)\right)dt.$$

□

Corollary 1. *Under the same conditions as in Theorem 3, if h is super-additive, i.e., $h(x)+h(y) \le h(x+y)$ for each $x,y \in [0,1]$, then the inequality*

$$\left|\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\le h(1)\left(\frac{v-u}{k}\right)\left(\frac{|\alpha'(u)| + |\alpha'(v)|}{2}\right)\int_0^k W_{k,u,v}(t)\,dt, \tag{2.8}$$

holds for each positive integer k, where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Corollary 2. *Under the same conditions as in Theorem 3, if h is symmetric about $\frac{1}{2}$ then*

$$\left|\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$

$$\leq \left(\frac{v-u}{k}\right)\left(\left|\alpha'(u)\right|+\left|\alpha'(v)\right|\right)\int_0^k W_{k,u,v}(t)\,h\left(\frac{k+t}{2k}\right)dt$$
$$=\left(\frac{v-u}{k}\right)\left(\left|\alpha'(u)\right|+\left|\alpha'(v)\right|\right)\int_0^k W_{k,u,v}(t)\,h\left(\frac{k-t}{2k}\right)dt,$$

for each positive integer k, where $W_{k,u,v}(t)=\int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Remark 1. In Lemma 1, if $h(t)=t$ for all $t\in[0,1]$, then Inequality (2.2) reduces to

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq \left(\frac{v-u}{2k}\right)\left(\left|\alpha'(u)\right|+\left|\alpha'(v)\right|\right)\int_0^k W_{k,u,v}(t)\,dt. \quad (2.9)$$

Theorem 4. *Let $\alpha: J\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on J° and u, $v\in J^\circ$ with $u<v$. Suppose that $p:[u,v]\to[0,\infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha'\in L_1([u,v])$ and $|\alpha'|^q$ is h-convex on $[u,v]$ for $q>1$, then for any positive integer k,*

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq \left(\frac{v-u}{k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}}\left(\frac{|\alpha'(u)|^q+|\alpha'(v)|^q}{2}\right)^{\frac{1}{q}}$$
$$\times\left(\int_0^k W_{k,u,v}(t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt\right)^{\frac{1}{q}}, \quad (2.10)$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $W_{k,u,v}(t)=\int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Proof. As in the proof of Theorem 3, we have

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq \left(\frac{v-u}{2k}\right)\int_0^k W_{k,u,v}(t)\left(\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right|+\left|\alpha'\left(\theta_{k,u,v}(t)\right)\right|\right)dt.$$

Applying Holder's inequality, we get that

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq \left(\frac{v-u}{2k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^k W_{k,u,v}(t)\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^k W_{k,u,v}(t)\left|\alpha'\left(\theta_{k,u,v}(t)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg].$$

Using inequality (1.3) and the $h-$convexity of $|\alpha'|^q$ on $[u,v]$, we have

$$\begin{aligned}
&\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right| \\
&\leq\left(\frac{v-u}{2^{\frac{1}{q}}k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^k W_{k,u,v}(t)\left(\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right|^q+\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\leq\left(\frac{v-u}{k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}}\left(\frac{|\alpha'(u)|^q+|\alpha'(v)|^q}{2}\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_0^k W_{k,u,v}(t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 3. *Under the same conditions as in Theorem 4, if $h(t)=t$ for each $t\in[0,1]$, then inequality (2.10) reduces to*

$$\begin{aligned}
&\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right| \\
&\qquad\leq\left(\frac{v-u}{k}\right)\left(\frac{|\alpha'(u)|^q+|\alpha'(v)|^q}{2}\right)^{\frac{1}{q}}\left(\int_0^k W_{k,u,v}(t)\,dt\right). \qquad (2.11)
\end{aligned}$$

Corollary 4. *Under the same conditions as in Theorem 4, if $p(x)=\frac{4}{k(v-u)}$ for each $x\in[u,v]$, then inequality (2.10) reduces to*

$$\begin{aligned}
&\left|\alpha\left(\frac{u+v}{2}\right)-\frac{1}{v-u}\int_u^v p(x)\alpha(x)\,dx\right| \\
&\leq\frac{(v-u)}{2^{2-\frac{1}{q}}k^{\frac{2}{q}}}\left[\frac{|\alpha'(u)|^q+|\alpha'(v)|^q}{2}\right]^{\frac{1}{q}} \\
&\quad\times\left(\int_0^k (k-t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt\right)^{\frac{1}{q}}. \qquad (2.12)
\end{aligned}$$

Corollary 5. *Under the same conditions as in Theorem 4, if h is super-additive, then the inequality*

$$\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$

$$\leq (h(1))^{\frac{1}{q}} \left(\frac{v-u}{k}\right) \left(\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2}\right)^{\frac{1}{q}} \left(\int_0^k W_{k,u,v}(t)\,dt\right), \quad (2.13)$$

holds for each positive integer k, where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Corollary 6. *Suppose that the assumptions of Theorem 4 are satisfied. If h is symmetric about $\frac{1}{2}$ then for any positive integer k,*

$$\begin{aligned}
&\left|\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right| \\
&\leq \left(\frac{v-u}{k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}} \left(\left|\alpha'(u)\right|^q + \left|\alpha'(v)\right|^q\right)^{\frac{1}{q}} \\
&\quad \times \left(\int_0^k W_{k,u,v}(t)\,h\left(\frac{k+t}{2k}\right)dt\right)^{\frac{1}{q}} \\
&= \left(\frac{v-u}{k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}} \left(\left|\alpha'(u)\right|^q + \left|\alpha'(v)\right|^q\right)^{\frac{1}{q}} \\
&\quad \times \left(\int_0^k W_{k,u,v}(t)\,h\left(\frac{k-t}{2k}\right)dt\right)^{\frac{1}{q}}, \qquad (2.14)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Theorem 5. *Let $\alpha : J \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on J° and $u, v \in J^\circ$ with $u < v$. Suppose that $p : [u,v] \to [0,\infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha' \in L_1([u,v])$ and $\left|\alpha'\right|$ is quasi-convex on $[u,v]$, then for any positive integer k,*

$$\begin{aligned}
&\left|\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right| \\
&\leq \left(\frac{v-u}{2k}\right)\left[\max\left\{\left|\alpha'(v)\right|, \left|\alpha'\left(\frac{u+v}{2}\right)\right|\right\}\right. \\
&\quad \left. + \max\left\{\left|\alpha'(u)\right|, \left|\alpha'\left(\frac{u+v}{2}\right)\right|\right\}\right]\int_0^k W_{k,u,v}(t)\,dt. \qquad (2.15)
\end{aligned}$$

where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x)\,dx$.

Proof. As in the proof of Theorem 3, we have

$$\begin{aligned}
&\left|\int_u^v p(x)\alpha(x)\,dx - \alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right| \\
&\qquad \leq \left(\frac{v-u}{2k}\right)\int_0^k W_{k,u,v}(t)\left(\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right| + \left|\alpha'\left(\theta_{k,u,v}(t)\right)\right|\right)dt.
\end{aligned}$$

Using the quasi-convexity of $\left|\alpha'\right|$ on $[u,v]$, we get

$$\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{2k}\right)\left[\max\left\{\left|\alpha'(v)\right|,\left|\alpha'\left(\frac{u+v}{2}\right)\right|\right\}\right.$$
$$\left.+\max\left\{\left|\alpha'(u)\right|,\left|\alpha'\left(\frac{u+v}{2}\right)\right|\right\}\right]\int_0^k W_{k,u,v}(t)\,dt.$$

□

Remark 2. In Theorem 5,

(1) If $|\alpha'|$ is non-decreasing, then inequality (2.15) reduces to

$$\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{2k}\right)\left(\left|\alpha'(v)\right|+\left|\alpha'\left(\frac{u+v}{2}\right)\right|\right)\int_0^k W_{k,u,v}(t)\,dt. \quad (2.16)$$

(2) If $|\alpha'|$ is non-increasing, then inequality (2.15) reduces to

$$\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{2k}\right)\left(\left|\alpha'(u)\right|+\left|\alpha'\left(\frac{u+v}{2}\right)\right|\right)\int_0^k W_{k,u,v}(t)\,dt. \quad (2.17)$$

Theorem 6. *Let $\alpha: J\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on J° and u, $v\in J^\circ$ with $u<v$. Suppose that $p:[u,v]\to[0,\infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha'\in L_1([u,v])$ and $|\alpha'|^q$ is quasi-convex on $[u,v]$ for $q>1$, then for any positive integer k,*

$$\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{4k^{1-\frac{1}{q}}}\right)\left[\left(\max\left\{\left|\alpha'(v)\right|^q,\left|\alpha'\left(\frac{u+v}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{\left|\alpha'(u)\right|^q,\left|\alpha'\left(\frac{u+v}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]\left(\int_0^k W_{k,u,v}(t)\,dt\right), \quad (2.18)$$

where $W_{k,u,v}(t)=\int_u^{\theta_{k,u,v}(t)}p(x)\,dx$.

Proof. As in the proof of Theorem 4, we have

$$\left|\int_u^v p(x)\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$

$$\leq \left(\frac{v-u}{2^{\frac{1}{q}}k}\right)\left(\int_0^k W_{k,u,v}(t)\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^k W_{k,u,v}(t)\left(\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right|^q+\left|\alpha'\left(\theta^*_{k,u,v}(t)\right)\right|^q\right)dt\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the quasi-convexity of $|\alpha'|^q$on $[u,v]$, we get that

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{4k^{1-\frac{1}{q}}}\right)\left[\left(\max\left\{|\alpha'(v)|^q,\left|\alpha'\left(\frac{u+v}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\max\left\{|\alpha'(u)|^q,\left|\alpha'\left(\frac{u+v}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]\left(\int_0^k W_{k,u,v}(t)\,dt\right).$$

□

Remark 3. In Theorem 6,

(1) If $|\alpha'|$ is non-decreasing, then inequality (2.18) reduces to

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{4k^{1-\frac{1}{q}}}\right)\left(|\alpha'(v)|+\left|\alpha'\left(\frac{u+v}{2}\right)\right|\right)\int_0^k W_{k,u,v}(t)\,dt. \quad (2.19)$$

(2) If $|\alpha'|$ is non-increasing, then inequality (2.18) reduces to

$$\left|\int_u^v p(x)\,\alpha(x)\,dx-\alpha\left(\frac{u+v}{2}\right)\int_u^v p(x)\,dx\right|$$
$$\leq\left(\frac{v-u}{4k^{1-\frac{1}{q}}}\right)\left(|\alpha'(u)|+\left|\alpha'\left(\frac{u+v}{2}\right)\right|\right)\int_0^k W_{k,u,v}(t)\,dt. \quad (2.20)$$

3. Some Applications

We start this section with an application to continuous random variables whose probability density functions are continuous.

Proposition 1. *Let X be a continuous random variable taking its values in the finite interval $[c,d]$, where $0<c<d$, with a continuous probability density function $w:[c,d]\to[0,1]$ which is symmetric about $\frac{c+d}{2}$. Let $q>1$, and $r\geq 1+\frac{2}{q}$. If the* *r-moment of X is finite, i.e.,* $E_r(X) = \int_c^d t^r p(t)\,dt < \infty$, *and* $\tau \in (0,1]$ *then*

$$\left| E_r(X) - \left(\frac{c+d}{2}\right)^r \right| \le \frac{r(d-c)}{2^{1-\frac{1}{q}}} \left(\frac{c^{q(r-1)} + d^{q(r-1)}}{2(\tau+1)} \right)^{\frac{1}{q}}. \tag{3.1}$$

Proof. Let $\alpha(x) = \frac{1}{r}x^r$ for $x \in [c,d]$. Since $|\alpha'(x)|^q = x^{q(r-1)}$ is h-convex for $h(t) = t^\tau, t \in (0,1]$, applying Theorem 4, we have

$$\begin{aligned} &\left| \int_c^d p(x)\alpha(x)\,dx - \alpha\left(\frac{c+d}{2}\right) \int_c^d p(x)\,dx \right| \\ &\le \left(\frac{d-c}{k}\right) \left(\int_0^k W_{k,c,d}(t)\,dt \right)^{\frac{1}{p}} \left(\frac{|\alpha'(c)|^q + |\alpha'(d)|^q}{2} \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_0^k W_{k,c,d}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $W_{k,c,d}(t) = \int_c^{\theta_{k,c,d}(t)} p(x)\,dx$. The result follows using the facts that

$$\int_c^d p(x)\,dx = 1,$$

$$\int_c^d p(x)\alpha(x)\,dx = \frac{1}{r}E_r(X),$$

$$\int_0^k W_{k,c,d}(t)\,dt = \int_0^k \int_c^{\theta_{k,,c,d}(t)} p(x)\,dx\,dt \le \frac{k}{2},$$

and

$$\begin{aligned} &\int_0^k W_{k,c,d}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \\ &\qquad \le \frac{1}{2} \int_0^k \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt = \frac{k}{\tau+1}. \end{aligned}$$

□

The second application will be devoted to approximating an integral. Recall that a partition D of a finite interval $[c,d], c < d$, is a finite sequence of numbers $c = c_0 < c_1 < \cdots < c_n = d$.

Proposition 2. *Let* $\alpha : J \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* J° *and* $c, d \in J^\circ$ *with* $c < d$. *Suppose that* $p : [c,d] \to [0,\infty)$ *is a continuous mapping which is* *symmetric about* $\frac{c+d}{2}$. *Let* $D : c = c_0 < c_1 < \cdots < c_n = d$ *be a partition of* $[c,d]$. *If* $\alpha' \in L_1([c,d])$ *and* $\left|\alpha'\right|$ *is h-convex on* $[c,d]$, *then for any positive integer k*

$$\int_c^d p(x)\alpha(x)\,dx = A(\alpha,p,D)+E(\alpha,p,D),$$

where

$$A(\alpha,p,D) = \sum_{j=0}^{n-1}\alpha\left(\frac{c_j+c_{j+1}}{2}\right)P_{c_j,c_{j+1}},$$

$$P_{c_j,c_{j+1}} = \int_{c_j}^{c_{j+1}} p(x)\,dx,$$

and

$$\begin{aligned}|E(\alpha,p,D)| \leq \sum_{j=0}^{n-1}\left(\frac{c_{j+1}-c_j}{k}\right)\left[\frac{\left|\alpha'(c_j)\right|+\left|\alpha'(c_{j+1})\right|}{2}\right]\\ \times\int_0^k W_{k,c_j,c_{j+1}}(t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt.\end{aligned} \tag{3.2}$$

Proof. For each $j = 0,1,\cdots,n-1$, applying Theorem 3 over the interval $[c_j, c_{j+1}]$, we have

$$\begin{aligned}&\left|\int_{c_j}^{c_{j+1}} p(x)\alpha(x)\,dx-\alpha\left(\frac{c_j+c_{j+1}}{2}\right)\int_{c_j}^{c_{j+1}} p(x)\,dx\right|\\ &\leq\left(\frac{c_{j+1}-c_j}{k}\right)\left[\frac{\left|\alpha'(c_j)\right|+\left|\alpha'(c_{j+1})\right|}{2}\right]\\ &\quad\times\int_0^k W_{k,c_j,c_{j+1}}(t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt.\end{aligned} \tag{3.3}$$

where $W_{k,c_j,c_{j+1}} = \int_{c_j}^{\theta_{k,c_j,c_{j+1}}(t)} p(x)\,dx$. Note that

$$\begin{aligned}E(\alpha,p,D) &= \int_c^d p(x)\alpha(x)\,dx - A(\alpha,p,D)\\ &= \sum_{j=0}^{n-1}\int_{c_j}^{c_{j+1}} p(x)\alpha(x)\,dx-\sum_{j=0}^{n-1}\alpha\left(\frac{c_j+c_{j+1}}{2}\right)P_{c_j,c_{j+1}}\\ &= \sum_{j=0}^{n-1}\left[\int_{c_j}^{c_{j+1}} p(x)\alpha(x)\,dx-\left(\frac{\alpha(c_j)+\alpha(c_{j+1})}{2}\right)\int_{c_j}^{c_{j+1}} p(x)\,dx\right].\end{aligned}$$

Using the triangle inequality and inequality (3.3), we get that

$$|E(\alpha,p,D)| \leq \sum_{j=0}^{n-1}\left(\frac{c_{j+1}-c_j}{k}\right)\left[\frac{|\alpha'(c_j)|+|\alpha'(c_{j+1})|}{2}\right] \\ \times\int_0^k W_{k,c_j,c_{j+1}}(t)\left(h\left(\frac{k+t}{2k}\right)+h\left(\frac{k-t}{2k}\right)\right)dt.$$

□

REFERENCES

[1] S. Dragomir and R. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998, doi: 10.1016/S0893-9659(98)00086-X.

[2] D. Hwang, "Some inequalities for differentiable convex mapping with application to weighted midpoint formula and higher moments of random variables," *Applied Mathematics and Computation*, vol. 232, pp. 68–75, 2014, doi: 10.1016/j.amc.2014.01.050.

[3] D. A. Ion, "Some estimates on the hermite-hadamard inequality through quasi-convex functions," *Annals of the University of Craiova - Mathematics and Computer Science Series*, vol. 34, pp. 82–87, 2007.

[4] U. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004, doi: 10.1016/S0096-3003(02)00657-4.

[5] C. Pearce and J. Pečarić, "Inequalities for differentiable mappings with application to special means and quadrature formulæ," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 51–55, 2000, doi: 10.1016/S0893-9659(99)00164-0.

[6] F. Qi and B.-Y. Xi, *Some Hermite–Hadamard type inequalities for geometrically quasi-convex functions.*, 2014, vol. 124, no. 3, doi: 10.1007/s12044-014-0182-7.

[7] M. Sarikaya, A. Saglam, and H. Yildirim, "On some Hadamard-type inequalities for h-convex functions." *J. Math. Inequal.*, vol. 2, no. 3, pp. 335–341, 2008, doi: 10.7153/jmi-02-30.

[8] M. Sarikaya, E. Set, and M. Özdemir, "On some new inequalities of Hadamard type involving h-convex functions." *Acta Math. Univ. Comenian.(NS)*, vol. 79, no. 2, pp. 265–272, 2010.

[9] S. Varšanec, "On h-convexity." *J. Math. Anal. Appl.*, vol. 326, no. 1, pp. 303–311, 2007, doi: 10.1016/j.jmaa.2006.02.086.

Authors' addresses

Sofian Obeidat

(**Corresponding author**) Department of Basic Sciences, Deanship of Preparatory Year, University of Hail, Hail 2440, Saudi Arabia

E-mail address: obeidatsofian@gmail.com

M. A. Latif

Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa,Saudi Arabia

E-mail address: m_amer_latif@hotmail.com

S. S. Dragomir
Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
Current address: DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail address: sever.dragomir@vu.edu.au