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ON (p,φ_h)-CONVEX FUNCTION AND ITS APPLICATIONS

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Abstract. In this paper, we introduce the notion of (p,φ_h)-convex functions and present some properties and representation of such functions. Finally, a version of Hermite Hadamard-type inequalities for (p,φ_h)-convex functions are established.

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1. INTRODUCTION

Recently theory of convexity has received much attentions by many researchers. Consequently the classical concepts of convex sets and convex functions have been extended and generalized in several directions using novel and innovative ideas, see [16]. In [23], Varošneac introduced the notion of h-convex functions. It is worth to mention here that besides the classical convex functions, the class of h-convex functions also includes the class of some convex functions which are s-convex functions, Godunova-Levin functions and p-functions [7]. For some recent investigations on h-convex functions, see [19, 20]. The interrelationship between theory of convex functions and theory of inequalities has attracted many researchers. One of the most extensively studied inequality for convex functions is the Hermite–Hadamard inequality. This inequality provides the necessary and sufficient condition for a function to be convex. The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in literature (see, e.g. [7]). These inequalities state that if $f: I\to R$ is a convex function on the interval I of real numbers and a, $b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians who used many different convex functions. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained by using h-convex, φ_h-convex, p-convex etc., to mention a few (see [1–7, 13, 17, 21, 22]) and the references cited therein.

In this paper, we define new general version of p-convex functions name as the (p,φ_h)-convex function and we give some important properties of the (p,φ_h)-convex function. Then by using convexity we establish the Hermite–Hadamard type inequalities.

2. Preliminaries and lemmas

In this section, we recall some previously known concepts and derive some new results which play an important role in the development of our main results.

In [25], Zhang and Wan define a new convexity class called p-convex function and in [12], İşcan give a different version of the definition of p-convex function as follows:

Definition 1. [12, Theorem 3] Let $I \subset (0,\infty)$ be a real interval and $p \in R\setminus\{0\}$. A function $f: I \to R$ is said to be a p-convex function or belong the class $PC(I)$, if

$$f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

It is clear that p-convex functions includes convex functions and Harmonically convex functions as special cases. According to Definition 1, it can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively. For some results related to p-convex functions and its generalizations, we refer the reader to see [9, 12–15, 25].

Let us consider a function $\varphi: [a,b] \to [a,b]$ where $[a,b] \subset \mathbb{R}$. Youness have defined the φ-convex functions in [24], but we work here with the improved definition, according to [4]:

Definition 2. [4, Definition 1] A function $f: [a,b] \to R$ is said to be φ-convex on $[a,b]$ if for every two point $x \in [a,b]$, $y \in [a,b]$ and $t \in [0,1]$ the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)). \tag{2.1}$$

In [23] Varošneac introduced the notion of h-convex functions as follows:

Definition 3. [23, Definition 4] Let I be an interval in R and $h: (0,1) \to (0,\infty)$ be a given function. We say that a function $f: I \to R$ is h-convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{2.2}$$

for all $x, y \in I$ and $t \in (0,1)$.

This notion unifies and generalizes the known classes of convex functions, s-convex functions, Godunova-Levin functions and p-functions, which are obtained putting in (2.2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$ and $h(t) = 1$ respectively. Many properties of them can be found for instance, in [7, 8, 10, 11, 16, 19, 20].

In [17], Sarikaya has defined the following φ_h-convex functions which is a generalization of the above convex functions:

Definition 4. Let I be an interval in R and $h\colon (0,1)\to(0,\infty)$ be a given function. We say that a function $f\colon I\to R$ is φ_h-convex if

$$f\left(t\varphi(x)+(1-t)\varphi(y)\right)\le h(t)f(\varphi(x))+h(1-t)f(\varphi(y)) \tag{2.3}$$

for all $x,\ y\in I$ and $t\in(0,1)$. If the inequality (2.3) is reversed, then f is said to be φ_h-concave. In particular if (2.3) with $h(t)=t$, $h(t)=t^s$, $h(t)=\frac{1}{t}$ and $h(t)=1$, then f is said to be φ-convex, φ_s-convex, φ-Godunova-Levin function and φ-P-function, respectively(see e.g. [17, 18]).

Definition 5 ([7])**.** A function $h\colon I\to R$ is called a super-multiplicative function if

$$h(xy)\ge h(x)h(y) \tag{2.4}$$

for all $x,\ y\in I$. If the inequality sign in (2.4) is reversed, then h is multiplicative function. If $c>1$, then h is a sub-multilicative function and if the equality holds in (2.4), then h is called a multiplicative function.

3. Main results

In this section, we give new definitions and properties of the (p,φ_h)-convex function. Throughout this paper we assume that $(0,1)\subseteq J$, $I\subset(0,\infty)$, f and h are real non-negative functions defined on I and J, respectively, $p\in R\setminus\{0\}$ and $\varphi\colon[a,b]\to[a,b]$ is continuous where $[a,b]\subset\mathbb{R}$. We first give a definition of the new class of convex functions.

Definition 6. Let $h\colon J\to R$ be a non-negative and non-zero function. We say that $f\colon I\to R$ is a (p,φ_h)-convex or that f belongs to the class $fsx(p,\varphi_h,I)$, if f is non-negative and

$$f([t\varphi^p(x)+(1-t)\varphi^p(y)]^{\frac{1}{p}})\le h(t)f(\varphi(x))+h(1-t)f(\varphi(y)) \tag{3.1}$$

for all $x,y\in I$ and $t\in(0,1)$. Similarly, if the inequality sign in (3.1) is reversed, then f is said to be a (p,φ_h)-concave function or to the class $fsv(p,\varphi_h,I)$.

Remark 1. It can be obviously seen that if f is satisfies (2.1) with $p=1$, f is said to be φ_h-convex, $h(t)=t^s$ and $p=1$, f is said to be φ_s-convex, $h(t)=\frac{1}{t}$ and $p=1$ f is said to be φ-Godunova–Levin function, $h(t)=1$ and $p=1$, f is said to be φ-P-function.

Lemma 1. *Let $h\colon(0,1)\to(0,\infty)$. If $f,\ g\in fsx(p,\varphi_h,I)$ and $\lambda>0$, then $f+g$, $\lambda f\in fsx(p,\varphi_h,I)$. Similarly, if $f,\ g\in fsv(p,\varphi_h,I)$ and $\lambda>0$, then $f+g$, $\lambda f\in fsv(p,\varphi_h,I)$.*

Proof. Let identify $m(x) = f(x) + g(x)$, then we get

$$\begin{aligned}
& m\left(\left[t\varphi^p(x)+(1-t)\varphi^p(y)\right]^{\frac{1}{p}}\right) \\
&= f\left(\left[t\varphi^p(x)+(1-t)\varphi^p(y)\right]^{\frac{1}{p}}\right)+g\left(\left[t\varphi^p(x)+(1-t)\varphi^p(y)\right]^{\frac{1}{p}}\right) \\
&\le h(t)f(\varphi(x))+h(1-t)f(\varphi(y))+h(t)g(\varphi(x))+h(1-t)g(\varphi(y)) \\
&= h(t)\left[f(\varphi(x))+g(\varphi(x))\right]+h(1-t)\left[f(\varphi(y))+g(\varphi(y))\right] \\
&= h(t)m(\varphi(x))+h(1-t)m(\varphi(y))
\end{aligned}$$

and hence, $f+g \in fsx(p,\varphi_h,I)$. Similarly, it is proved that $\lambda f \in fsx(p\varphi_h,I)$. □

Lemma 2. *Let $h_1, h_2 \colon (0,1) \to (0,\infty)$ be such that $h_2(t) \le h_1(t)$ for all $t \in (0,1)$. If $f \in fsx(p,\varphi_{h_2},I)$, then $f \in fsx(p,\varphi_{h_1},I)$.*

Proof. If $f \in fsx(p,\varphi_{h_2},I)$, then for any $x, y \in I$ and $t \in (0,1)$ we have

$$\begin{aligned}
f\left(\left[t\varphi^p(x)+(1-t)\varphi^p(y)\right]^{\frac{1}{p}}\right) &\le h_2(t)f(\varphi(x))+h_2(1-t)f(\varphi(y)) \\
&\le h_1(t)f(\varphi(x))+h_1(1-t)f(\varphi(y))
\end{aligned}$$

and hence $f \in fsx(p,\varphi_{h_1},I)$. □

Lemma 3. *Let I be an interval such that $0 \in I$ and let $p > 0$. Then, we have the following*

(1) If $f \in fsx(p,\varphi_h,I)$, $f(0)=0$, $\varphi(0)=0$ and h is super-multiplicative, then the inequality

$$f\left(\left[\alpha\varphi^p(a)+\beta\varphi^p(b)\right]^{\frac{1}{p}}\right) \le h(\alpha)f(\varphi(x))+h(\beta)f(\varphi(y))$$

holds for all $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha+\beta \le 1$.

(2) If $f \in fsv(p,\varphi_h,I)$, $f(0)=0$, $\varphi(0)=0$ and h is sub-multiplicative, then the inequality

$$f\left(\left[\alpha\varphi^p(a)+\beta\varphi^p(b)\right]^{\frac{1}{p}}\right) \ge h(\alpha)f(\varphi(x))+h(\beta)f(\varphi(y))$$

holds for all $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha+\beta \le 1$.

Proof.

(1) Let $\alpha, \beta > 0$, $\alpha+\beta=\gamma<1$ and let a and b be numbers such that $a=\frac{\alpha}{\gamma}$ and $b=\frac{\beta}{\gamma}$. Then, we have $a+b=1$. Since $f \in fsx(p,\varphi_h,I)$, $f(0)=0$, $\varphi(0)=0$, and h is super-multiplicative, we obtain

$$\begin{aligned}
f\left(\left[\alpha\varphi^p(x)+\beta\varphi^p(y)\right]^{\frac{1}{p}}\right) &= f\left(\left[a\gamma\varphi^p(x)+b\gamma\varphi^p(y)\right]^{\frac{1}{p}}\right) \\
&\le h(a)f\left(\gamma^{\frac{1}{p}}\varphi(x)\right)+h(b)f\left(\gamma^{\frac{1}{p}}\varphi(y)\right) \\
&= h(a)f\left(\left[\gamma\varphi^p(x)+(1-\gamma)\varphi^p(0)\right]^{\frac{1}{p}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+h(b)f\left(\left[\gamma\varphi^p(y)+(1-\gamma)\varphi^p(0)\right]^{\frac{1}{p}}\right)\\
&\le h(a)h(\gamma)f(\varphi(x))+h(b)h(\gamma)f(\varphi(y))\\
&\le h(a\gamma)f(\varphi(x))+h(b\gamma)f(\varphi(y))\\
&= h(\alpha)f(\varphi(x))+h(\beta)f(\varphi(y))
\end{aligned}
$$

for all $x,y\in I$.

(2) Similarly, let $\alpha,\beta>0$, $\alpha+\beta=\gamma<1$, and let a and b be numbers such that $a=\frac{\alpha}{\gamma}$ and $b=\frac{\beta}{\gamma}$. Then, we have $a+b=1$. Since $f\in fsv(p,\varphi_h,I)$, $f(0)=0$, $\varphi(0)=0$ and h is sub-multiplicative, we have

$$
\begin{aligned}
f\left(\left[\alpha\varphi^p(x)+\beta\varphi^p(y)\right]^{\frac{1}{p}}\right)&=f\left(\left[a\gamma\varphi^p(x)+b\gamma\varphi^p(y)\right]^{\frac{1}{p}}\right)\\
&\ge h(a)f\left(\gamma^{\frac{1}{p}}\varphi(x)\right)+h(b)f\left(\gamma^{\frac{1}{p}}\varphi(y)\right)\\
&= h(a)f\left(\left[\gamma\varphi^p(x)+(1-\gamma)\varphi^p(0)\right]^{\frac{1}{p}}\right)\\
&\quad+h(b)f\left(\left[\gamma\varphi^p(y)+(1-\gamma)\varphi^p(0)\right]^{\frac{1}{p}}\right)\\
&\ge h(a)h(\gamma)f(\varphi(x))+h(b)h(\gamma)f(\varphi(y))\\
&\ge h(a\gamma)f(\varphi(x))+h(b\gamma)f(\varphi(y))\\
&= h(\alpha)f(\varphi(x))+h(\beta)f(\varphi(y))
\end{aligned}
$$

for all $x,y\in I$.

□

Lemma 4. *Let $f\colon [a,b]\to R$, $h\colon (0,1)\to(0,\infty)$ and $\varphi\colon [a,b]\to[a,b]$. Then the following statements are equivalent:*

(i) *The function f is a (p,φ_h)-convex on $[a,b]$.*

(ii) *For every x, $y\in[a,b]$, the mapping $g\colon [0,1]\to R$, $g(t)=f\left(\left[t\varphi^p(x)+(1-t)\varphi^p(y)\right]^{\frac{1}{p}}\right)$ is h-convex on $[a,b]$.*

Proof. Assume that f is (p,φ_h)-convex function and let us consider two points x, $y\in[a,b]$, $\lambda\in(0,1)$ and t_1, $t_2\in[0,1]$, then we obtain

$$
\begin{aligned}
g(\lambda t_1+(1-\lambda)t_2)&=f\left(\left[(\lambda t_1+(1-\lambda)t_2)\varphi^p(x)+(1-\lambda t_1-(1-\lambda)t_2)\varphi^p(y)\right]^{\frac{1}{p}}\right)\\
&=f\left(\left[\lambda(t_1\varphi^p(x)+(1-t_1)\varphi^p(y))+(1-\lambda)(t_2\varphi^p(x)+(1-t_2)\varphi^p(y))\right]^{\frac{1}{p}}\right)\\
&\le h(\lambda)f\left(\left[(t_1\varphi^p(x)+(1-t_1)\varphi^p(y))\right]^{\frac{1}{p}}\right)\\
&\quad+h(1-\lambda)f\left(\left[t_1\varphi^p(x)+(1-t_1)\varphi^p(y)\right]^{\frac{1}{p}}\right)\\
&=h(\lambda)g(t_1)+h(1-\lambda)g(t_2)
\end{aligned}
$$

which gives that g is a h-convex function.

Conversely, if g is a h-convex function, then for x, $y \in [a,b]$, $\lambda \in (0,1)$ and t_1 and $t_2 = 0$, we get

$$\begin{aligned} f\left([\lambda\varphi^p(x)+(1-\lambda)\varphi^p(y)]^{\frac{1}{p}}\right) &= g(\lambda 1+(1-\lambda)0) \\ &\leq h(\lambda)g(1)+h(1-\lambda)g(0) \\ &= h(\lambda)f\left([\varphi^p(x)]^{\frac{1}{p}}\right)+h(1-\lambda)f\left([\varphi^p(y)]^{\frac{1}{p}}\right) \\ &= h(\lambda)f(\varphi(x))+h(1-\lambda)f(\varphi(y)) \end{aligned}$$

which shows that f is (p,φ_h)-convex function. This completes the proof. □

Theorem 1. *Let $h\colon (0,1)\to(0,\infty)$ and f belongs to the class $fsx(p,\varphi_h,I)$. Let $\varphi(a), \varphi(b)\in I$ with $\varphi(a)<\varphi(b)$. Then the following inequalities hold:*

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)} x^{p-1}f(x)\,dx \\ &\leq [f(\varphi(a))+f(\varphi(b))]\int_0^1 h(t)\,dt. \end{aligned} \tag{3.2}$$

Proof. By using the (p,φ_h)-convexity of f, we have

$$\begin{aligned} & f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\ &= f\left(\left[\frac{t\varphi^p(a)+(1-t)\varphi^p(b)}{2}+\frac{(1-t)\varphi^p(a)+t\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\ &\leq h\left(\frac{1}{2}\right)\left[f\left([t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}\right)+f\left([(1-t)\varphi^p(a)+t\varphi^p(b)]^{\frac{1}{p}}\right)\right]. \end{aligned} \tag{3.3}$$

Integrating both side of (3.3) over the interval $(0,1)$, it follows that

$$\begin{aligned} f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) &\leq h\left(\frac{1}{2}\right)\cdot\left[\int_0^1 f\left([t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}\right)dt\right. \\ &\quad \left.+\int_0^1 f\left([(1-t)\varphi^p(a)+t\varphi^p(b)]^{\frac{1}{p}}\right)dt\right]. \end{aligned}$$

In the first integral, we substitute $x=[t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}$. Meanwhile, in the second integral, we also use the substitution $x=([(1-t)\varphi^p(a)+t\varphi^p(b)])^{\frac{1}{p}}$, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)}x^{p-1}f(x)\,dx.$$

In order to prove second inequality of (3.2), we start from the (p,φ_h)-convexity of f, meaning that for every $t\in(0,1)$, one has

$$f\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)\le h(t)f(\varphi(a))+h(1-t)f(\varphi(b))$$

integrating both sides of the above inequality over $(0,1)$, we obtain

$$\int_0^1 f\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)dt\le f(\varphi(a))\int_0^1 h(t)\,dt+f(\varphi(b))\int_0^1 h(1-t)\,dt.$$

The previous substitution in the first side of this inequality leads to

$$\frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)} x^{p-1}f(x)\,dx\le\left[f(\varphi(a))+f(\varphi(b))\right]\int_0^1 h(t)\,dt$$

which gives the second inequality of (3.2). This completes the proof. □

Remark 2. If $p=1$, then inequality (3.2) gives the Hermite–Hadamard inequality for φ_h-convex proved by Sarikaya in [17, Theorem 1].

Remark 3. If $p=1$, $h(t)=t$, $t\in(0,1)$, then in (3.2) coincide with the Hermite–Hadamard type inequalities for φ-convex functions proved by Cristescu in [4, Theorem 3].

Remark 4. If $p=1$, $h(t)=t^s\,(s\in(0,1))$, $t\in(0,1)$, then in (3.2) coincide with the Hermite–Hadamard type inequalities for φ_s-convex functions proved by Sarikaya in [17, Corollary 1].

Theorem 2. *Suppose that f and g are functions such that $f\in fsx(p,\varphi_{h_1},I)$, $g\in fsx(p,\varphi_{h_2},I)$, $fg\in L_1([a,b])$ and $\varphi(a),\varphi(b)\in I$ with $\varphi(a)<\varphi(b)$ and $h_1h_2\in L_1([0,1])$ with a, $b\in I$ and $a<b$. We then have*

$$\frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)} x^{p-1}f(x)g(x)\,dx$$
$$\le M_\varphi(a,b)\int_0^1 h_1(t)h_2(t)\,dt+N_\varphi(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt$$

where

$$M_\varphi(a,b)=f(\varphi(a))g(\varphi(a))+f(\varphi(b))g(\varphi(b))$$
$$N_\varphi(a,b)=f(\varphi(b))g(\varphi(a))+f(\varphi(a))g(\varphi(b)).$$

Proof. Since $f\in fsx(p,\varphi_{h_1},I)$ and $g\in fsx(p,\varphi_{h_2},I)$, we have

$$f\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)\le h_1(t)f(\varphi(a))+h_1(1-t)f(\varphi(b))$$

$$g\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)\le h_2(t)g(\varphi(a))+h_2(1-t)g(\varphi(b))$$

for all $t \in [0,1]$. Because f and g are non-negative, we get the inequality

$$\begin{aligned} & f\left([t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}\right) g\left([t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}\right) \\ & \quad \le h_1(t)h_2(t)f(\varphi(a))g(\varphi(a))+h_1(1-t)h_2(t)f(\varphi(b))g(\varphi(a)) \\ & \qquad +h_1(t)h_2(1-t)f(\varphi(a))g(\varphi(b))+h_1(1-t)h_2(1-t)f(\varphi(b))g(\varphi(b)). \end{aligned}$$

Integrating both sides of the above inequality over $(0,1)$ we obtain the inequality

$$\begin{aligned} & \int_0^1 f\left([t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}\right) g\left([t\varphi^p(a)+(1-t)\varphi^p(b)]^{\frac{1}{p}}\right) dt \\ & \quad \le f(\varphi(a))g(\varphi(a))\int_0^1 h_1(t)h_2(t)\,dt+f(\varphi(b))g(\varphi(a))\int_0^1 h_1(1-t)h_2(t)\,dt \\ & \qquad +f(\varphi(a))g(\varphi(b))\int_0^1 h_1(t)h_2(1-t)\,dt \\ & \qquad +f(\varphi(b))g(\varphi(b))\int_0^1 h_1(1-t)h_2(1-t)\,dt. \end{aligned}$$

Setting $x^p = t\varphi^p(a)+(1-t)\varphi^p(b)$, we get

$$\begin{aligned} & \frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)} x^{p-1}f(x)g(x)\,dx \\ & \qquad \le M_\varphi(a,b)\int_0^1 h_1(t)h_2(t)\,dt+N_\varphi(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt. \end{aligned}$$

□

Theorem 3. *Let $f \in fsx(p,\varphi_h,I)$, $g \in fsx(p,\varphi_h,I)$ be functions such that $fg \in L_1([a,b])$ and $h \in L_1([a,b])$ and let $\varphi(a), \varphi(b) \in I$ with $\varphi(a) < \varphi(b)$. Then we have*

$$\begin{aligned} & \frac{1}{2h^2\left(\frac{1}{2}\right)} f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\ & \qquad -\frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)} x^{p-1}f(x)g(x)\,dx \\ & \qquad \le N_\varphi(a,b)\int_0^1 h^2(t)\,dt+M_\varphi(a,b)\int_0^1 h(t)h(1-t)\,dt. \end{aligned}$$

Proof. Since

$$\frac{\varphi^p(a)+\varphi^p(b)}{2} = \frac{t\varphi^p(a)+(1-t)\varphi^p(b)}{2}+\frac{(1-t)\varphi^p(a)+t\varphi^p(b)}{2},$$

we have

$$f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)$$

$$
\begin{aligned}
&= f\left(\left[\frac{t\varphi^p(a)+(1-t)\varphi^p(b)}{2}+\frac{(1-t)\varphi^p(a)+t\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)\\
&\quad\cdot g\left(\left[\frac{t\varphi^p(a)+(1-t)\varphi^p(b)}{2}+\frac{(1-t)\varphi^p(a)+t\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)\\
&\le h\left(\frac{1}{2}\right)\left[f\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)+f\left(\left[(1-t)\varphi^p(a)+t\varphi^p(b)\right]^{\frac{1}{p}}\right)\right]\\
&\quad\cdot h\left(\frac{1}{2}\right)\left[g\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)+g\left(\left[(1-t)\varphi^p(a)+t\varphi^p(b)\right]^{\frac{1}{p}}\right)\right]\\
&\le h^2\left(\frac{1}{2}\right)f\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)g\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)\\
&\quad+h^2\left(\frac{1}{2}\right)f\left(\left[(1-t)\varphi^p(a)+t\varphi^p(b)\right]^{\frac{1}{p}}\right)g\left(\left[(1-t)\varphi^p(a)+t\varphi^p(b)\right]^{\frac{1}{p}}\right)\\
&\quad+h^2\left(\frac{1}{2}\right)\left[h(t)f(\varphi(a))+h(1-t)f(\varphi(b))\right]\\
&\qquad\cdot\left[h(1-t)g(\varphi(a))+h(t)f(\varphi(b))\right]\\
&\quad+h^2\left(\frac{1}{2}\right)\left[h(1-t)f(\varphi(a))+h(t)f(\varphi(b))\right]\\
&\qquad\cdot\left[h(t)g(\varphi(a))+h(1-t)f(\varphi(b))\right]\\
&= h^2\left(\frac{1}{2}\right)f\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)g\left(\left[t\varphi^p(a)+(1-t)\varphi^p(b)\right]^{\frac{1}{p}}\right)\\
&\quad+h^2\left(\frac{1}{2}\right)f\left(\left[(1-t)\varphi^p(a)+t\varphi^p(b)\right]^{\frac{1}{p}}\right)g\left(\left[(1-t)\varphi^p(a)+t\varphi^p(b)\right]^{\frac{1}{p}}\right)\\
&\quad+h^2\left(\frac{1}{2}\right)\left[(h(t)h(1-t)+h(1-t)h(t))\right]M_\varphi(a,b)\\
&\quad+h^2\left(\frac{1}{2}\right)\left[\left(h^2(t)+h^2(1-t)\right)\right]N_\varphi(a,b).
\end{aligned}
$$

Integrating above inequality over $[0,1]$, we obtain

$$
\begin{aligned}
&\frac{1}{2h^2\left(\frac{1}{2}\right)}f\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{\varphi^p(a)+\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)\\
&\qquad-\frac{p}{\varphi^p(b)-\varphi^p(a)}\int_{\varphi(a)}^{\varphi(b)}x^{p-1}f(x)g(x)\,dx\\
&\qquad\qquad\le N_\varphi(a,b)\int_0^1 h^2(t)\,dt+M_\varphi(a,b)\int_0^1 h(t)h(1-t)\,dt.
\end{aligned}
$$

This completes the proof. □

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