Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 13–27

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3285

FIRST ORDER SWEEPING PROCESS WITH SUBSMOOTH SETS

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Received 28 March, 2020

Abstract. In this paper, we introduce a perturbed first order non-convex sweeping process for a class of subsmooth moving sets depending on the time and the state. In the first result we study the existence of solution and the compactness of the attainable set, the perturbation considered here is an upper semi-continuous set-valued mapping with nonempty closed convex values unnecessarily bounded. In the second result, we prove the existence of solution for the autonomous problem under assumptions that do not require the convexity of the values of the perturbation and that weaken the assumption on the upper semi-continuity. Then, we deduce a solution of the time optimality problem and we describe the attainable set.

2010 *Mathematics Subject Classification:* 34A60; 49J53

Keywords: swepping process, subsmooth, perturbation, normal cone, almost convex, attainable set, time optimal problem

1. INTRODUCTION

The perturbed state-dependent sweeping process is an evolution differential inclusion governed by the normal cone to a mobile set depending on both time and state variables, of the following form:

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + G(t,u(t)), \text{ a.e } t \in [T_0,T] \\ u(t) \in C(t,u(t)),\ \forall t \in [T_0,T], \quad u(T_0) = u_0, \end{cases} \tag{1.1}$$

where $N_{C(t,u(t))}(u(t))$ is the normal cone to $C(t,u(t))$ at $u(t)$ and G is a set-valued mapping playing the role of a perturbation to the problem, that is an external force applied on the system. This type of problems was initiated by J. J. Moreau (see [18]) for time-dependent sets $C(t)$ and $G \equiv \{0\}$. After, many generalizations have been obtained, see for example [1, 2, 5, 7–10, 19] and the references therein. When the moving set C depends also on the state, one obtain a generalization of the classical sweeping process known as the state-dependent sweeping process. This problem has been studied in [12, 13, 16] when $C(t,u(t))$ are convex or non-convex sets.

In the study of existence of solutions for differential inclusions, the use of convexity assumptions on the perturbation is widely known, it is the property required in order to pass to a weak limit along a sequence, be it a minimizing sequence or a sequence of successive approximations, preserving the properties that are needed. Because of its generality, this approach need not always provide the best results, since it does not take into account possible additional information for example the presence of symmetries in the problem. In [11], a generalization of convexity has been defined, that is the almost convexity of sets, the authors have shown the existence of solution to the upper semi-continuous differential inclusions $\dot{u}(t) \in G(u(t))$, $u(0) = u_0$. This almost convexity condition has been used successfully by [2–4] to study the perturbed first order Moreau's sweeping process, the right-hand side contains a set-valued perturbation with almost convex values.

In this work, we extend the results in [2] in many direction. In the setting of a finite dimensional space, we provide the existence of solution and the compactness of the attainable sets for the problem (1.1), when the moving sets $C(t,u)$ are equi-uniform-subsmooth and the perturbation G is a set-valued mapping with nonempty closed convex values, upper semi-continuous and such that its element of minimum norm satisfies a linear growth condition.

On the other hand, under the almost convexity of the values of the perturbation and weakening the assumption of the upper semi-continuity, we provide the existence of solution for the autonomous problem:

$$\begin{cases} -\dot{u}(t) \in N_{C(u(t))}(u(t)) + G(u(t)), \text{ a.e } t \in [T_0,T] \\ u(t) \in C(u(t)), \ \forall t \in [T_0,T], \quad u(T_0) = u_0 \in C(u_0). \end{cases} \tag{1.2}$$

And, we deduce the existence of solutions to the time optimal control problem

$$\dot{u}(t) \in -N_{C(u(t))}(u(t)) + h(u(t),z(t)), \text{ for } z(t) \in Z(t), \tag{1.3}$$

where the set $G(u(t)) = h(u(t),Z(u(t)))$ is compact and almost convex, then we use the minimal time function to describe the attainable set of this problem.

The paper is organized as follows. In Section 2, we introduce notation and preliminaries needed throughout the paper. Section 3 is devoted to the study of (1.1) when the perturbation is convex. In Section 4, we prove the existence of solutions of (1.2) under the almost convexity and the upper semi-continuity of $co(G)$. Then, we present an application to an optimal time problem.

2. PRELIMINARIES

Let $\mathbb{R}^n$ be a n-dimensional Euclidean space and $I = [T,T_0]$, $T > T_0 \geq 0$ be an interval. We denote by $\mathbb{B}$ the unit closed ball of $\mathbb{R}^n$, $B(x,r)$ the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, $C_{\mathbb{R}^n}(I)$ the Banach space of all continuous mapping from I and $L^1_{\mathbb{R}^n}(I)$ the space of all Lebesgue integrable $\mathbb{R}^n$-valued mappings defined on I. We say that the mapping $u : I \to \mathbb{R}^n$ is absolutely continuous if there is $f \in L^1_{\mathbb{R}^n}(I)$ such that $u(t) = u(T_0) + \int_{T_0}^t f(s)\, ds$, for all $t \in I$.

Let S be a nonempty closed subset of $\mathbb{R}^n$, $\mathbb{I}_S$ is the characteristic function of the set S, that is, $\mathbb{I}_S(x) = 1$ if $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. We denote by $d_S(\cdot)$ or $d(\cdot,S)$ the usual distance function associated with S, i.e., $d_S(x) = \inf_{y\in S}\|y-x\|$ and by $Proj_S(x)$ the projection of x on S defined by $Proj_S(x) = \{y \in S : d_S(x) = \|y-x\|\}$. The convex hull of S is denoted by $co(S)$ and the closed convex hull by $\overline{co}(S)$ is characterized by

$$\overline{co}(S) = \left\{x \in \mathbb{R}^n, \forall x' \in \mathbb{R}^n, \langle x', x\rangle \leq \delta^*(x', S)\right\},$$

where $\delta^*(x', S) = \sup_{y\in S}\langle x', y\rangle$ is the support function of S at $x' \in \mathbb{R}^n$. Let $\tilde{t} \in I$, we denoted by

$$Acc_{u_0}(\tilde{t}) = \{z \in \mathbb{R}^n : z = u(\tilde{t}) \text{ such that } u(\cdot) \in S_{\tilde{t}}(u_0)\},$$

the attainable set of (1.1) at the time $\tilde{t}$, where $S_{\tilde{t}}(u_0)$ is the set of the trajectories of (1.1) on the interval $[T_0, \tilde{t}]$ and by $\mathfrak{T} : Acc_{u_0} \to I$ the minimal time mapping defined by

$$\mathfrak{T}(z) = \inf\{t \in I : z \in Acc_{u_0}(t)\}.$$

For a locally Lipschitzian function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, the Clarke subdifferential $\partial\varphi(x)$ of φ at x is defined by (see [15])

$$\partial\varphi(x) = \{\xi \in \mathbb{R}^n : \varphi^0(x, \nu) \geq \langle \nu, \xi\rangle, \text{ for all } \nu \in \mathbb{R}^n\},$$

where $\varphi^0(x,\nu) = \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{\varphi(y+t\nu)-\varphi(y)}{t}$ is the generalized directional derivative of φ at x in the direction ν. The Clarke normal cone $N_S(x)$ at $x \in S$ is defined from T_S^C by polarity, that is,

$$N_S(x) = \{\xi \in \mathbb{R}^n : \langle \xi, \nu\rangle \leq 0, \text{ for all } \nu \in T_S^C(x)\},$$

where $T_S^C(x)$ is the Clarke tangent cone at $x \in S$ given by

$$T_S^C(x) = \left\{\nu \in \mathbb{R}^n : \ d_S^0(x, \nu) = 0\right\},$$

where $d_S^0(x, \nu)$ is the generalized directional derivative of d_S at x in the direction ν.

A vector $\xi \in \mathbb{R}^n$ is said to be in the Fréchet subdifferential of φ at x (see [15, 17]) denoted by $\partial^F\varphi(x)$, provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \xi, y-x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon\|y-x\|, \quad \text{for all } y \in B(x, \delta).$$

We always have the inclusion $\partial^F\varphi(x) \subset \partial\varphi(x)$, for all $x \in S$. The Fréchet normal cone $N_S^F(x)$ at $x \in S$ is given by $N_S^F(x) = \partial^F\psi_S(x)$, where ψ_S is the indicator function of S, that is, $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$, otherwise. So we have the inclusion $N_S^F(x) \subset N_S(x)$, for all $x \in S$. On the other hand, the Fréchet normal cone is also related ([17]) to the Fréchet subdifferential, since for all $x \in S$

$$\partial^F d_S(x) = N_S^F(x) \cap \mathbb{B}. \tag{2.1}$$

Another important property is that, whenever $y \in Proj_S(x)$, one has

$$x - y \in N_S^F(y) \quad \text{then} \quad x - y \in N_S(y). \tag{2.2}$$

Now, we introduce the definition of equi-uniform subsmoothness for a family of sets. We begin with some basic definitions from subsmoothness while referring the reader to [6].

Definition 1. Let S be a closed subset of $\mathbb{R}^n$. The set S is called subsmooth at $x_0 \in S$, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x_1, x_2 \in B(x_0, \delta) \cap S$ and $\xi_i \in N_S(x_i) \cap \mathbb{B},\ \ i \in \{1,2\}$, one has

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|. \tag{2.3}$$

The set S is called subsmooth if it is subsmooth at every $x_0 \in S$. We say that S is uniformly-subsmooth if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (2.3) holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_S(x_i) \cap \mathbb{B},\ \ i \in \{1,2\}$.

The following subdifferential regularity of the distance function also holds true for subsmooth sets (see [6]).

Proposition 1. *Let S be a closed set of $\mathbb{R}^n$. If S is subsmooth at $x \in S$, then*

$$\partial d_S(x) = \partial^F d_S(x) \quad \text{and} \quad N_S(x) = N_S^F(x). \tag{2.4}$$

Now we introduce the definition of equi-uniformly-subsmoothness.

Definition 2. Let $(S(q))_{q \in Q}$ be a family of closed sets of $\mathbb{R}^n$ with parameter $q \in Q$. This family is called equi-uniformly- subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $q \in Q$, the inequality (2.3) holds for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_{S(q)}(x_i) \cap \mathbb{B},\ \ i \in \{1,2\}$.

For the proof of the next proposition, we refer the reader to [16].

Proposition 2. *Let $\{S(t,x) : (t,x) \in I \times \mathbb{R}^n\}$ be a family of closed and nonempty sets of $\mathbb{R}^n$, which is equi-uniformly-subsmooth and let a real $\eta \geq 0$. Assume that there exist a real constant $L \geq 0$ and a continuous function $v : I \to \mathbb{R}$ such that, for any $x, x', y, y' \in \mathbb{R}^n$ and $s, t \in I$*

$$\left| d_{S(t,x)}(y) - d_{S(s,x')}(y') \right| \leq \left\| y - y' \right\| + |v(t) - v(s)| + L \left\| x - x' \right\|.$$

Then,

(i) *for all $(t,x,y) \in Gph S$, we have $\eta \partial d_{S(t,x)}(y) \subset \eta \mathbb{B}$;*

(ii) *for any sequence $\left(t_n, x_n\right)_n$ in $I \times \mathbb{R}^n$ converging to (t,x), $(y_n)_n$ converging to $y \in S(t,x)$ with $y_n \in S(t_n, x_n)$ and any $\xi \in \mathbb{R}^n$, we have*

$$\limsup_{n \to +\infty} \delta^* \left(\xi, \eta \partial d_{S(t_n,x_n)}(y_n) \right) \leq \delta^* \left(\xi, \eta \partial d_{S(t,x)}(y) \right).$$

Next we give the definition of the almost convex sets.

Definition 3 ([11]). For a vector space X, a set $Q \subset X$ is called almost convex if for every $\xi \in co(Q)$ there exist λ_1 and λ_2, $0 \leq \lambda_1 \leq 1 \leq \lambda_2$, such that $\lambda_1 \xi \in Q$ and $\lambda_2 \xi \in Q$.

Trivially, any convex set is almost convex, since $Q = co(K)$. If K is a convex set not containing the origin, $Q = \partial K$ is almost convex, and if the convex set K contains the origin, one takes $Q = \{0\} \cup \partial K$.

The following lemma is a direct consequence of discrete Gronwall's inequality proved in [14] (by using the inequality $1 + b_k \leq \exp(b_k)$).

Lemma 1. *Let $A > 0$ and let (a_n) and (b_n) be two nonnegative sequences such that*

$$a_n \leq A + \sum_{k=0}^{n-1} b_k a_k, \quad \textit{for all } \ n \in \mathbb{N}.$$

Then,

$$a_n \leq A \exp\left(\sum_{k=0}^{n-1} b_k\right).$$

3. SWEEPING PROCESS WITH CONVEX PERTURBATION

In this section, we prove the existence of solution for (1.1), where the perturbation is an upper semicontinuous set-valued mapping with nonempty closed convex values unnecessarily bounded and without any compactness assumptions.

Theorem 1. *Let $C : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed values such that:*

($\mathcal{H}_1$) *for all $(t,x) \in I \times \mathbb{R}^n$, the sets $C(t,x)$ are equi-uniformly-subsmooth;*

($\mathcal{H}_2$) *there are two constants $L_1 \geq 0$, $L_2 \in [0,1[$ such that, for all $s,t \in I$, and any $x_1, x_2, y \in \mathbb{R}^n$*

$$\left|d_{C(t,x_1)}(y) - d_{C(s,x_2)}(y)\right| \leq L_1|t-s| + L_2\left\|x_1 - x_2\right\|.$$

And let $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed convex values, upper semi-continuous such that:

($\mathcal{H}_3$) *for some real $\alpha \geq 0$, $d\big(0, G(t,x)\big) \leq \alpha\big(1 + \|x\|\big)$, $\forall (t,x) \in I \times \mathbb{R}^n$.*

Then, for every $u_0 \in C\big(T_0, u_0\big)$,

(1) *(1.1) admits an absolutely continuous solution $u : I \to \mathbb{R}^n$.*

(2) *For $\tilde{t} \in I$ fixed the attainable set $Acc_{u_0}(\tilde{t})$ is compact on $\mathbb{R}^n$.*

Proof. (1) For each integer $n \geq 1$, we consider the following partition of I by $I_k^n = [t_k^n, t_{k+1}^n[$, $t_k^n = T_0 + ke_n$, $e_n = \dfrac{T - T_0}{n}$, $k \in \{0, 1, \cdots, n-1\}$ and $I_n^n = \{t_n^n\} = \{T\}$. For all $(t,x) \in I \times \mathbb{R}^n$, we denote by $g(t,x)$ the element of minimal norm of the closed convex set $G(t,x)$, i.e., $g(t,x) = Proj_{G(t,x)}(0)$.

Step 1. We construct the sequence (x_k^n) as follows: set $x_0^n = u_0 \in C(t_0^n, x_0^n)$, as $C(t_1^n, x_0^n)$ is closed, we can choose

$$x_1^n \in Proj_{C(t_1^n, x_0^n)}(x_0^n + e_n g(t_0^n, x_0^n)),$$

according to (2.2), we get $x_1^n \in C\big(t_1^n, x_0^n\big)$, $x_0^n + e_n g(t_0^n, x_0^n) - x_1^n \in N_{C\left(t_1^n, x_0^n\right)}(x_1^n)$, and $\big\|g(t_0^n, x_0^n)\big\| \leq \alpha\big(1 + \|x_0^n\|\big)$. By $(\mathcal{H}_2)$ and the last inequalities, we obtain

$$\begin{aligned}
\big\|x_1^n - x_0^n\big\| &\leq \|x_1^n - \big(x_0^n + e_n g(t_0^n, x_0^n)\big)\| + \big\|e_n g(t_0^n, x_0^n)\big\| \\
&= d_{C\left(t_1^n, x_0^n\right)}\Big(x_0^n + e_n g(t_0^n, x_0^n)\Big) + \big\|e_n g(t_0^n, x_0^n)\big\| \\
&\leq \Big|d_{C\left(t_1^n, x_0^n\right)}(x_0^n) - d_{C\left(t_0^n, x_0^n\right)}(x_0^n)\Big| + 2e_n\big\|g(t_0^n, x_0^n)\big\| \\
&\leq L_1 e_n + 2\alpha e_n(1 + \|x_0^n\|). \qquad (3.1)
\end{aligned}$$

By induction, we define x_{k+1}^n satisfying

$$x_{k+1}^n \in C(t_{k+1}^n, x_k^n); \qquad (3.2)$$

$$x_k^n + e_n g\big(t_k^n, x_k^n\big) - x_{k+1}^n \in N_{C\left(t_{k+1}^n, x_k^n\right)}(x_{k+1}^n). \qquad (3.3)$$

Using $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, we obtain

$$\begin{aligned}
\|x_{k+1}^n - x_k^n\| &\leq \big\|x_{k+1}^n - (x_k^n + e_n g(t_k^n, x_k^n))\big\| + \big\|e_n g(t_k^n, x_k^n)\big\| \\
&= d_{C\left(t_{k+1}^n, x_k^n\right)}(x_k^n + e_n g(t_k^n, x_k^n)) + \big\|e_n g(t_k^n, x_k^n)\big\| \\
&\leq \Big|d_{C\left(t_{k+1}^n, x_k^n\right)}(x_k^n) - d_{C\left(t_k^n, x_{k-1}^n\right)}(x_k^n)\Big| + 2e_n\big\|g(t_k^n, x_k^n)\big\| \\
&\leq L_1 e_n + L_2\|x_k^n - x_{k-1}^n\| + 2\alpha e_n(1 + \|x_k^n\|).
\end{aligned}$$

Then, for $0 \leq k \leq n-1$, we have

$$\|x_{k+1}^n - x_k^n\| \leq (L_1 + 2\alpha)e_n \sum_{j=0}^{k} L_2^j + 2\alpha e_n \sum_{j=0}^{k} L_2^{k-j}\|x_j^n\|,$$

since $L_2 \in [0,1[$, we get

$$\|x_{k+1}^n - x_k^n\| \leq \frac{L_1 + 2\alpha}{1 - L_2} e_n + 2\alpha e_n \sum_{j=0}^{k} L_2^{k-j}\|x_j^n\|. \qquad (3.4)$$

On the other hand, we have

$$\begin{aligned}
\|x_k^n - x_0^n\| &\leq \|x_k^n - x_{k-1}^n\| + \|x_{k-1}^n - x_{k-2}^n\| + \cdots + \|x_1^n - x_0^n\| \\
&\leq \frac{L_1 + 2\alpha}{1 - L_2} e_n + 2\alpha e_n \sum_{j=0}^{k-1} L_2^{k-j}\|x_j^n\| + \frac{L_1 + 2\alpha}{1 - L_2} e_n + 2\alpha e_n \sum_{j=0}^{k-2} L_2^{k-j}\|x_j^n\| \\
&\quad + \cdots + e_n(L_1 + 2\alpha) + 2\alpha e_n\|x_0^n\| \\
&\leq \frac{L_1 + 2\alpha}{1 - L_2} e_n(k-1) + 2\alpha e_n\|x_0^n\| \sum_{j=0}^{k-1} L_2^j + 2\alpha e_n\|x_1^n\| \sum_{j=0}^{k-1} L_2^j
\end{aligned}$$

$$+2\alpha e_n\|x_2^n\|\sum_{j=0}^{k-1}L_2^j+\cdots+2\alpha e_n\|x_{k-1}^n\|\sum_{j=0}^{k-1}L_2^j$$

$$\leq T\frac{L_1+2\alpha}{1-L_2}+\frac{2\alpha T}{1-L_2}\sum_{j=0}^{k-1}\|x_j^n\|.$$

So,

$$\|x_k^n\|\leq\|x_0^n\|+T\frac{L_1+2\alpha}{1-L_2}+\frac{2\alpha T}{1-L_2}\sum_{j=0}^{k-1}\|x_j^n\|.$$

By Lemma 1, for all $k\in\{0,1,\cdots,n-1\}$, we can write

$$\|x_k^n\|\leq\Big(\|x_0^n\|+T\frac{L_1+2\alpha}{1-L_2}\Big)\exp\Big(\frac{2\alpha T}{1-L_2}\Big)=\beta. \tag{3.5}$$

Step 2. Construction of sequence $\big(u_n(\cdot)\big)_{n\geq 0}$.
For every $n\geq 1$ and for any $t\in I_k^n$ with $k\in\{0,1,\cdots,n-1\}$, we define

$$u_n(t)=\frac{t_{k+1}^n-t}{e_n}\,x_k^n+\frac{t-t_k^n}{e_n}\,x_{k+1}^n.$$

Thus, $u_n(t_k^n)=x_k^n$ and on $]t_k^n,t_{k+1}^n[$

$$\dot u_n(t)=\frac{x_{k+1}^n-x_k^n}{e_n}. \tag{3.6}$$

By (3.2) and (3.3) we obtain

$$u_n(t_{k+1}^n)\in C(t_{k+1}^n,u_n(t_k^n)) \tag{3.7}$$

$$\dot u_n(t)\in -N_{C\left(t_{k+1}^n,u_n(t_k^n)\right)}(u_n(t_{k+1}^n))+g(t_k^n,u_n(t_k^n))\ \text{a.e.}\ \ t\in I_k^n, \tag{3.8}$$

and by (3.4) and (3.5) we get

$$\frac{\|x_{k+1}^n-x_k^n\|}{e_n}\leq\frac{L_1+2\alpha}{1-L_2}+2\alpha\sum_{j=0}^{k}L_2^{k-j}\beta\leq\frac{1}{1-L_2}\Big(L_1+2\alpha+2\alpha\beta\Big). \tag{3.9}$$

The relations (3.6) and (3.9) give

$$\|\dot u_n(t)\|\leq\frac{1}{1-L_2}\Big(L_1+2\alpha+2\alpha\beta\Big)=\gamma. \tag{3.10}$$

For each $t\in I$ and $n\geq 1$, let define the function

$$\delta_n(t)=\begin{cases} t_k^n & \text{if}\quad t\in I_k^n,\\ t_{n-1}^n & \text{if}\quad t=T.\end{cases}$$

$$\theta_n(t)=\begin{cases} t_{k+1}^n & \text{if}\quad t\in I_k^n;\\ T & \text{if}\quad t=T,\end{cases}$$

and we have

$$\lim_{n\to+\infty}\big|\delta_n(t)-t\big|=\lim_{n\to+\infty}\big|\theta_n(t)-t\big|=0. \tag{3.11}$$

Furthermore, the definitions of $\delta_n(\cdot)$ and $\theta_n(\cdot)$ combined with (3.5), (3.7) and (3.8) yield

$$u_n\big(\theta_n(t)\big) \in C\big(\theta_n(t), u_n\big(\delta_n(t)\big)\big), \quad \text{for all } t \in I, \tag{3.12}$$

$$\dot{u}_n(t) \in -N_{C\left(\theta_n(t), u_n(\delta_n(t))\right)}\big(u_n(\theta_n(t))\big) + g(\delta_n(t), u_n(\delta_n(t))) \text{ a.e. } t \in I, \tag{3.13}$$

$$\big\|g(\delta_n(t), u_n(\delta_n(t)))\big\| \le \alpha(1+\beta) = \eta \quad \text{for all } t \in I. \tag{3.14}$$

Step 3. Convergence of the sequences.
From (3.5) it follows that $(u_n(\theta_n(t))$ is relatively compact and by (3.10) we have

$$\big\|u_n(\theta_n(t)) - u_n(t)\big\| \le \int_t^{\theta_n(t)} \big\|\dot{u}_n(s)\big\|\, ds \le \gamma\,\big(\theta_n(t) - t\big),$$

then

$$\lim_{n\to+\infty} \big\|u_n(\theta_n(t)) - u_n(t)\big\| = 0. \tag{3.15}$$

So, $(u_n(t))_{n\ge1}$ is relatively compact for all $t \in I$, on the other hand $(u_n(\cdot))_{n\ge1}$ is equicontinuous according to (3.10). Using Ascoli-Arzelà's theorem $\big(u_n\big)_{n\ge1}$ is relatively compact in $C_{\mathbb{R}^n}(I)$, so we can extract a subsequence of $(u_n)_{n\ge1}$ (that we do not relabel) which converges uniformly to some mapping $u \in C_{\mathbb{R}^n}(I)$ and $(\dot{u}_n)_{n\ge1}$ converges weakly in $L^1_{\mathbb{R}^n}(I)$ to mapping $z \in L^1_{\mathbb{R}^n}(I)$ with $\|z(t)\| \le \gamma$ a.e. $t \in I$. Fixing $t \in I$ and taking any $\xi \in \mathbb{R}^n$, the above weak convergence in $L^1_{\mathbb{R}^n}(I)$ yields

$$\lim_{n\to+\infty} \int_{T_0}^{T} \langle \mathbb{I}_I(s)\,\xi, \dot{u}_n(s)\rangle\, ds = \int_{T_0}^{T} \langle \mathbb{I}_I(s)\,\xi, z(s)\rangle\, ds$$

or equivalently

$$\lim_{n\to+\infty} \langle \xi, u_0 + \int_{T_0}^{t} \dot{u}_n(s)\, ds\rangle = \langle \xi, u_0 + \int_{T_0}^{t} z(s)\, ds\rangle.$$

Then, $\lim_{n\to+\infty} \int_{T_0}^{t} \dot{u}_n(s)\, ds = \int_{T_0}^{t} z(s)\, ds$, since $u_n(\cdot)$ is an absolutely continuous mapping, we get

$$\lim_{n\to+\infty} \big(u_n(t) - u_0\big) = \lim_{n\to+\infty} \int_{T_0}^{t} \dot{u}_n(s)\, ds = \int_{T_0}^{t} z(s)\, ds,$$

so $u(\cdot)$ is an absolutely continuous mapping, and $z = \dot{u}$.

Let put $\big(g(\delta_n(\cdot), u_n(\delta_n(\cdot))\big)_n = \big(h_n(\cdot)\big)_n$, so $(h_n)_{n\ge1}$ is a measurable because $G(\cdot,\cdot)$ is upper semi-continuous and $\|h_n(t)\| \le \eta$, then $(h_n(\cdot)) \in L^\infty_{\mathbb{R}^n}(I)$, taking a subsequence if necessary we conclude that $\big(h_n(\cdot)\big)_{n\ge1}$ weakly converges to a mapping $h \in L^\infty_{\mathbb{R}^n}(I)$ with $\|h(t)\| \le \eta$ a.e.

Step 4. We prove in this step that u is a solution of (1.1).
Fix any $t \in I$, by $(\mathcal{H}_2)$ and (3.12), we have

$$d(u_n(t), C(t, u(t))) \le \|u_n(\theta_n(t)) - u_n(t)\| + d(u_n(\theta_n(t)), C(t, u(t)))$$
$$\le \|u_n(\theta_n(t)) - u_n(t)\| + \|d(u_n(\theta_n(t)), C(t, u(t))) - d(u_n(\theta_n(t)), C(\theta_n(t), u(\delta_n(t))))\|$$

$\leq \|u_n(\theta_n(t)) - u_n(t)\| + L_1(\theta_n(t) - t) + L_2\|u_n(\delta_n(t)) - u(t)\|.$

Using (3.11) and (3.15) and by passing to the limit in the last inequality, we get $u(t) \in C(t,u(t))$ for all $t \in I$ thanks to the closedness of $C(t,u(t))$. We have

$$\big\| - \dot{u}_n(t) + h_n(t)\big\| \leq (\gamma + \eta) = M \text{ a.e. } t \in I,$$

that is, $-\dot{u}_n(t) + h_n(t) \in M\mathbb{B}$, then

$$-\dot{u}_n(t) + h_n(t) \in N_{C\left(\theta_n(t), u_n(\delta_n(t))\right)}\left(u_n(\theta_n(t))\right) \cap M\mathbb{B},$$

we get by relation (2.1) and Proposition 1

$$-\dot{u}_n(t) + h_n(t) \in M\partial d_{C\left(\theta_n(t), u_n(\delta_n(t))\right)}\left(u_n(\theta_n(t))\right) \text{ a.e. } t \in I. \tag{3.16}$$

Note that $\big(- \dot{u}_n + h_n, h_n\big)$ weakly converges in $L^1_{\mathbb{R}^n \times \mathbb{R}^n}(I)$ to $\big(- \dot{u} + h, h\big)$ according to Mazur's lemma, there exists a sequence $\big(\xi_n, \zeta_n\big)_{n \geq 1}$ which strongly converges in $L^1_{\mathbb{R}^n \times \mathbb{R}^n}(I)$ to $\big(- \dot{u} + h, h\big)$ such that

$$\xi_n \in co\{-\dot{u}_q + h_q,\ q \geq n\} \quad \text{and} \quad \zeta_n \in co\{h_q,\ q \geq n\}. \tag{3.17}$$

Extracting a subsequence if necessary, we may assume that $\big(\xi_n(\cdot), \zeta_n(\cdot)\big)_{n \geq 1}$ converges almost every to $\big(- \dot{u}(\cdot) + h(\cdot), h(\cdot)\big)$. Then, there is, a Lebesgue negligible set $N \subset I$ such that for every $t \in I \backslash N$

$$-\dot{u}(t) + h(t) \in \bigcap_{n \geq 0} \overline{\{\xi_q(t),\ q \geq n\}} \subset \bigcap_{n \geq 0} \overline{co}\{-\dot{u}_q(t) + h_q(t),\ q \geq n\}, \tag{3.18}$$

and

$$h(t) \in \bigcap_{n \geq 0} \overline{\{\zeta_q(t),\ q \geq n\}} \subset \bigcap_{n \geq 0} \overline{co}\{h_q(t),\ q \geq n\}. \tag{3.19}$$

Fix any $t \in I \backslash N$ and $\mu \in \mathbb{R}^n$. Then, relations (3.16) and (3.18) give

$$\langle \mu, -\dot{u}(t) + h(t)\rangle \leq \limsup_{n \to +\infty} \delta^*\Big(\mu, M\partial d_{C\left(\theta_n(t), u_n(\delta_n(t))\right)}\left(u_n(\theta_n(t))\right)\Big).$$

By using Proposition 2, we obtain

$$\langle \mu, -\dot{u}(t) + h(t)\rangle \leq \delta^*\Big(\mu, M\partial d_{C(t,u(t))}\big(u(t)\big)\Big).$$

Since $M\partial d_{C\left(t,u(t)\right)}\big(u(t)\big)$ is closed and convex, we conclude that

$$-\dot{u}(t) + h(t) \in M\partial d_{C(t,u(t))}\big(u(t)\big) \subset N_{C\left(t,u(t)\right)}\big(u(t)\big) \text{ a.e. } t \in I \backslash N. \tag{3.20}$$

Further, by relation (3.19) we have

$$\langle \mu, h(t)\rangle \leq \limsup_{n \to +\infty} \delta^*\Big(\mu, G(\delta_n(t), u_n(\delta_n(t))\Big).$$

Since G is upper semi-continuous with closed values, we obtain $h(t) \in G\big(t, u(t)\big)$. Consequently, $\dot{u}(t) \in -N_{C(t,u(t))}\big(u(t)\big) + G\big(t, u(t)\big)$ a.e. $t \in I$.

(2) To prove that the attainable set $Acc_{u_0}(\tilde{t})$ is compact it suffices to show that $S_{\tilde{t}}(u_0)$ is compact for $\tilde{t} \in I$. By the part (1), we get $S_{\tilde{t}}(u_0) \neq \emptyset$. Let $(u_n)_n \subset S_{\tilde{t}}(u_0)$, then, for each $n \in \mathbb{N}$, u_n is an absolutely continuous solution of (1.1) with

$$\|\dot{u}_n(t)\| \leq \gamma \ a.e. \ t \in [T_0, \tilde{t}]. \tag{3.21}$$

and

$$\|u_n(t)\| \leq \|u_0\| + \int_{T_0}^{t} \|\dot{u}_n(s)\| \, ds \leq \|u_0\| + \gamma(t - T_0). \tag{3.22}$$

Then, $(u_n(t))_n$ is relatively compact in $\mathbb{R}^n$, in addition, it is equi-continuous according to (3.21). By Arzelà-Ascoli theorem $(u_n)_n$ is relatively compact in $C_{\mathbb{R}^n}([T_0, \tilde{t}])$, so, we can extract a subsequence of $(u_n)_n$ (that we do not relabel) which converges uniformly to some mapping u on $[T_0, \tilde{t}]$ and $(\dot{u}_n)_n$ converges in $L^1_{\mathbb{R}^n}([T_0, \tilde{t}])$ to $\dot{u}(\cdot)$ with $\|\dot{u}(t)\| \leq \gamma$ a.e. $t \in [T_0, \tilde{t}]$ and $u(t) = u_0 + \int_{T_0}^{t} \dot{u}(s) ds$. For the rest of the demonstration we can follow the proof of part (1) to get

$$\dot{u}(t) \in -N_{C(t,u(t))}\big(u(t)\big) + G\big(t, u(t)\big) \ \text{ a.e. } t \in [T_0, \tilde{t}].$$

□

4. Sweeping process with almost convex perturbation

In the following theorem we prove the existence of solution of (1.2), when the perturbation G takes almost convex values and with weaker assumption on upper semi-continuity. In the previous results, one takes the perturbation upper semi-continuous, an analysis of the proof, shows that we need only the upper semi-continuity of the $co(G)$.

Theorem 2. *Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed values satisfying:*

($\mathcal{A}_1$) *for all $x \in \mathbb{R}^n$ the sets $C(x)$ are equi-uniformly-subsmooth;*
($\mathcal{A}_2$) *there is a constant $L_2 \in [0, 1[$ such that, for all $x_1, x_2, y \in \mathbb{R}^n$*

$$\big|d\big(y, C(x_1)\big) - d\big(y, C(x_2)\big)\big| \leq L_2 \|x_1 - x_2\|.$$

And let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a measurable set-valued mapping with compact and almost convex values such that

($\mathcal{A}_3$) *the set-valued mapping $co(G(\cdot))$ is upper semi-continuous on $\mathbb{R}^n$;*
($\mathcal{A}_4$) *for some real $\alpha \geq 0$, $d\big(0, co\big(G(x)\big)\big) \leq \alpha(1 + \|x\|)$ for all $x \in \mathbb{R}^n$.*

Then, for every $u_0 \in C\big(u_0\big)$,

(1) *(1.2) admits an absolutely continuous solution;*
(2) *for $\tilde{t} \in I$ fixed, the attainable set of (1.2) at $\tilde{t}$, $Acc_{u_0}(\tilde{t})$, coincides with $Acc^{co}_{u_0}(\tilde{t})$ the attainable set at $\tilde{t}$ of the convexified problem.*

Proof. (1) **Step 1.** We start by proving the existence of two integrable functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ defined on I, such that $0 \le \lambda_1(t) \le 1 \le \lambda_2(t)$ and for $t \in I$

$$\lambda_1(t)m(u(t)) \in G(u(t)),\ \ \lambda_2(t)m(u(t)) \in G(u(t)), \tag{4.1}$$

where $m(u(t)) = Proj_{co(G(u(t)))}(0)$. Since for every $t \in I$, $G(u(t))$ is almost convex, there exists two nonempty sets $\Lambda_1(t)$ and $\Lambda_2(t)$ such that

$$\Lambda_1(t) = \big\{\lambda_1 \in [0,1] \ :\ \lambda_1 m(u(t)) \in G(u(t))\big\},$$

$$\Lambda_2(t) = \big\{\lambda_2 \in [1,+\infty[\ :\ \lambda_2 m(u(t)) \in G(u(t))\big\}.$$

Consider $Gph(\Lambda_1)$ the graph of Λ_1 defined by

$$\begin{aligned} Gph(\Lambda_1) &= \{(t,\lambda_1) \in I \times [0,1] :\ \lambda_1 m(u(t)) \in G(u(t))\} \\ &= \{(t,\lambda_1) \in I \times [0,1] :\ d\big(\lambda_1 m(u(t)), G(u(t))\big) = 0\} \\ &= \varphi^{-1}\big(\{0\}\big) \cap \big(I \times [0,1]\big), \end{aligned}$$

where $\varphi : (t,\lambda_1) \to d\big(\lambda_1 m(u(t)), G(u(t))\big)$ is measurable. Then $Gph(\Lambda_1)$ is measurable. So, there exists an integrable selection $\lambda_1(\cdot)$ of $\Lambda_1(\cdot)$ satisfying $0 \le \lambda_1(t) \le 1$ and $\lambda_1(t)m(u(t)) \in G(u(t))$ for $t \in I$. The existence of an integrable selection $\lambda_2(\cdot)$ of $\Lambda_2(\cdot)$, satisfying $\lambda_2(t) \ge 1$ and $\lambda_2(t)m(u(t)) \in G(u(t))$, can be proved using the same reasoning as above with the fact that $G(u(t))$ is bounded.

Step 2. (a) Let $[a,b] \subset I$ be an interval and assume that there exist two integrable functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ define on $[a,b]$, such that $0 \le \lambda_1(\tau) \le 1 \le \lambda_2(\tau)$, for all $\tau \in I$. Using the same procedure as in the proof of Theorem 4.1. in [2], we conclude the existence of two measurable subsets of $[a,b]$, having characteristic functions $\mathbb{I}_1$ and $\mathbb{I}_2$ such that $\mathbb{I}_1(\cdot) + \mathbb{I}_2(\cdot) = \mathbb{I}_{[a,b]}(\cdot)$ and an absolutely continuous function $s : [a,b] \to [a,b]$ with $s(b) - s(a) = b - a$, such that

$$\dot{s}(\tau) = \frac{1}{\lambda_1(\tau)}\,\mathbb{I}_1(\tau) + \frac{1}{\lambda_2(\tau)}\,\mathbb{I}_2(\tau).$$

(b) Using Theorem 1, there exists an absolutely continuous solution $x : I \to \mathbb{R}^n$ of the convexified problem

$$\begin{cases} \dot{u}(t) \in -N_{C(u(t))}(u(t)) + co\big(G(u(t))\big), \ \text{a.e. } t \in I; \\ u(t) \in C(u(t)),\ \forall t \in I;\ u(T_0) = u_0 \in C(u_0). \end{cases}$$

Consider the set $\Omega = \{\tau \in I : 0 \in co\big(G(x(\tau))\big)\}$. Since $co(G(\cdot))$ is an upper semi-continuous set-valued mapping with compact values and $x(\cdot)$ is continuous, we obtain the closedness of Ω.

If Ω is empty, in this case $\lambda_1(\tau) > 0$ on I, so, we can apply part (a) to interval I. Set $s(\tau) = T_0 + \int_{T_0}^{\tau} \dot{s}(\omega)d\omega$, s is increasing and we have $(s(T_0), s(T)) = (T_0, T)$, so s defined from I into itself. Let $\gamma : I \to I$ be its inverse, then $(\gamma(T_0), \gamma(T)) =$

(T_0,T), $\dfrac{d}{d\tau}\, s(\gamma(\tau)) = \dot{s}(\gamma(\tau))\,\dot{\gamma}(\tau) = 1$, and

$$\dot{\gamma}(\tau) = \frac{1}{\dot{s}(\gamma(\tau))} = \lambda_1\big(\gamma(\tau)\big)\mathbb{I}_1\big(\gamma(\tau)\big) + \lambda_2\big(\gamma(\tau)\big)\mathbb{I}_2\big(\gamma(\tau)\big).$$

Consider $\tilde{x} : I \to \mathbb{R}^n$ the mapping defined by $\tilde{x}(\tau) = x(\gamma(\tau))$, then we get

$$\frac{d}{d\tau}\tilde{x}(\tau) = \dot{x}\big(\gamma(\tau)\big)\big(\lambda_1\big(\gamma(\tau)\big)\mathbb{I}_1\big(\gamma(\tau)\big) + \lambda_2\big(\gamma(\tau)\big)\mathbb{I}_2\big(\gamma(\tau)\big)\big),$$

using the properties of the normal cone and (4.1), we obtain

$$\begin{aligned}\frac{d}{d\tau}\tilde{x}(\tau) &\in (-N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + m(x(\gamma(\tau))))(\lambda_1\big(\gamma(\tau)\big)\mathbb{I}_1\big(\gamma(\tau)\big) + \lambda_2(\gamma(\tau))\mathbb{I}_2(\gamma(\tau)))\\ &\in -N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + G(x(\gamma(\tau))) = -N_{C(\tilde{x}(\tau))}(\tilde{x}(\tau)) + G(\tilde{x}(\tau)).\end{aligned}$$

If Ω is nonempty, let $l = \sup\{\tau,\ \tau \in \Omega\} \in \Omega$. The complement of Ω is open relatively to I, it consists of at most countably many overlapping open intervals $]a_i,b_i[$, with the possible exception of the form $[a_{i_i},b_{i_i}[$ with $a_{i_i} = T_0$ and one of the form $]a_{i_f},b_{i_f}]$ with $a_{i_f} = l$. For each i, we can apply the part (a) to the $]a_i,b_i[$, so, there exist two measurable subsets of $]a_i,b_i[$ with characteristic functions $\mathbb{I}_1^i(\cdot)$ and $\mathbb{I}_2^i(\cdot)$ such that $\mathbb{I}_1^i(\cdot)+\mathbb{I}_2^i(\cdot) = \mathbb{I}_{]a_i,b_i[}(\cdot)$. Setting $\dot{s}(\tau) = \frac{1}{\lambda_1(\tau)}\mathbb{I}_1^i(\tau) + \frac{1}{\lambda_2(\tau)}\mathbb{I}_2^i(\tau)$, we obtain $\int_{a_i}^{b_i}\dot{s}(\omega)d\omega = b_i - a_i$. For all $\tau \in [T_0,l]$, set

$$\dot{s}(\tau) = \frac{1}{\lambda_2(\tau)}\,\mathbb{I}_\Omega(\tau) + \sum_i\big(\frac{1}{\lambda_1(\tau)}\,\mathbb{I}_1^i(\tau) + \frac{1}{\lambda_2(\tau)}\,\mathbb{I}_2^i(\tau)\big),$$

where the sum is over all intervals of the complementary of Ω contained in $[T_0,l]$, in addition to that $\lambda_2(\tau) \geq 1$ and $\displaystyle\int_{T_0}^{l}\dot{s}(\omega)d\omega = p \leq l - T_0$. Setting $s(t) = T_0 + \displaystyle\int_{T_0}^{\tau}\dot{s}(\omega)\,d\omega$, then $s(\cdot)$ is an invertible map from $[T_0,l]$ to $[T_0,p]$. Let define $\gamma : [T_0,p] \to [T_0,l]$ to be the inverse function of $s(\cdot)$. Extend $\gamma(\cdot)$ as an absolutely continuous map $\tilde{\gamma}(\cdot)$ on $[T_0,l]$, setting $\dot{\tilde{\gamma}}(\tau) = 0$ for $\tau \in [p,l]$. Let us show that the mapping $\tilde{x}(\tau) = x\big(\tilde{\gamma}(\tau)\big)$ is a solution of the problem (1.2) for all $\tau \in [T_0,l]$.

On $[T_0,p]$, we have $\tilde{\gamma}(\tau) = \gamma(\tau)$, γ is invertible and

$$\dot{\gamma}(\tau) = \lambda_2\big(\gamma(\tau)\big)\mathbb{I}_\Omega\big(\gamma(\tau)\big) + \sum_i\big(\lambda_1\big(\gamma(\tau)\big)\mathbb{I}_1^i\big(\gamma(\tau)\big) + \lambda_2\big(\gamma(\tau)\big)\mathbb{I}_2^i\big(\gamma(\tau)\big)\big),$$

as $\dfrac{d}{d\tau}\tilde{x}(\tau) = \dot{x}\big(\gamma(\tau)\big)\dot{\gamma}(\tau)$, we get

$$\begin{aligned}\frac{d}{d\tau}\tilde{x}(\tau) &\in \dot{\gamma}(\tau)\big(-N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + m(x(\gamma(\tau)))\big)\\ &\in -N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + G(x(\gamma(\tau))) = -N_{C(\tilde{x}(\tau))}(\tilde{x}(\tau)) + G(\tilde{x}(\tau)).\end{aligned}$$

On $]p,l]$, we have $\gamma(p) = l$ and $\dot{\tilde{\gamma}}(\tau) = 0$, then we get $\tilde{\gamma}(\tau) = \tilde{\gamma}(p) = \gamma(p)$, we obtain $\tilde{x}(\tau) = x(\tilde{\gamma}(\tau)) = x(\gamma(p)) = l$, then $\tilde{x}$ is constant on $]p,l]$, and we have $\frac{d}{d\tau}\tilde{x}(\tau) = 0 \in$

$co\big(G(\tilde{x}(\tau))\big)$. Using (4.1), we conclude that for $\tau \in]p,l]$

$$\frac{d}{d\tau}\tilde{x}(\tau) = 0 \in -N_{C(\tilde{x}(\tau))}(\tilde{x}(\tau)) + G(\tilde{x}(\tau)).$$

On $[l,T]$, the set Ω is empty, and $\lambda_1(\tau) > 0$, then we can repeat the arguments of part (a). We conclude, that $\tilde{x}$ is an absolutely continuous solution of the problem (1.2).

(2) For the proof of this part, we refer the reader to Theorem 3 in [2].

This completes the proof of the theorem. □

5. Application to an optimal time problem

Now consider the control system (1.3) under the almost convexity assumption.

Theorem 3. *Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed values satisfies the hypothesis $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ in Theorem 2, $Z : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with compact valued upper semi-continuous on $\mathbb{R}^n$ such that $0 \in Z(x)$ for all $x \in R^n$. And let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping satisfying the following assumption*

- $(\mathcal{H}_1^h)$ *there is a nonnegative constant α, such that $\|h(x,y)\| \le \alpha(1+\|x\|)$, for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$;*
- $(\mathcal{H}_2^h)$ *for all $x \in \mathbb{R}^n$, $h(x,0) = 0$;*
- $(\mathcal{H}_3^h)$ *Let G be a measurable set-valued mapping with compact and almost convex values such that $G(x) = \{h\big(x,z\big)\}_{z\in Z(x)}$.*

Let u_0, u_1 be given in $\mathbb{R}^n$, and assume that fore some $t \in I$, $u_1 \in Acc_{u_0}(t)$ the attainable set of the problem (1.3). Then,

1. *the problem of reaching u_1 from u_0 in a minimum time admits a solution;*
2. *for all $t \in I$*

$$Acc_{u_0}(t) = \{u \in \mathbb{R}^n : \mathfrak{T}(u) \le t\}.$$

Proof. (1) By the assumptions on h and Z, we conclude that $co(G(\cdot))$ is an upper semicontinuous set-valued mapping values and

$$d\big(0, co\big(G(x)\big)\big) \le \alpha(1+\|x\|), \quad \text{for all } x \in \mathbb{R}^n.$$

According to Theorem 2 the problem (1.3) admits a solution.

Let $\bar{t} = \inf\{\tau \in [T_0,t] : u_1 \in Acc_{u_0}(t)\}$, so by the lower bound property, there exists a decreasing sequence $(\bar{t}_n)$ in $[T_0,t]$ converging to $\bar{t}$ and for each n, $u_n(\cdot)$ is solution of the problem

$$\begin{cases} \dot{u}(t) \in -N_{C(u(t))}(u(t)) + G(u(t)) \quad \text{a.e. } t \in [T_0,\bar{t}_n]; \\ u(t) \in C(u(t)), \quad \forall t \in [T_0,\bar{t}_n];\ u(T_0) = u_0 \in C(u_0), \end{cases}$$

such that $u_1 = u_n(\bar{t}_n)$. We define the sequence $(y_n(\cdot))$ by $y_n(s) = u_n(s)$ for all $s \in [T_0,\bar{t}]$. So,

$$\big(y_n(s)\big) \subset Acc_{u_0}(s) = Acc_{u_0}^{co}(s).$$

By the compactness of $Acc^{co}_{u_0}(s)$, we can extract a subsequence if necessary and conclude that $\big(y_n(s)\big)$ converges to $u(s) \in Acc^{co}_{u_0}(\bar{t})$ and $y(\bar{t}) = u_1 \in Acc^{co}_{u_0}(\bar{t}) = Acc_{u_0}(\bar{t})$. Consequently, u is the solution of the problem (1.3) that reaches u_1 in the minimum time, and $\bar{t}$ is the value of the minimum time.

(2) We have $u_1 \in Acc_{u_0}(t)$, this means that there exists $u(\cdot)$ solution of the problem (1.3) such that $u_1 = u(t)$. We define the mapping

$$\tilde{u}(s) = \begin{cases} u(s) & \text{for } s \in [T_0, t] \\ u_1 & \text{for } s \in [t, T]. \end{cases}$$

For $s \in [T_0, t]$, we can write

$$\dot{\tilde{u}}(s) \in -N_{C\left(\tilde{u}(s)\right)}(\tilde{u}(s)) + G\big(\tilde{u}(s)\big) \text{ a.e.} \tag{5.1}$$

For $s \in [t, T]$, $\dot{\tilde{u}}(s) = 0$, by hypotheses $(\mathcal{H}^h_2)$ and since $0 \in N_{C(\tilde{u}(s))}(\tilde{u}(s))$ we get

$$\dot{\tilde{u}}(s) = 0 \in -N_{C(\tilde{u}(s))}(\tilde{u}(s)) + G(\tilde{u}(s)) \text{ a.e.} \tag{5.2}$$

By relation (5.1) and (5.2), we conclude that $\tilde{u}(\cdot)$ is a solution of (1.3) for all $t \in I$. On the other hand, for all $t < s$, $\tilde{u}(s) = u_1 \in Acc_{u_0}(s)$. Then

$$Acc_{u_0}(t) \subseteq Acc_{u_0}(s). \tag{5.3}$$

Now let $z \in \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}$, according to relation (5.3), $Acc_{u_0}\big(\mathfrak{T}(u)\big) \subseteq Acc_{u_0}(t)$, then $z \in Acc_{u_0}(t)$. Hence

$$\{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\} \subseteq Acc_{u_0}(t). \tag{5.4}$$

In addition, using the definition of the attainable set we get

$$Acc_{u_0}(t) \subseteq \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}. \tag{5.5}$$

By (5.4) and (5.5) we conclude that

$$Acc_{u_0}(t) = \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}.$$

□

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