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SINGULAR DEGENERATE NORMAL OPERATORS FOR FIRST-ORDER

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Abstract. In this paper, all normal extensions of the minimal operator associated with the singular degenerate linear differential expressions for first order in the Hilbert space of vector-functions are investigated in terms of boundary values. As well, it analyses the spectrum parts of any normal extension.

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1. INTRODUCTION

A densely defined closed operator $T : D(T) \subset H \to H$ in a Hilbert space H is said formally normal if $D(T) \subset D(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T)$. This definition is equal to $(Tx,Ty) = (T^*x,T^*y)$ for all $x,y \in D(T) \subset D(T^*)$. A formally normal operator T in H is a normal operator such that $D(T) = D(T^*)$. An unbounded normal operator was given the first satisfactory description by Neuman [25]. The fundamental results in the normal extension of unbounded formally normal operators in a Hilbert space are attributable to Kilpi [14–16] and Davis [7]. Besides, Biriuk and Coddington [4], Coddington[6], Stochel and Szafraniec [20–22] developed it as a general theory, and advanced it. Some results of this idea can also be found in [10–13, 27].

Differential operator analysis is partly inspired by problems in physics, geometry, applied mathematics, quantum mechanics, quantum field theory, and so on [17, 24, 26]. In addition, several authors [1, 2, 9, 18, 19, 23] have studied the theory of the degenerate Cauchy problem in Banach spaces. Carrol and Showalter [5] performed the first research in this area, which has many important theoretical results and applications.

In this study, we give the boundary conditions for any normal extension of first order singular operator with a degeneration. We consider singularities to occur only at the end points of the domain of the Hilbert valued functions in $L^2(H_1,(-\infty,a)) \oplus$

$L^2(H_2,(b,+\infty))$. The difference of this study from [27] is that the differential expression is degenerated.

Throughout the article, we use the notations $Range(T)$, $Ker(T)$, $\sigma_c(T)$, $\sigma_p(T)$ for the range of T, the kernel of T, the continuous spectrum of T and the point spectrum of T, respectively.

2. Normal extensions of the minimal operator

Let H_1 and H_2 be separable complex Hilbert spaces, $A_k : H_k \to H_k$ be linear bounded positive selfadjoint operators with closed range and $KerA_k \neq \{0\}$ and $B_k : H_k \to H_k$ be linear bounded selfadjoint operators for $k = 1,2$. Consider the Hilbert space of vector-functions

$$\mathcal{H} := L^2(H_1,(-\infty,a)) \oplus L^2(H_2,(b,+\infty)),\ a<b.$$

with the inner product given by for all $u(t) = (u_1(t),u_2(t)), v(t) = (v_1(t),v_2(t)) \in \mathcal{H}$

$$(u(t),v(t))_{\mathcal{H}} := \int_{-\infty}^{a} (u_1(t),v_1(t))_{H_1} dt + \int_{b}^{+\infty} (u_2(t),v_2(t))_{H_2} dt$$

where $u_1(t),v_1(t) \in L^2(H_1,(-\infty,a))$ and $u_2(t),v_2(t) \in L^2(H_2,(b,+\infty))$. We deal with the linear degenerate differential expression for $u(t) = (u_1(t),u_2(t)) \in \mathcal{H}$

$$l(u(t)) = \begin{cases} (A_1u_1(t))' + B_1u_1(t), & t<a \\ (A_2u_2(t))' + B_2u_2(t), & t>b \end{cases} \tag{2.1}$$

where $\dim Range(A_1) = \dim Range(A_2) > 0$ and $(-1)^k B_k \geqslant 0$, for $k=1,2$. Formally adjoint of this differential expression in the space $\mathcal{H}$ is in the form

$$l^+(v(t)) = \begin{cases} -(A_1v_1(t))' + B_1v_1(t), & t<a \\ -(A_2v_2(t))' + B_2v_2(t), & t>b \end{cases} \tag{2.2}$$

where $v(t) = (v_1(t),v_2(t)) \in \mathcal{H}$.

Now, consider the operator L_0' on the dense linear manifold in $\mathcal{H}$

$$D_0' := \{u(t) \in \mathcal{H} : u(t) = \sum_{i=1}^{n} \varphi_i(t)f_i,\ \varphi_i(t)f_i = (\varphi_{i1}(t)f_{i1}, \varphi_{i2}(t)f_{i2}),$$
$$\varphi_{i1}(t) \in C_0^\infty(-\infty,a),\ \varphi_{i2}(t) \in C_0^\infty(b,\infty),\ f_{ik} \in H_k,\ k=1,2,\ n \in \mathbb{N}\}$$

and $L_0'u(t) := l(u(t))$. L_0' has closure because the domain of $L_0'^*$ contains D_0'. We say that the closure operator of L_0' in $\mathcal{H}$ is called the minimal operator associated with the linear differential expression (2.1) and denote by L_0.

Additionally, the minimal operator L_0^+ in $\mathcal{H}$ generated by the linear differential expression (2.2) can be given. The adjoint operator of L_0^+ (L_0) in $\mathcal{H}$ is called the maximal operator generated by (2.1) ((2.2)) and is denoted by L (L^+) [3,8]. We note that if T is a closable operator on a Hilbert space H and $S : H \to H$ is a bounded operator, then $\overline{T+S} = \overline{T} + \overline{S}$ and $D\left(\overline{T+S}\right) = D\left(\overline{T}\right)$ where $\overline{T}$ is the smallest closed extension of T. In this fact and [27] it can be easily seen that $L_0 \subset L, L_0^+ \subset L^+$,

$$D(L_0) = \{u = (u_1, u_2) \in \mathcal{H} : ((A_1u_1)', (A_2u_2)') \in \mathcal{H}, u_1(a) \in KerA_1,\ u_2(b) \in KerA_2\}$$

and $D(L) = \{u = (u_1, u_2) \in \mathcal{H} : ((A_1u_1)', (A_2u_2)') \in \mathcal{H}\}$.

Also, we denote $H_{k0} := KerA_k$, $H_{k1} := RangeA_k$, $A_{k1} := A|_{H_{k1}} : H_{k1} \to H_{k1}$ and $B_{kj} := B|_{H_{kj}} : H_{kj} \to H_{kj}$, for $k = 1,2$, *and* $j = 0,1$.

Theorem 1. *The following statements are equivalent:*

(i) *L_0 is a formally normal operator on $\mathcal{H}$.*
(ii) $A_kB_k = B_kA_k, k = 1,2$.

Proof. $(i) \to (ii)$ Let L_0 be a formally normal operator on $\mathcal{H}$, then for each $u(t) = (u_1(t), u_2(t)), v(t) = (v_1(t), v_2(t)) \in D(L_0) \subset D(L_0^*)$ the following equality holds

$$\begin{aligned} &(L_0u(t), L_0v(t))_{\mathcal{H}} - (L_0^*u(t), L_0^*v(t))_{\mathcal{H}} \\ &= 2\Big[\big((A_1u_1(t))', B_1v_1(t)\big)_{L^2(H_1,(-\infty,a))} + \big(B_1u_1(t), (A_1v_1(t))'\big)_{L^2(H_1,(-\infty,a))} \\ &\quad + \big((A_2u_2(t))', B_2v_2(t)\big)_{L^2(H_2,(b,+\infty))} + \big(B_2u_2(t), (A_2v_2(t))'\big)_{L^2(H_2,(b,+\infty))}\Big] = 0. \end{aligned}$$

Also, if it is used before the inequation for $e^{it}u(t), e^{it}v(t) \in D(L_0)$, then

$$((B_1A_1 - A_1B_1)u_1(t), v_1(t))_{L^2(H_1,(-\infty,a))} + ((B_2A_2 - A_2B_2)u_2(t), v_2(t))_{L^2(H_2,(b,+\infty))} = 0$$

Since $D(L_0)$ is dense in $\mathcal{H}$, we have

$$A_kB_k = B_kA_k \ \ \text{for}\, \mathrm{k} = 1,2.$$

$(ii) \to (i)$ In this case, since B_k are bounded linear operator

$$B_k(A_ku_k(t))' = (B_kA_ku_k(t))'$$

is true for $k = 1,2$. Therefore,from the first equation and this result it is can be obtained that L_0 is a formally normal operator. □

Theorem 2. *Let $A_kB_k = B_kA_k$, $k = 1,2$ be hold. L_n is any normal extension of the minimal operator L_0 iff the domain of L_n is equal to a restriction of $D(L)$ satisfied the following boundary condition*

$$\begin{cases} A_2^{1/2}u(b) = WA_1^{1/2}u(a), \\ A_1^{1/2}u(a) \in Ker(-B_1)^{1/2}, A_2^{1/2}u(b) \in Ker(B_2)^{1/2} \end{cases}$$

where $W : H_{11} \to H_{21}$ is a unitary operator. Moreover, the unitary operator W is defined uniquely by the extension L_n, i.e. $L_n = L_W$.

Proof. Let L_n be a normal extension of the minimal operator L_0. In this case, we have

$$(L_n u(t), u(t))_{\mathcal{H}} - (u(t), L_n^* u(t))_{\mathcal{H}} = \|A_1^{1/2} u(a)\|_{H_1}^2 - \|A_2^{1/2} u(b)\|_{H_2}^2 = 0.$$

for all $u \in D(L_n) = D(L_n^*)$. Therefore, there is an isometric operator $WA_1^{1/2}u(a) = A_2^{1/2}u(b)$, $u \in D(L_n)$. Moreover,

$$H_a = \{u(a) \in H_1 : u(t) \in D(L_n)\},$$

$$H_b = \{u(b) \in H_2 : u(t) \in D(L_n)\}$$

are subspaces of H_1 and H_2, respectively. We claim that $H_a = H_1$ and $H_b = H_2$. Otherwise, there exist a nonzero element $f_1 \in H_a^{\perp}$ or $f_2 \in H_b^{\perp}$ and it can be defined two functions such that $u_1(t) = \left(e^t A_{11}^{-1} f_1, 0\right)$ and $u_2(t) = \left(0, e^{-t} A_{21}^{-1} f_2\right)$ in $\mathcal{H}$. Moreover, for all $u \in D(L_n)$

$$(L_n u(t), u_k(t))_{\mathcal{H}} - (u(t), L_n^* u_k(t))_{\mathcal{H}} = (A_1 u(a), u_k(a))_{H_1} - (A_2 u(b), u_k(b))_{H_2} = 0$$

is true. This result implies $u_k \in D(L_n^*)$, but $u_k \notin D(L_n)$, $k = 1, 2$. Because L_n is a normal operator, it is a contradiction. Therefore, $W : H_{12} \to H_{21}$ is a unitary operator and determined uniquely by the extension of L_n.

On the other hand, L_n is a normal extension operator of L_0, for every $u(t) \in D(L_n)$ the following inequalities hold

$$\begin{aligned}
&(L_n u(t), L_n u(t))_{\mathcal{H}} - (L_n^* u(t), L_n^* u(t))_{\mathcal{H}} \\
&= 2\left[\left(B_1 A_1^{1/2} u(a), A_1^{1/2} u(a)\right)_{H_1} - \left(B_2 A_2^{1/2} u(b), A_2^{1/2} u(b)\right)_{H_2}\right] \\
&= -2\|(-B_1 A_1)^{1/2} u(a)\|_{H_1}^2 - 2\|(B_2 A_2)^{1/2} u(b)\|_{H_2}^2 = 0.
\end{aligned}$$

Hence, $u(a) \in Ker(-B_1)^{1/2}$ and $u(b) \in Ker(B_2)^{1/2}$.

Conversely, suppose that $W : H_{11} \to H_{21}$ is a unitary operator and the boundary conditions

$$A_2^{1/2} u(b) = W A_1^{1/2} u(a), u(a) \in Ker(-B_1)^{1/2}, u(b) \in Ker(B_2)^{1/2}$$

are satisfied. In this case, the adjoint operator $L_W{}^*$ is generated by the differential-operator expression (2.2) and $D(L_W) = D(L_W{}^*)$. Also, the other condition of normal extensions in $\mathcal{H}$ can be easily obtained. □

Corollary 1. *If any of B_1 or B_2 is a one to one operator, then there isn't any normal extension of L_0 in $\mathcal{H}$.*

Corollary 2. *If there exists a normal extension of L_0 in $\mathcal{H}$, then*

$$\dim Ker(-B_{11})^{1/2} = \dim Ker(B_{21})^{1/2} > 0.$$

3. THE SPECTRUM PARTS OF NORMAL EXTENSIONS

In this section, the spectrum parts of any normal extension L_W of minimal operator L_0 generated by (2.1) in $\mathcal{H}$ and boundary conditions with the unitary operator $W : H_{11} \to H_{21}$ and

$$\begin{cases} A_2^{1/2}u(b) = WA_1^{1/2}u(a), \\ A_1^{1/2}u(a) \in Ker(-B_1)^{1/2}, A_2^{1/2}u(b) \in Ker(B_2)^{1/2} \end{cases}$$

will be investigated. Also, it will be considered that $0 \in \sigma_p((-B_{11})^{1/2}) \cap \sigma_p((B_{21})^{1/2})$.

Theorem 3. *Letting L_W be a normal extension of L_0, then the point spectrum $\sigma_p(L_W)$ of L_W is the following form*

$$\sigma_p(L_W) = \sigma_p(B_{10}) \cup \sigma_p(B_{20}).$$

Proof. Consider a problem for the point spectrum for the normal extension L_W, i.e.

$$L_W u(t) = \lambda u(t), u(t) \in D(L_W), \lambda \in \mathbb{C}.$$

It can be written $u(t) = (u_1(t), u_2(t)) = (u_{10}(t) + u_{11}(t), u_{20}(t) + u_{21}(t))$, $u_{ki}(t) \in H_{ki}$, $k = 1,2$, $i = 0,1$. Also, since the restriction operator A_{k1} on H_{k1}, $k = 1,2$ have a bounded inverse, $u'_{k1}(t)$ exists. The solution of this problem in $\mathcal{H}$

$$\begin{aligned} u_{11}(t) &= e^{-A_{11}^{-1}(B_1-\lambda)(t-a)} f_1, t < a, f_1 \in H_{11}, \\ u_{21}(t) &= e^{-A_{21}^{-1}(B_2-\lambda)(t-b)} f_2, t > b, f_2 \in H_{21}, \\ A_2^{1/2} f_2 &= WA_1^{1/2} f_1, \\ B_1 u_{10}(t) &= \lambda u_{10}(t), B_2 u_{20}(t) = \lambda u_{20}(t). \end{aligned}$$

Also, $A_1^{1/2} f_1 \in Ker(B_1)$ and $A_2^{1/2} f_2 \in Ker(B_2)$, from Theorem 1 it must be

$$\begin{aligned} u_{11}(t) &= e^{\lambda A_{11}^{-1}(t-a)} f_1, t < a, f_1 \in H_{11}, \\ u_{21}(t) &= e^{\lambda A_{21}^{-1}(t-b)} f_2, t > b, f_2 \in H_{21}, \\ A_2^{1/2} f_2 &= WA_1^{1/2} f_1, \\ B_1 u_{10}(t) &= \lambda u_{10}(t), B_2 u_{20}(t) = \lambda u_{20}(t). \end{aligned}$$

But $(u_{11}(t), u_{21}(t)) \in \mathcal{H}$ iff $f_k = 0$, $k = 1,2$, as a result of this

$$\sigma_p(L_W) = \sigma_p(B_{10}) \cup \sigma_p(B_{20}).$$

□

Theorem 4. *Letting L_W be any normal extension of L_0, then the continuous spectrum $\sigma_c(L_W)$ of L_W is in form*

$$\sigma_c(L_W) = i\mathbb{R} \cup \sigma_c(B_{10}) \cup \sigma_c(B_{20}).$$

Proof. It is known that the spectrum of normal operators [8],

$$\sigma(L_W) \subset \sigma(ReL_W) + i\sigma(ImL_W),$$

where ReL_W and ImL_W are real part and imaginary part of L_W. If $\lambda = \lambda_r + i\lambda_i \in \sigma_c(L_W)$, then $\lambda_r \in \sigma(ReL_W)$ and $\lambda_i \in \sigma(ImL_W)$.

Let us deal with the solution of the following problem of the normal extension L_W,

$$L_W u(t) = \lambda u(t) + f(t),\, u(t) \in D(L_W),\, f(t) \in \mathcal{H},\, \lambda \in \mathbb{C}.$$

For $u(t) = (u_1(t), u_2(t)) = (u_{11}(t) + u_{10}(t), u_{21}(t) + u_{20}(t)),\ u_{ki}(t) \in H_{ki}$, $k = 1,2,\, i = 0,1$

$$\begin{aligned}
A_1 u'_{11}(t) + B_1 u_{11}(t) &= \lambda u_{11}(t) + f_{11}(t),\, u_{11}(t), f_{11}(t) \in L^2(H_{11}, (-\infty, a)),\\
A_2 u'_{21}(t) + B_2 u_{21}(t) &= \lambda u_{21}(t) + f_{21}(t),\, u_{21}(t), f_{21}(t) \in L^2(H_{21}, (b, \infty)),\\
A_2^{1/2} u_{21}(b) &= W A_1^{1/2} u_{11}(a),\, u_{11}(a) \in Ker(-B_1)^{1/2},\, u_{21}(b) \in Ker(B_2)^{1/2},\\
B_{10} u_{10}(t) &= \lambda u_{10}(t) + f_{10}(t),\, u_{10}(t), f_{10}(t) \in L^2(H_{10}, (-\infty, a)),\\
B_{20} u_{20}(t) &= \lambda u_{20}(t) + f_{10}(t),\, u_{20}(t), f_{20}(t) \in L^2(H_{20}, (b, \infty)).
\end{aligned}$$

In this case, when $\lambda_r \neq 0$ or $\lambda_r \in \sigma_c(B_{10}) \cup \sigma_c(B_{20})$ there isn't any solution of this problem. Also, because of the boundary conditions and Theorem 1 the general solution is in the following form

$$u_{11}(t) = e^{i\lambda_i A_{11}^{-1}(t-a)} f_1 - \int_t^a e^{-A_{11}^{-1}(B_1 - i\lambda_i)(t-s)} A_{11}^{-1} f_{11}(s)\, ds,\ t < a,\, f_1 \in H_{11},$$

$$u_{21}(t) = e^{i\lambda_i A_{21}^{-1}(t-b)} f_2 + \int_b^t e^{-A_{21}^{-1}(B_2 - i\lambda_i)(t-s)} A_{21}^{-1} f_{21}(s)\, ds,\ t > b,\, f_2 \in H_{21}.$$

On the other hand, there exists a function

$$f(t) = (e^{(1+i\lambda_i)A_{11}^{-1}(t-a)} f_1^*, e^{(-1+i\lambda_i)A_{21}^{-1}(t-b)} f_2^*),$$

$A_{k1}^{-1} f_k^* \in Ker(B_{k1})$, $k = 1,2$ in $\mathcal{H}$, then

$$\begin{aligned}
u_{11}(t) &= e^{i\lambda_i A_{11}^{-1}(t-a)} f_1 - \int_t^a e^{i\lambda_i A_{11}^{-1}(t-a)} e^{A_{11}^{-1}(s-a)} A_{11}^{-1} f_1^* ds\\
&= e^{i\lambda_i A_{11}^{-1}(t-a)} f_1 + e^{i\lambda_i A_{11}^{-1}(t-a)} (e^{A_{11}^{-1}(t-a)} - E) f_1^*,\ t < a,\, f_{1\lambda_i} \in H_{11},\\
u_{21}(t) &= e^{i\lambda_i A_{21}^{-1}(t-b)} f_2 + \int_b^t e^{i\lambda_i A_{21}^{-1}(t-b)} e^{-A_{21}^{-1}(s-b)} A_{21}^{-1} f_2^* ds\\
&= e^{i\lambda_i A_{21}^{-1}(t-b)} f_{2\lambda_i} - e^{i\lambda_i A_{21}^{-1}(t-b)} (e^{-A_{11}^{-1}(t-b)} - E) f_2^*\ t > b,\, f_2 \in H_{21}.
\end{aligned}$$

Also, $u(t)=(u_{11}(t),u_{21}(t))\in D(L_W)$ iff $f_1^*=f_1$, $f_2^*=-f_2$ and $A_2^{1/2}f_2^*=WA_1^{1/2}f_1^*$. Because of this result for $\alpha\in\mathbb{C}\backslash\{1\}$, we have

$$f_\alpha(t)=(e^{(1+i\lambda_i)A_{11}^{-1}(t-a)}f_1^*,\alpha e^{(-1+i\lambda_i)A_{21}^{-1}(t-b)}f_2^*),$$

where $A_{k1}^{-1}f_k^*\in Ker(B_{k1})$, for $k=1,2$. There is no solution $L_Wu(t)=i\lambda_iu(t)+f_\alpha(t)$ in $D(L_W)$. This result completes of the proof. □

COMPETING INTERESTS

The authors declare that they have no competing interests.

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