POSITIVE SOLUTIONS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON AN INFINITE INTERVAL

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Abstract. This paper is concerned with the existence of positive solutions of second-order impulsive differential equations with integral boundary conditions on an infinite interval. As an application, an example is given to demonstrate our main results.

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1. INTRODUCTION

Consider the following second-order impulsive integral boundary value problem (IBVP) with integral boundary conditions,

\[
\begin{aligned}
\frac{1}{p(t)} \left( p(t)x'(t) \right)' + f(t,x(t),x'(t)) &= 0, \quad \forall t \in J'_+ \\
\Delta x|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, 2, ..., n \\
\Delta x'|_{t=t_k} &= -T_k(x(t_k)), \quad k = 1, 2, ..., n \\
\end{aligned}
\]

where \( J = [0,\infty) \), \( J_+ = (0,\infty) \), \( J'_+ = J_+ \setminus \{t_1, ..., t_n\} \), \( J_0 = [0,t_1) \), \( J_i = (t_i,t_{i+1}] \), \( i = 1, 2, ..., n \), \( \Delta x|_{t=t_k} \) and \( \Delta x'|_{t=t_k} \) denote the jump of \( x(t) \) and \( x'(t) \) at \( t = t_k \), i.e.,

\[
\begin{aligned}
\Delta x|_{t=t_k} &= x(t_k^+) - x(t_k^-), \quad \Delta x'|_{t=t_k} = p(t_k)[x'(t_k^+) - x'(t_k^-)], \\
\end{aligned}
\]

where \( x(t_k^+) \), \( x'(t_k^+) \) and \( x(t_k^-) \), \( x'(t_k^-) \) denote the right-hand limit and left-hand limit of \( x(t) \) and \( x'(t) \) at \( t = t_k \), \( k = 1, 2, ..., n \), respectively.

Throughout this paper, we assume that the following conditions hold;

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for the following nonlinear second-order double impulsive integral boundary value multiple positive solutions.

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solutions for double impulsive IBVP on an infinite interval expect that in [24, 27].

Due to the fact that an infinite interval is noncompact, the discussion about boundary
dynamic equations have been extensively studied, see [11,12,21–23].

and the papers [3, 6, 7, 9–11, 13–17, 19].

[5], Lakshmikantham, et al. [1], Samoilenko and Prestyuk [18], Benchohra, et al. [4]
equations with fixed time of pulses; see the monographs by Bainov and Simeirov

cesses. A significant development was observed in theory of impulsive differential
equations encountered in physics, chemical technology, population dynamics, biotecnology, economics etc. (see [2] and the references there in)
have become more important in recent years due to the appearance of some important
in recent years due to the appearance of some mathematical models of the actual pro-
processes. The existence and multiplicity of positive solutions for linear and nonlinear second-
order impulsive dynamic equations have been extensively studied, see [11,12,21–23].

Due to the fact that an infinite interval is noncompact, the discussion about boundary value
problems on the half-line more complicated, in particular, for impulsive IBVP on an infinite interval, few works were done, see [8,25]. There is not work on positive solutions for double impulsive IBVP on an infinite interval expect that in [24, 27].

In [27], Zhang, Yang and Feng studied the following double impulsive IBVP:

\[
\begin{align*}
\Delta x(t) &= I_k(x(t)), \\
\Delta x'(t) &= I_k(x(t)), \\
x(0) &= \int_0^\infty g(t)x(t)dt,
\end{align*}
\]

Using the fixed point theorem in cones, they obtained criteria for existence of the multiple positive solutions.

In [24], Yu, Wang and Guo discussed the existence and multiple positive solutions for the following nonlinear second-order double impulsive integral boundary value
A function (1.1) if
\[ x(t) = \begin{cases} 1 & \text{for } t \in I, \quad t \neq t_k, \\ 0 & \text{otherwise}, \end{cases} \]
\[ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots \]
\[ \Delta \phi(x)|_{t=t_k} = T_k(x(t_k)), \quad k = 1, 2, \ldots \]
\[ x(0) = \int_0^t g(t)x(t)dt, \quad x'(\infty) = 0. \]

Motivated by the above works, in this study, we consider the existence of two positive solutions for the second-order double impulsive integral boundary value problem (1.1). Our boundary conditions are more general. Hence, these results can be considered as a contribution to this field.

The present paper is organized as follows. In Section 2, we present some preliminaries and lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2. Preliminaries and Lemmas

In this section, we will employ several lemmas to prove the main results in this paper.

Set
\[ PC(J) = \left\{ x : J \rightarrow \mathbb{R} : x \in C(J'), x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k^+) = x(t_k), \quad 1 \leq k \leq n \right\}. \]
\[ PC^1(J) = \left\{ x \in PC(J) : x' \in C(J'), x'(t_k^+) \text{ and } x'(t_k^-) \text{ exist and } x'(t_k^+) = x'(t_k) \right\}. \]
\[ BPC^1(J) = \left\{ x \in PC^1(J) : \lim_{t \to +\infty} x(t) \text{ exists, and } \sup_{t \in J} |x'(t)| < \infty \right\}. \]

It is easy to see that $BPC^1(J)$ is a Banach space with the norm
\[ \|x\| = \sup_{t \in J} \{ |x(t)| + |x'(t)| \}. \]

A function $x \in PC^1(J) \cap C^2(J_+')$ is called a positive solution of the impulsive IBVP (1.1) if $x(t) > 0$ for all $t \in J$ and $x(t)$ satisfies (1.1).

We define a cone $K \subset BPC^1(J)$ as follows:
\[ K = \left\{ x \in BPC^1(J) : x(t) > 0, \quad t \in J_+ \right\}. \]

$K$ is a positive cone in $BPC^1(J)$.

By $\theta$ and $\varphi$ we denote the solutions of the corresponding homogeneous equation
\[ \frac{1}{p(t)}(p(t)\theta'(t))' = 0, \quad t \in (0, \infty), \quad (2.1) \]
under the initial conditions,
\[ \theta(0) = b_1, \quad \lim_{t \to 0^+} p(t)\theta'(t) = a_1, \]
\[ \lim_{t \to +\infty} \theta(t) = b_2, \quad \lim_{t \to +\infty} p(t)\varphi'(t) = -a_2. \quad (2.2) \]
Using the initial conditions (2.2), we can deduce from equation (2.1) for \( \theta(t) \) and \( \varphi(t) \), the following equations:

\[
\theta(t) = b_1 + a_1 \int_0^t \frac{ds}{p(s)}, \quad (2.3)
\]
\[
\varphi(t) = b_2 + a_2 \int_t^\infty \frac{ds}{p(s)}. \quad (2.4)
\]

Let \( G(t,s) \) be the Green Function for (1.1) is given by

\[
G(t,s) = \frac{1}{D} \begin{cases} 
\theta(t)\varphi(s), & 0 \leq t \leq s < \infty, \\
\theta(s)\varphi(t), & 0 \leq s \leq t < \infty,
\end{cases}
\]

where \( \theta(t) \) and \( \varphi(t) \) are given in (2.3) and (2.4) respectively.

**Lemma 1.** Suppose that (H1) – (H6) are satisfied. Then \( x \in PC^1(J) \cap C^2(J_+) \) is a solution of the impulsive IBVP (1.1) if and only if \( x(t) \) is a solution of the following integral equation

\[
x(t) = \int_0^t G(t,s) p(s)f(s,x(s),x'(s))ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(x(s))\psi(s)ds \\
+ \frac{\theta(t)}{D} \int_0^\infty g_2(x(s))\psi(s)ds + \sum_{k=1}^n G(t,t_k)I_k(x(t_k)) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{t=t_k}I_k(x(t_k)),
\]

where \( G(t,s) \) is given by (2.5).

**Remark 1.** Under the conditions (H1) and (H6), the Green function \( G(t,s) \) in equation (2.5) possesses the following properties:

1. \( G(t,s) \) is continuous on \( J_+ \times J_+ \).
2. for each \( s \in J_+ \), \( G(t,s) \) is continuously differentiable on \( J_+ \) except \( t = s \),
3. \( \frac{\partial G(t,s)}{\partial t}|_{t=s^+} - \frac{\partial G(t,s)}{\partial t}|_{t=s^-} = \frac{1}{p(s)}, \)
4. \( G(t,s) \leq G(s,s) < \infty \), and \( G_s(t,s) \leq G_s(t,s)|_{t=s} < \infty, \)
5. \( |G_t(t,s)| \leq \frac{c}{p(t)}G(s,s), \) and \( |G_{tt}(t,s)| \leq \frac{c}{p(t)}G_s(t,s)|_{t=s}, \) where

\[
c = \max\{a_1, a_2\} \frac{1}{\min\{b_1, b_2\}},
\]

6. \( \bar{G}(s) = \lim_{t \to s} G(t,s) = \frac{b_2}{D} \theta(s) \leq G(s,s) < \infty, \)
7. \( \bar{G}'(s) = \lim_{t \to s} G_s(t,s) = \frac{b_2}{D} \theta'(s) \leq G_s(t,s)|_{t=s} < \infty, \)
8. for any \( t \in [a,b] \subset (0,\infty) \) and \( s \in [0,\infty) \), we have

\[
G(t,s) \geq wG(s,s),
\]
where
\[ w = \min \left\{ \frac{b_1 + a_1 B(0, a)}{b_1 + a_1 B(0, \infty)}, \frac{b_2 + a_2 B(b, \infty)}{b_2 + a_2 B(0, \infty)} \right\}. \]  

(2.7)

Obviously, \( 0 < w < 1 \).

Define
\[ (Tx)(t) = \int_0^\infty G(t, s)p(s)f(s, x(s), x'(s))ds + \frac{\Phi(t)}{D} \int_0^\infty g_1(x(s))\psi(s)ds \]
\[ + \frac{\Theta(t)}{D} \int_0^\infty g_2(x(s))\psi(s)ds + \sum_{k=1}^n G(t, t_k)I_k(x(t_k)) \]  
\[ + \sum_{k=1}^n p(t_k)G_s(t, s)|_{s=t_k} I_k(x(t_k)), \]  

(2.8)

where \( G \) is defined by as in (2.5).

Obviously, the impulsive IBVP (1.1) has a solution \( x \) if and only if \( x \in K \) is a fixed point of the operator \( T \) defined by (2.8).

It is convenient to list the following condition which is to be used in our theorems:

\[(H7) \quad 0 < \int_0^\infty G(s, s)p(s)k(s)ds < \infty.\]

As we know that the Ascoli-Arzela Theorem does not hold in infinite interval \( J \), we need the following compactness criterion:

**Lemma 2** (\([20]\)). Let \( M \subset BPC^1(J) \). Then \( M \) is relatively compact in \( BPC^1(J) \) if the following conditions hold.

(i) \( M \) is uniformly bounded in \( BPC^1(J) \).

(ii) The function belonging to \( M \) are equicontinuous on any compact interval of \([0, \infty)\).

(iii) The functions from \( M \) are equiconvergent, that is, for any given \( \varepsilon > 0 \), there exist a \( T = T(\varepsilon) > 0 \) such that \( |f(t) - f(\infty)| < \varepsilon \) for any \( t > T \), \( f \in M \).

The main tool of this work is a fixed point theorem in cones.

**Lemma 3** (\([26]\)). Let \( X \) be an Banach space and \( K \) is a positive cone in \( X \). Assume that \( \Omega_1, \Omega_2 \) are open subsets of \( X \) with \( 0 \in \Omega_1 \), \( \overline{\Omega}_1 \subset \Omega_2 \). Let \( T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K \) be a completely continuous operator such that

(i) \( \|Tx\| \leq \|x\| \) for all \( x \in K \cap \partial \Omega_1 \).

(ii) There exists a \( \Phi \in K \) such that \( x \neq Tx + \lambda \Phi \), for all \( x \in K \cap \partial \Omega_2 \) and \( \lambda > 0 \).

Then \( T \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

**Lemma 4.** If \((H1)-(H7)\) are satisfied, then for any bounded open set \( \Omega \subset BPC^1(J) \), \( T : K \cap \overline{\Omega} \rightarrow K \) is a completely continuous operator.
Proof. For any bounded open set \( \Omega \subset BPC^1(J) \), there exists a constant \( M > 0 \) such that \( \|x\| \leq M \) for any \( x \in \overline{\Omega} \).

First, we show \( T : K \cap \overline{\Omega} \to K \) is well defined. Let \( x \in K \cap \overline{\Omega} \). From (H2), (H3) and (H4), we have

\[
S_M = \sup\{S_1, S_2, S_3, S_4, S_5\},
\]

where

\[
S_1 = \sup\{h(x, y) : |x| + |y| \leq M\} < \infty, \quad S_2 = \sup\{l_k(x) : 0 \leq x \leq M\},
\]

\[
S_3 = \sup\{l_k(x) : 0 \leq x \leq M\}, \quad S_4 = \sup\{g_1(x) : 0 \leq x \leq M\},
\]

\[
S_5 = \sup\{g_2(x) : 0 \leq x \leq M\}.
\]

Let \( t_1, t_2 \in J, t_1 < t_2 \), then

\[
\int_0^\infty |G(t_1, s) - G(t_2, s)||p(s)k(s)ds \leq 2 \int_0^\infty G(s, s)p(s)k(s)ds < \infty. \tag{2.10}
\]

Hence, by the Lebesgue dominated convergence theorem, we have for any \( t_1, t_2 \in J, x \in K \cap \overline{\Omega} \), we have

\[
|\langle Tx \rangle(t_1) - \langle Tx \rangle(t_2)|
\]

\[
\leq \int_0^\infty |G(t_1, s) - G(t_2, s)||p(s)f(s, x(s), x'(s))|ds
\]

\[
+ \frac{\|\varphi(t_1) - \varphi(t_2)\|}{D} \int_0^\infty g_1(x(s))|\psi(s)|ds + \frac{\|\theta(t_1) - \theta(t_2)\|}{D} \int_0^\infty g_2(x(s))|\psi(s)|ds
\]

\[
+ \sum_{k=1}^n |G(t_1, t_k) - G(t_2, t_k)||l_k(x(t_k))
\]

\[
+ \sum_{k=1}^n |p(t_k)G_s(t, s)\big|_{t=t_k} - p(t_k)G_s(t, s)\big|_{t=t_k}|l_k(x(t_k))
\]

\[
\leq S_M \left\{ \int_0^\infty |G(t_1, s) - G(t_2, s)||p(s)k(s)ds
\right.
\]

\[
+ \frac{\|\varphi(t_1) - \varphi(t_2)\|}{D} \int_0^\infty |\psi(s)|ds + \sum_{k=1}^n |G(t_1, t_k) - G(t_2, t_k)|
\]

\[
+ \frac{1}{D} \sum_{t_{k-1} \leq t \leq t_k} p(t_k)\theta'(t_k)|\varphi(t_1) - \varphi(t_2)|
\left. + \frac{1}{D} \sum_{t_{k-1} \leq t \leq t_k} p(t_k)|\varphi'(t_k)|\theta(t_1) - \theta(t_2)|
\right.
\]

\[
+ \frac{1}{D} \sum_{t_{k-1} \leq t \leq t_k} p(t_k)|\theta'(t_k)|\varphi(t_1) - \theta'(t_k)|\varphi(t_2)|
\}
\]

\[
\to 0 \quad \text{as} \quad t_1 \to t_2, \quad \tag{2.11}
\]

\[
|\langle Tx \rangle'(t_1) - \langle Tx \rangle'(t_2)|
\]
Therefore, suppose that $0 < \alpha_1 < 1$ and $\alpha_2 < 1$. Then, for any $t \in (0, T)$, we have
\[
\frac{\alpha_1}{D} \int_0^t V(s)ds + \frac{\alpha_2}{Dp(t)} \int_t^T \Phi(s)ds < 1.
\]
Thus, \( \lim_{t \to 0} T(x(t)) = 0 \).

(2.12)

Thus, \( T \in BPC^1(J) \). We can show that \( T \in BPC^1(J) \).

Then by \( (H5), (H7) \), the properties \( (5), (6), (7) \) of Remark 1 and the Lebesgue dominated convergence theorem, we have
\[
\lim_{t \to \infty} T(x(t)) = \int_0^\infty G(s) \frac{\Phi(s)}{D} \int_t^\infty g_1(x(s)) \phi(s)ds + \frac{\Phi(\infty)}{D} \int_0^\infty g_2(x(s)) \psi(s)ds + \int_0^\infty H(s) \theta(s)ds,
\]
and
\[
\frac{\theta(t)}{D} \int_0^\infty g_2(x(s)) \phi(s)ds + \sum_{k=1}^n G(t_k) \theta(t_k) < \infty.
\]

Therefore, \( \sup_{t \in J} |T(x(t))| < \infty \). Hence \( T : K \cap \overline{\Omega} \to K \) is well defined.

Next, we prove that \( T \) is continuous. Let \( x_n \to x \) in \( K \cap \overline{\Omega} \), then \( \|x_n\| \leq M \), \( n = 1, 2, \ldots \). We will show that \( T x_n \to T x \). For any \( \varepsilon > 0 \), by \( (H7) \) there exists a constant \( A_0 > 0 \) such that
\[
S_m \int_{A_0}^\infty G(s) \frac{\Phi(s)}{D} \int_t^\infty g_1(x(s)) \phi(s)ds \leq \frac{\varepsilon}{12}.
\]
On the other hand, by the continuity of \( f(t,u,v) \) on \((0,A_0] \times J_+ \times \mathbb{R} \), the continuities of \( g_1, g_2 \) on \( J_+ \) and the continuities of \( I_k, T_k \) on \( J_+ \), for the above \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that, for any \( u, v, u_1, v_1 \), satisfying \( |u| + |v| < M \), and \( |u_1| + |v_1| < M \),

\[
|u - u_1| + |v - v_1| < \delta,
\]

and

\[
|f(s,u,v) - f(s,u_1,v_1)| < \frac{\epsilon}{6} \left( \int_0^{A_0} G(s,s)p(s) ds \right)^{-1},
\]

\[
|g_1(u) - g_1(u_1)| < \frac{\epsilon}{6} \left( \frac{\phi(0)}{D} \int_0^{\infty} \psi(s) ds \right)^{-1},
\]

\[
|g_2(u) - g_2(u_1)| < \frac{\epsilon}{6} \left( \frac{\theta(\infty)}{D} \int_0^{\infty} \psi(s) ds \right)^{-1}, \tag{2.16}
\]

\[
|I_k(u(t_k)) - I_k(u_1(t_k))| < \frac{\epsilon}{6} \left( \sum_{k=1}^{n} p(t_k) G_s(t,s) \right)_{t=t_k}^{-1},
\]

\[
|T_k(u(t_k)) - T_k(u_1(t_k))| < \frac{\epsilon}{6} \left( \sum_{k=1}^{n} G(t_k,t_k) \right)^{-1}.
\]

From the fact that \( ||x_n - x|| \to 0 \) as \( n \to \infty \), for above \( \delta \), there exists a sufficiently large number \( N \) such that, when \( n > N \), we have, for \( t \in (0,A_0] \),

\[
|x_n(t) - x(t)| + |x_n'(t) - x'(t)| \leq ||x_n - x|| < \delta. \tag{2.17}
\]

By (2.15)-(2.16), we have, for \( n > N \),

\[
|(T_{x_n})(t) - (Tx)(t)| \leq \left| \int_0^{A_0} G(s,s)p(s) \left[ f(s,x_n(s),x'_n(s)) - f(s,x(s),x'(s)) \right] ds \right|
\]

\[
+ \frac{\phi(0)}{D} \int_0^{\infty} \left[ g_1(x_n(s)) - g_1(x(s)) \right] \psi(s) ds
\]

\[
+ \frac{\theta(\infty)}{D} \int_0^{\infty} \left[ g_2(x_n(s)) - g_2(x(s)) \right] \psi(s) ds
\]

\[
+ \sum_{k=1}^{n} G(t_k,t_k) \left[ I_k(x_n(t_k)) - I_k(x(t_k)) \right]
\]

\[
+ \sum_{k=1}^{n} p(t_k) G_s(t,s)_{t=t_k} \left[ I_k(x_n(t_k)) - I_k(x(t_k)) \right] \right| 
\]

\[
\leq \int_0^{A_0} G(s,s)p(s) \left[ f(s,x_n(s),y_n(s)) - f(s,x(s),y(s)) \right] ds
\]

\[
+ 2S_M \int_0^{\infty} G(s,s)p(s)k(s) ds
\]

\[
+ \frac{\phi(0)}{D} \int_0^{\infty} \left| g_1(x_n(s)) - g_1(x(s)) \right| \psi(s) ds.
\]
we have

\[
\frac{1}{D} \int_0^\infty |g_2(x_n(s)) - g_2(x(s))| \psi(s) ds + \sum_{k=1}^n G(t_k, t_k)|I_k(x_n(t_k)) - I_k(x(t_k))| + \sum_{k=1}^n p(t_k)G_s(t, s)|_{t=t_k} |I_k(x_n(t_k)) - I_k(x(t_k))| \\
\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\]

Similarly, we can see that when \(|x_n - x| \to 0\) as \(n \to \infty\), \(|(Tx_n)'(t) - (Tx)'(t)| \to 0\) as \(n \to \infty\). This implies that \(T\) is a continuous operator.

Finally we show that \(T : K \cap \bar{\Omega} \to K\) is a compact operator. In fact for any bounded set \(D \subset \bar{\Omega}\), there exists a constant \(R > 0\) such that \(|x| \leq R\) for any \(x \in K \cap D\). Hence, we have

\[
|(Tx)(t)| \leq \int_0^\infty G(s, s)p(s)f(s, x(s), x'(s))ds + \frac{\varphi(0)}{D} \int_0^\infty g_1(x(s))\psi(s)ds \\
+ \frac{\theta(\infty)}{D} \int_0^\infty g_2(x(s))\psi(s)ds \\
+ \sum_{k=1}^n G(t_k, t_k)|\mathcal{J}_k(x(t_k))| + \sum_{k=1}^n p(t_k)G_s(t, s)|_{t=t_k} |\mathcal{I}_k(x(t_k))| \\
\leq S_R \left( \int_0^\infty G(s, s)p(s)k(s)ds + \frac{\varphi(0)}{D} \int_0^\infty \psi(s)ds \\
+ \frac{\theta(\infty)}{D} \int_0^\infty \psi(s)ds + \sum_{k=1}^n G(t_k, t_k) + \sum_{k=1}^n p(t_k)G_s(t, s)|_{t=t_k} \right) < \infty.
\]

From (2.14), we get \(|(Tx)'(t)| < \infty\) for \(t \in J\). Therefore, \(T(K \cap D)\) is uniformly bounded in \(BPC^1(J)\).

Given \(r > 0\), for any \(t_1, t_2 \in J, x \in K \cap D\), as the proof of (2.11), (2.12), we can get \(||(Tx)(t_1) - (Tx)(t_2)|| \to 0\) and \(||(Tx)'(t_1) - (Tx)'(t_2)|| \to 0\) as \(t_1 \to t_2\), i.e., \(||(Tx)(t_1) - (Tx)(t_2)|| \to 0\) as \(t_1 \to t_2\). Thus \(F = \{Tx : x \in K \cap D\}\) is equicontinuous on \([0, r]\). Since \(r > 0\) arbitrary, \(F\) is locally equicontinuous on \(J_1\). By (H5), (H7) the properties (5), (6), (7) and the Lebesgue dominated converges theorem, we get

\[
||(Tx)(t) - (Tx)(\infty)|| \leq S_R \left( \int_0^\infty |G(t, s) - \mathcal{G}(s)|p(s)k(s)ds \\
+ \frac{\varphi(t) - \varphi(\infty)}{D} \int_0^\infty \psi(s)ds + \frac{\theta(t) - \theta(\infty)}{D} \int_0^\infty \psi(s)ds \\
+ \sum_{k=1}^n |G(t, t_k) - \mathcal{G}(t_k)| + \frac{1}{D} \sum_{t \leq t_k} p(t_k)\theta'(t_k)|\varphi(t) - \varphi(\infty)|
\]
\[ + \frac{1}{D} \sum_{t \leq t_k} p(t_k) \left| \theta(t) \phi(t_k) - \theta(t_k) \phi'(\infty) \right| \]
\[ \to 0 \text{ as } t \to \infty \]

and
\[
| (Tx)'(t) - (Tx)'(\infty) | \leq S_K \left\{ a_2 \frac{1}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_0^t \theta(s) p(s) k(s) ds \\
+ a_1 \frac{1}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_t^\infty \phi(s) p(s) k(s) ds \\
+ a_2 \frac{1}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_0^\infty \psi(s) ds \\
+ a_1 \frac{1}{D} \sum_{t \leq t_k} [\phi(t_k) + p(t_k) \phi'(t_k)] \\
+ a_2 \frac{1}{D} \sum_{t \leq t_k} [\theta(t_k) + p(t_k) \theta'(t_k)] \right\} \to 0 \text{ as } t \to \infty. \tag{2.18}
\]

Hence \( T(K \cap D) \) is equiconvergent in \( BPC^1(J) \). By Lemma 2, we have that \( F \) is relatively compact in \( BPC^1(J) \). Therefore, \( T : K \cap \bar{\Omega} \to K \) is completely continuous. \( \square \)

3. MAIN RESULTS

For convenience and simplicity in the following discussion, we use following notations:
\[
\begin{align*}
 f_0 &= \lim_{|x|+|y| \to 0} \inf_{t \in [a,b]} \frac{f(t,x,y)}{|x| + |y|}, \\
 g_{ih} &= \lim_{x \to 0} \inf_{x+y = 1} g_i(x), \quad (1 \leq i \leq 2), \\
 g_{ih} &= \lim_{x \to 0} \inf_{x+y = 1} g_i(x), \quad (1 \leq i \leq 2), \\
 h^q &= \lim_{|x|+|y| \to q} \sup_{|x| + |y|} \frac{h(x,y)}{|x| + |y|}, \\
 g_i^q &= \lim_{x \to q} \sup_{x+y \to q} \frac{g_i(x)}{x}, \quad (1 \leq i \leq 2), \\
 I_0(k) &= \lim_{x \to 0} \inf_{x} \frac{I_k(x)}{x}, \\
 I_\infty(k) &= \lim_{x \to \infty} \inf_{x} \frac{I_k(x)}{x}, \\
 \bar{I}_0(k) &= \lim_{x \to 0} \inf_{x} \frac{\bar{I}_k(x)}{x}, \\
 \bar{I}_\infty(k) &= \lim_{x \to \infty} \inf_{x} \frac{\bar{I}_k(x)}{x},
\end{align*}
\]
Define the open sets 

\[ I^q(k) = \limsup_{x \to q} \frac{I_k(x)}{x}, \quad T^q(k) = \limsup_{x \to q} \frac{T_k(x)}{x}. \]

**Theorem 1.** Assume that the conditions (H1)-(H7) are satisfied. Then the impulsive IBVP (1.1) has at least two positive solutions satisfying \( 0 < \|x_1\| < q < \|x_2\| \) if for \([a, b] \subset (0, \infty)\), the following conditions hold:

(A1) \( w \left( \frac{f_0}{a} \int_a^b g(s) \frac{p(s)ds}{D} + \frac{\min \{ \varphi(\infty), \theta(0) \}}{D} (g_{10} + g_{20}) \int_a^b \psi(s)ds \right. \)

\[ + \sum_{k=1}^n G(t_k,t_k)T_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} T_0(k) \right) > 1, \]

\( w \left( \frac{f_0}{a} \int_a^b g(s) \frac{p(s)ds}{D} + \frac{\min \{ \varphi(\infty), \theta(0) \}}{D} (g_{10} + g_{20}) \int_a^b \psi(s)ds \right. \)

\[ + \sum_{k=1}^n G(t_k,t_k)T_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} T_0(k) \right) > 1, \]

(A2) There exists a \( q > 0 \) such that

\[ 1 + c \sup_{r \in J} \frac{1}{p(r)} \left\{ \frac{g_1 q}{h} \int_0^\infty G(s,s) p(s)k(s)ds + \frac{\max \{ \varphi(0), \theta(\infty) \}}{D} (g_{10} + g_{20}) \right. \]

\[ \times \int_0^\infty \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)T_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} T_0(k) \right\} < 1, \]

for all \( 0 < |x| + |x'| \leq q, \ a.e. \ t \in [0, \infty). \)

**Proof.** By the definition of \( f_0, I_0, T_0, g_{10} \) and \( g_{20} \) for any \( \varepsilon > 0 \), there exist \( r \in (0, q) \) such that,

\[ f(t, x, y) \geq (1 - \varepsilon)f_0(|x| + |y|), \quad (|x| + |y| \leq r, \ t \in [a, b]) \]

\[ g_1(x) \geq (1 - \varepsilon)g_{10} x, \quad g_2(x) \geq (1 - \varepsilon)g_{20} x, \]

\[ I_k(x) \geq (1 - \varepsilon)I_0(k) x, \quad T_k(x) \geq (1 - \varepsilon)T_0(k) x, \]

\[ (1 - \varepsilon)w \left( \frac{f_0}{a} \int_a^b g(s) \frac{p(s)ds}{D} + \frac{\min \{ \varphi(\infty), \theta(0) \}}{D} (g_{10} + g_{20}) \int_a^b \psi(s)ds \right. \]

\[ + \sum_{k=1}^n G(t_k,t_k)T_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} T_0(k) \right) \geq 1. \]

Define the open sets

\[ \Omega_r = \{ x \in BPC^1(J) : \|x\| < r \}. \]

Let \( \Phi = 1 \) then, \( \Phi \in K \). Now we prove that

\[ x \neq T x + \lambda \Phi, \ \forall x \in K \cap \partial \Omega_r, \ \lambda > 0 \quad (3.1) \]
We assume that \( x_0 = T x_0 + \lambda_0 \Phi \) where \( x_0 \in K \cap \partial \Omega_r \) and \( \lambda_0 > 0 \). Let \( \mu = \min_{t \in [a, b]} x_0(t) \), then for any \( t \in [a, b] \), we have

\[
x_0(t) = T x_0(t) + \lambda_0 \Phi
\]

\[
= \int_0^\infty G(t, s)p(s)f(s, x_0(s), x_0'(s))ds
\]

\[
+ \frac{\Phi(t)}{D} \int_0^\infty g_1(x_0(s))\psi(s)ds + \frac{\theta(t)}{D} \int_0^\infty g_2(x_0(s))\psi(s)ds
\]

\[
+ \sum_{k=1}^n G(t, t_k)I_k(x_0(t_k)) + \sum_{k=1}^n p(t_k)G_s(t, s)|_{s=t_k}I_k(x_0(t_k)) + \lambda_0
\]

\[
> w(1 - \varepsilon) \left\{ \int_a^b G(s, s)p(s)ds + \frac{\min\{\Phi(\infty), \theta(0)\}}{D} (g_1 + g_2) \right\} + \lambda_0
\]

\[
\geq \mu + \lambda_0.
\]

This implies \( \mu > \mu + \lambda_0 \) a contradiction. Therefore (3.1) holds.

By the definition of \( f_\infty, L_\infty, I_\infty, g_{1, x} \) and \( g_{2, x} \) for any \( \varepsilon > 0 \), there exist \( R > q \) such that

\[
f(t, x, y) \geq (1 - \varepsilon)f_\infty(|x| + |y|), \quad (|x| + |y| \geq R, \quad t \in [a, b]),
\]

\[
g_1(x) \geq (1 - \varepsilon)g_{1, x}, \quad (\forall |x| \geq R),
\]

\[
l_k(x) \geq (1 - \varepsilon)l_\infty(k)x, \quad \tilde{I}_k(x) \geq (1 - \varepsilon)\tilde{I}_\infty(k)x,
\]

\[
(1 - \varepsilon)w \left( \int_0^\infty G(s, s)p(s)ds + \frac{\min\{\Phi(\infty), \theta(0)\}}{D} (g_1 + g_2) \right) \int_0^\infty \psi(s)ds
\]

\[
+ \sum_{k=1}^n G(t, t_k)I_\infty(k) + \sum_{k=1}^n p(t_k)G_s(t, s)|_{s=t_k}I_\infty(k) \geq 1.
\]

Define the open sets

\[
\Omega_R = \{ x \in BPC^1(J) : \|x\| < R \}.
\]

As the proof of (3.1), we can get that

\[
x \neq Tx + \lambda \Phi, \quad \forall x \in K \cap \partial \Omega_R, \quad \lambda > 0.
\]

On the other hand, for any \( \varepsilon > 0 \), choose \( q \) in (A2) such that

\[
(1 + \varepsilon) \left[ 1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h^q \int_0^\infty G(s, s)p(s)k(s)ds + \frac{\max\{\theta(0), \theta(\infty)\}}{D} (g_1^q + g_2^q) \right\}
\]


Define the open sets
\[ \Omega_q = \{ x \in BPC^1(J) : ||x|| < q \}. \] (3.5)

By the definition of \( h^i, I^q, T^l, g^i_1 \) and \( g^i_2 \), for the above \( \epsilon > 0 \), there exists \( \delta > 0 \), when \( |x|, |x| + |y| \in (q - \delta, q + \delta) \); thus, we get
\[
\begin{align*}
    h(x, y) &\leq (1 + \epsilon)h^i(|x| + |y|), \\
g_i(x) &\leq (1 + \epsilon)g_i^1 x, \quad 1 \leq i \leq 2 \\
I_k(x) &\leq (1 + \epsilon)I^q(k)x, \\
T_k(x) &\leq (1 + \epsilon)T^q(k)x.
\end{align*}
\]

Then for any \( x \in K \cap \partial \Omega_q \) and \( t \in J \) we obtain that
\[
\begin{align*}
    |(Tx)(t)| + |(Tx)'(t)| &\leq \int_0^\infty G(s, s)p(s)f(s, x(s), x'(s))ds \\
    &+ \frac{\max \{ \varphi(0), \Theta(\infty) \}}{D} \int_0^\infty |g_1(x(s))| + g_2(x(s))|\psi ds \\
    &+ \sum_{k=1}^n G(t_k, t_k)T_k(x(t_k)) + \sum_{k=1}^n p(t_k)G_s(t, s)|_{t=t_k}I_k(x(t_k)) \\
    &+ c \sup_{t \in J} \frac{1}{p(t)} \int_0^\infty G(s, s)p(s)f(s, x(s), y(s))ds \\
    &+ \frac{\max \{ a_1, a_2 \}}{D} \sup_{t \in J} \frac{1}{p(t)} \int_0^\infty |g_1(x(s))| + g_2(x(s))|\psi ds \\
    &+ \sum_{k=1}^n p(t_k)G_s(t, s)|_{t=t_k}I_k(x(t_k)) \\
    &\leq \left[ 1 + c \sup_{t \in J} \frac{1}{p(t)} \int_0^\infty G(s, s)p(s)\kappa(h(x(s), x'(s)))ds \\
    &+ \frac{\max \{ \varphi(0), \Theta(\infty) \}}{D} + \frac{\max \{ a_1, a_2 \}}{D} \sup_{t \in J} \frac{1}{p(t)} \right] \\
    &\times \int_0^\infty [g_1(x(s)) + g_2(x(s))]|\psi|ds \\
    &+ \left[ 1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left[ \sum_{k=1}^n G(t_k, t_k)T_k(x(t_k)) \\
    &+ \sum_{k=1}^n p(t_k)G_s(t, s)|_{t=t_k}I_k(x(t_k)) \right].
\end{align*}
\]
Using a similar proof of Theorem 1, we can get the following theorem.

**Theorem 2.** Assume that the conditions (H1)-(H7) are satisfied. Then the impulsive IBVP (1.1) has at least two positive solutions satisfying $0 < \|x_1\| < q < \|x_2\|$ if for $[a, b] \subset (0, \infty)$, the following conditions hold;

\[
\begin{align*}
(\text{A3}) & \quad \left[ 1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h_1^0 \int_0^{\infty} G(s, s) p(s) k(s) ds + \max \{ \phi(0), \theta(\infty) \} \frac{D}{D} (g_1^0 + g_2^0) \right. \\
& \hspace{1cm} \left. \times \int_0^{\infty} \psi(s) ds + \sum_{k=1}^{n} G(t_k, t_k) \bar{T}^0(k) + \sum_{k=1}^{n} p(t_k) G_s(t, s) |_{t=\bar{t}_k} T^0(k) \right\} < 1, \nonumber \\
(\text{A4}) & \quad \left[ 1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h_2^\infty \int_0^{\infty} G(s, s) p(s) k(s) ds + \max \{ \phi(0), \theta(\infty) \} \frac{D}{D} (g_1^\infty + g_2^\infty) \right. \\
& \hspace{1cm} \left. \times \int_0^{\infty} \psi(s) ds + \sum_{k=1}^{n} G(t_k, t_k) \bar{T}^\infty(k) + \sum_{k=1}^{n} p(t_k) G_s(t, s) |_{t=\bar{t}_k} T^\infty(k) \right\} < 1, \\
& \quad \text{There exists a } q > 0 \text{ such that} \\
& \quad w \left\{ f_\bar{b} \int_a^b G(s, s) p(s) ds + \min \{ \phi(\infty), \theta(0) \} \frac{D}{D} (g_1 + g_2) \right. \\
& \hspace{1cm} \left. + \sum_{k=1}^{n} G(t_k, t_k) \bar{T}_q(k) + \sum_{k=1}^{n} p(t_k) G_s(t, s) |_{t=\bar{t}_k} I_q(k) \right\} > 1, \\
& \quad \text{for all } 0 < |x| + |x'| \leq q, \text{ a.e. } t \in [0, \infty).
\end{align*}
\]

4. Example

To illustrate how our main results can be used in practice we present the following example.
Consider the following boundary value problem:

\[
\begin{cases}
  e^{-t}(e^{t}x'(t))' + f(t,x(t),x'(t)) = 0, & t \in J_+, t \neq \frac{1}{2}, \\
  \Delta x|_{t=\frac{1}{2}} = 100x(\frac{1}{2}), \\
  \Delta x'|_{t=\frac{1}{2}} = 1 - e^{-\frac{1}{4}}x^{-1}(\frac{1}{2}), \\
  x(0) = \frac{1}{40\pi} \int_{0}^{\infty} \frac{x^2(s)}{1+s^2} ds, \\
  \lim_{t \to \infty} e^t x'(t) = \frac{1}{8000\pi} \int_{0}^{\infty} \frac{x^2(s)}{1+s^2} ds,
\end{cases}
\]

(4.1)

where

\[
f(t,x(t),x'(t)) = \frac{e^{-t}}{e^t - 2}, \quad a_1 = 1, a_2 = 0, \quad b_1 = 1, \quad b_2 = 1, \quad p(t) = e^t, \quad \psi(t) = \frac{1}{1+t^2},
\]

\[
I_k(x(t)) = \frac{x(t)}{100}, \quad \overline{I}_k(x(t)) = \frac{x^{-1}(t)}{2 - e^{-\frac{1}{2}}}, \quad g_1(x(s)) = \frac{x^2(s)}{40\pi}, \quad g_2(x(s)) = \frac{x^4(s)}{8000\pi}.
\]

Set \( k(t) = \frac{e^{-t}}{e^t - 2}, \) \( h(x(t), y(t)) = 1 \) and \( q = 10. \) It follows from a direct calculation that

\[
g_1^q = \frac{1}{4\pi}, \quad g_2^q = \frac{1}{8\pi}, \quad g_{1\infty} = \infty, \quad g_{2\infty} = \infty, \quad g_{10} = 0, \quad g_{20} = 0, \quad I^q(k) = \frac{1}{100},
\]

\[
\overline{I}^q(k) = \frac{1}{100(1-e^{-\frac{1}{2}})}, \quad I_0(k) = \frac{1}{100}, \quad I_0(k) = \infty, \quad L_{e}(k) = \frac{1}{100}, \quad \overline{I}_{e}(k) = 0.
\]

Furthermore, \( f_0 = \infty, \) \( f_{\infty} = 1, \) \( \int_{0}^{\infty} G(s,s)p(s)k(s)ds = 1 \) and \( \int_{0}^{\infty} \psi(s)ds = \frac{\pi}{2}. \) Thus \( (A1) \) and \( (A2) \) are satisfied. Therefore, by Theorem 1, the impulsive IBVP (4.1) has at least two positive solutions \( x_1, x_2 \) satisfying \( 0 < \|x_1\| < 10 < \|x_2\|. \)

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